

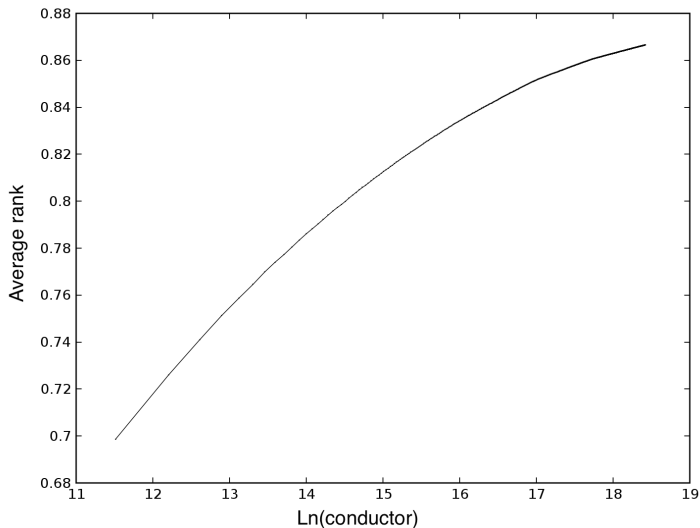
# Databases of elliptic curves ordered by height and distributions of Selmer groups and ranks

Jennifer S. Balakrishnan, Wei Ho, Nathan Kaplan, Simon Spicer,  
William Stein, and James Weigandt

ANTS-XII, Kaiserslautern

# Average rank versus conductor [BMSW 2007]

from the 136,832,795 curves of conductor  $< 10^8$  in the Stein-Watkins db:

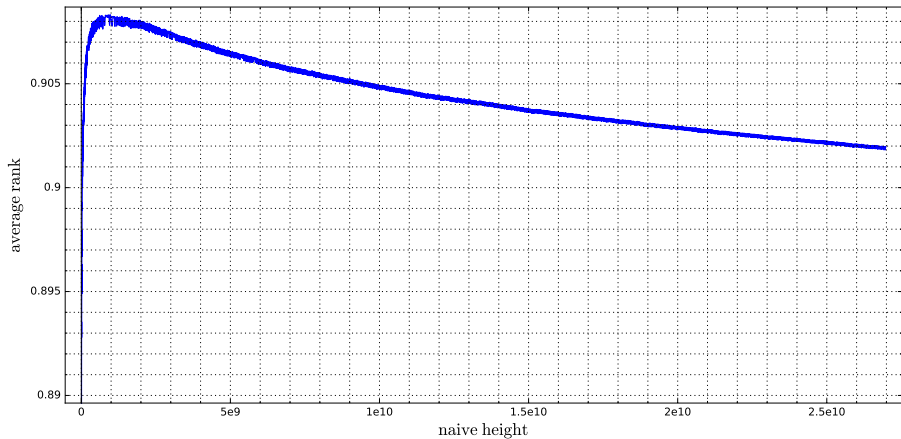


## Average rank versus height

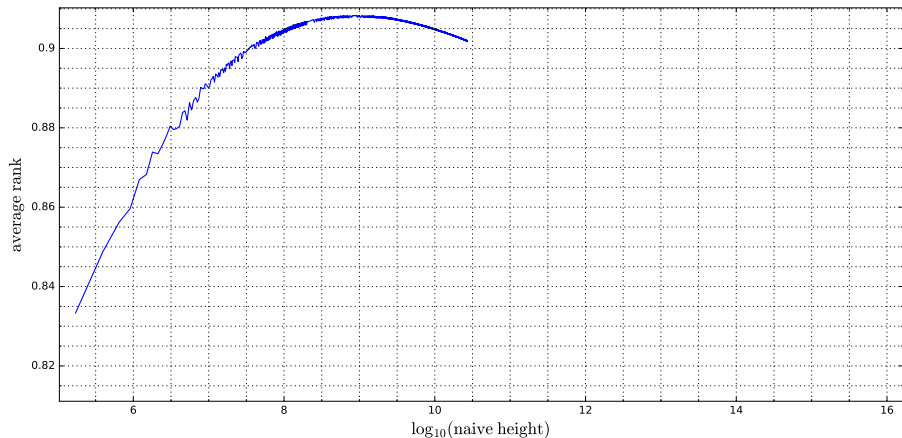
from the 238,764,310 curves of height  $< 2.7 \cdot 10^{10}$  in new database:

# Average rank versus height

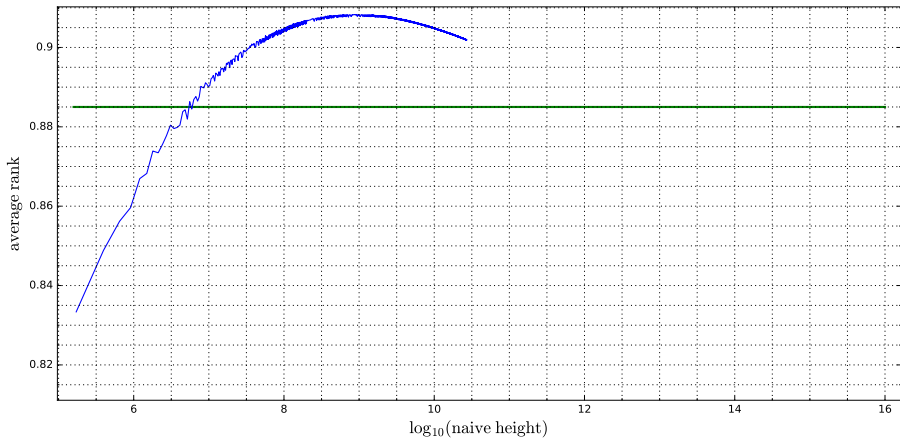
from the 238,764,310 curves of height  $< 2.7 \cdot 10^{10}$  in new database:



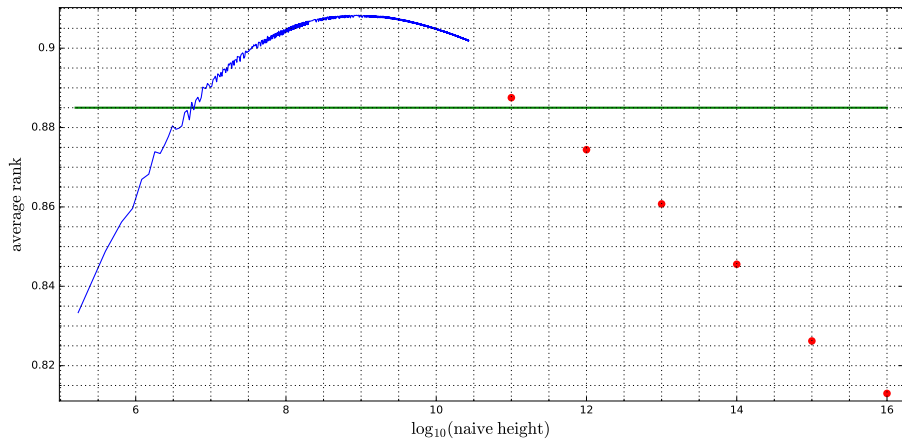
# Average rank ( $\log_{10}$ scale)



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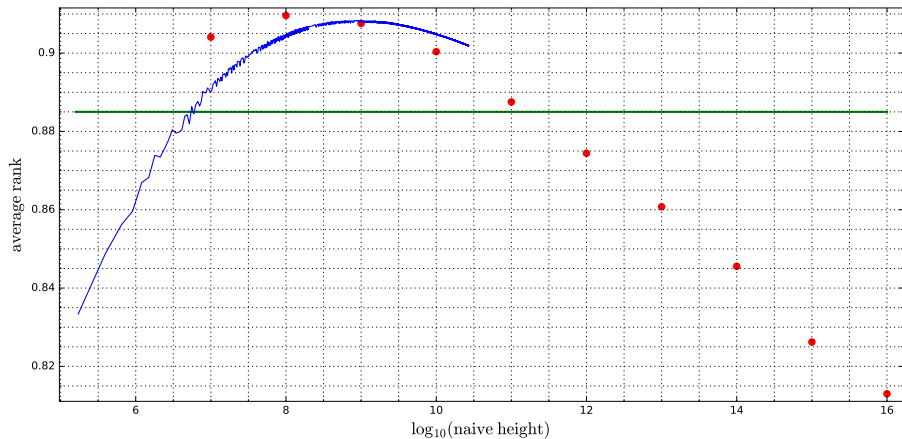


# Average rank ( $\log_{10}$ scale)



red points: samples of 100,000 curves

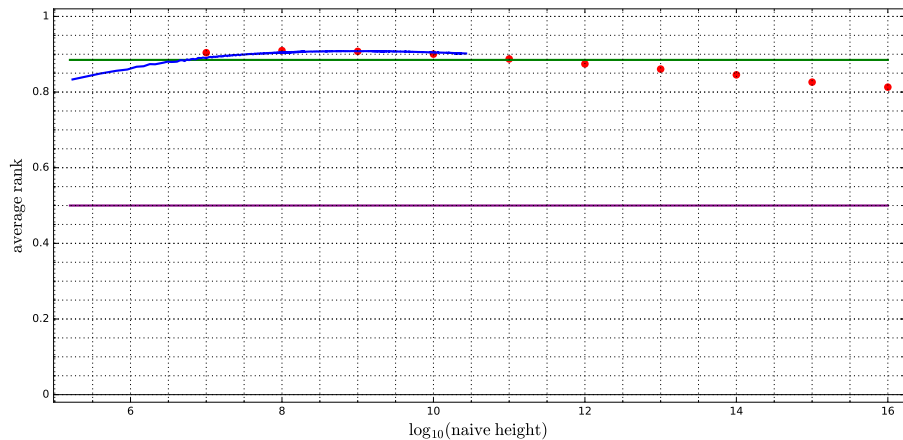
# Average rank ( $\log_{10}$ scale)



running average versus average in a range/sample



# Average rank ( $\log_{10}$ scale) – rescaled



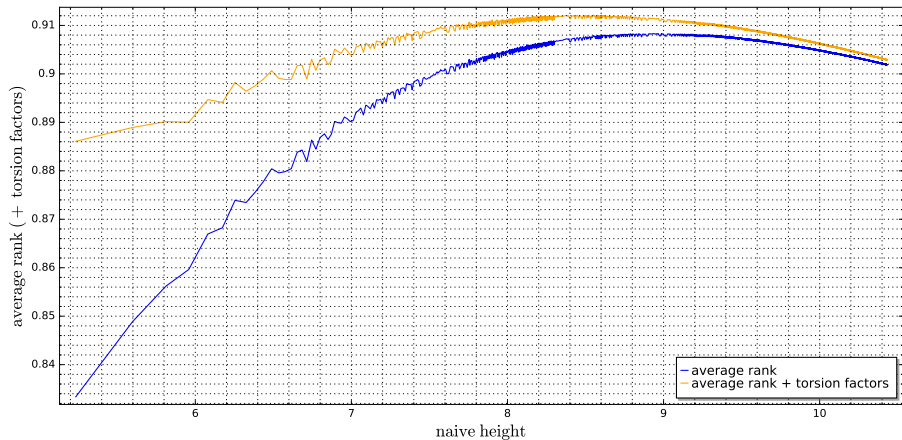
# Distribution of rank

Rank	# Curves	% Curves
0	78,039,852	32.685%
1	113,128,980	47.381%
2	40,949,307	17.151%
3	6,259,159	2.615%
4	380,519	0.159%
5	6481	0.003%
6	12	0.0005%
total	238,764,310	

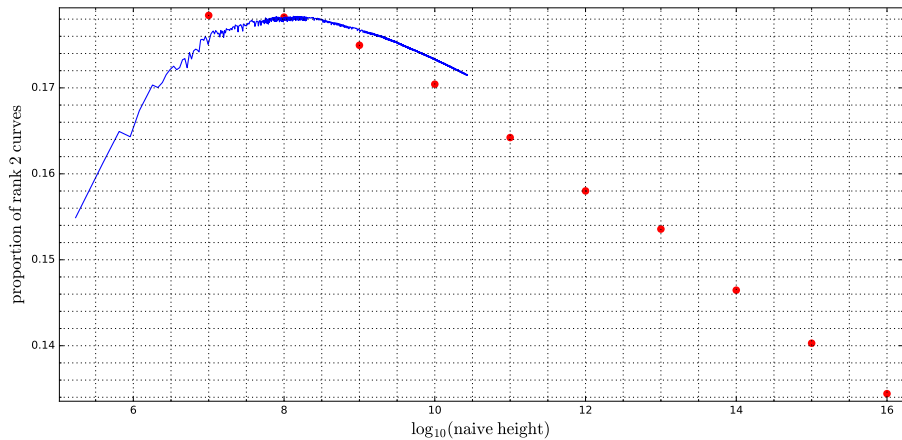
average rank = 0.901976

# Why does average rank increase at first?

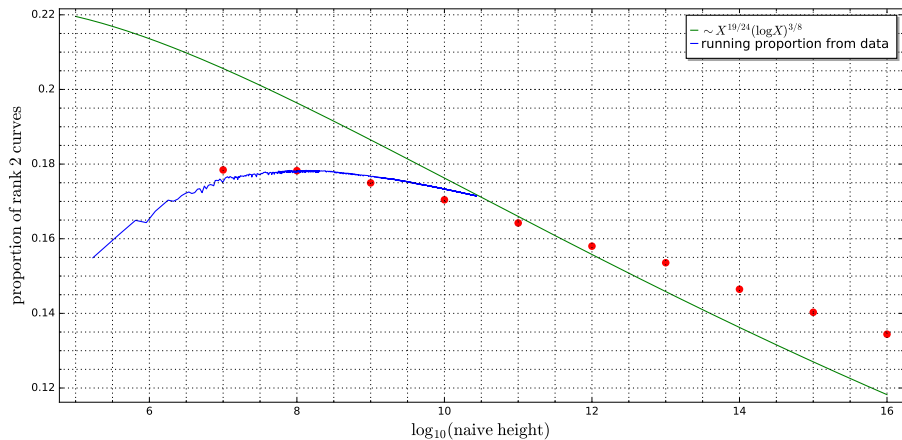
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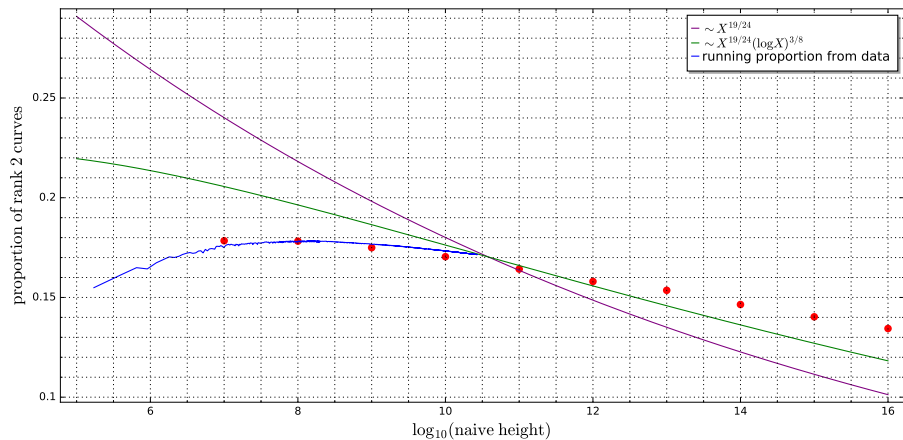
# Proportion of rank 2 curves



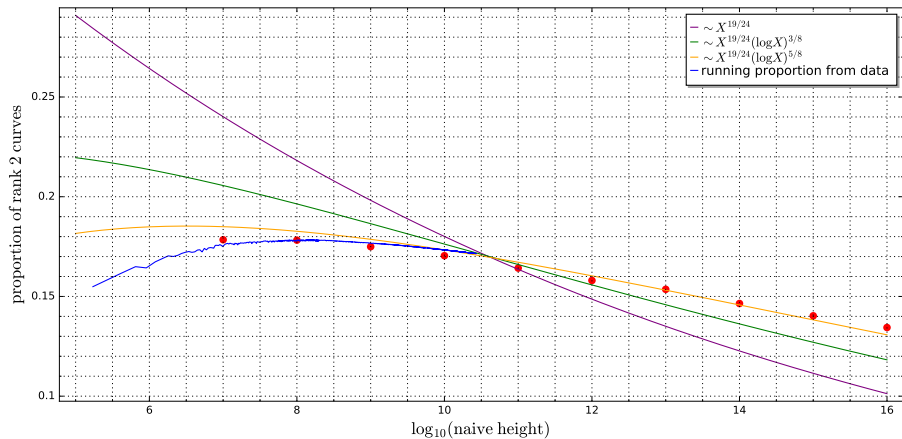
# Proportion of rank 2 curves: data versus conjecture



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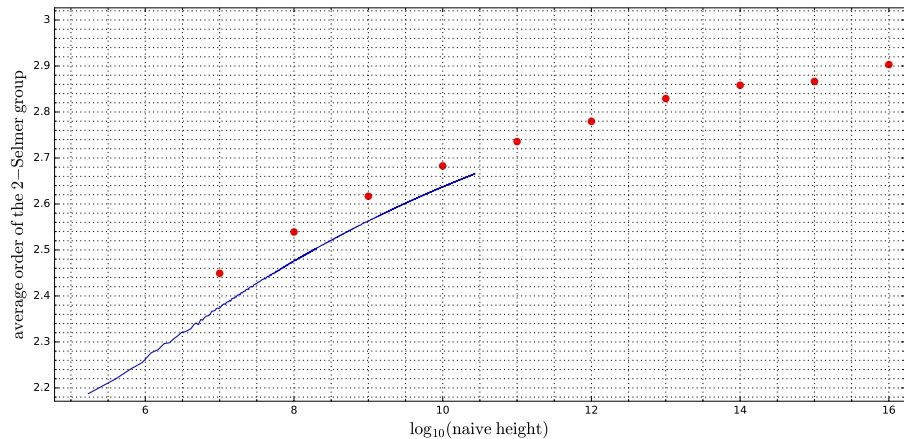


# Proportion of rank 2 curves: data versus conjecture

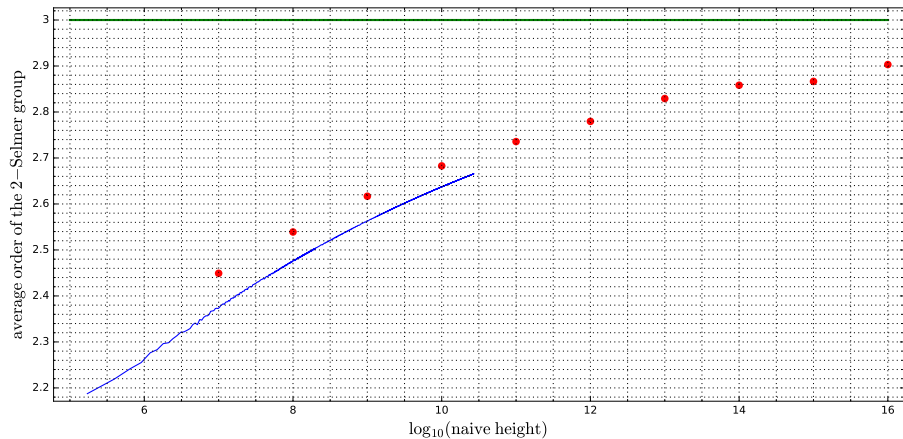




# Average order of the 2-Selmer group



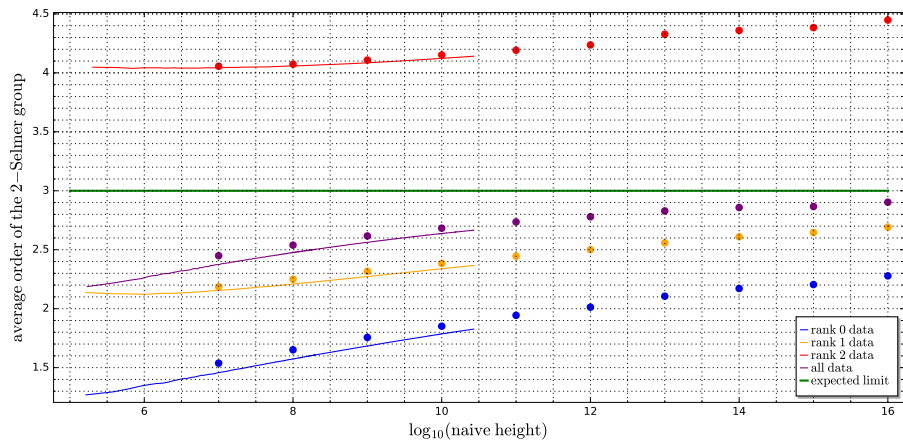
# Average order of the 2-Selmer group



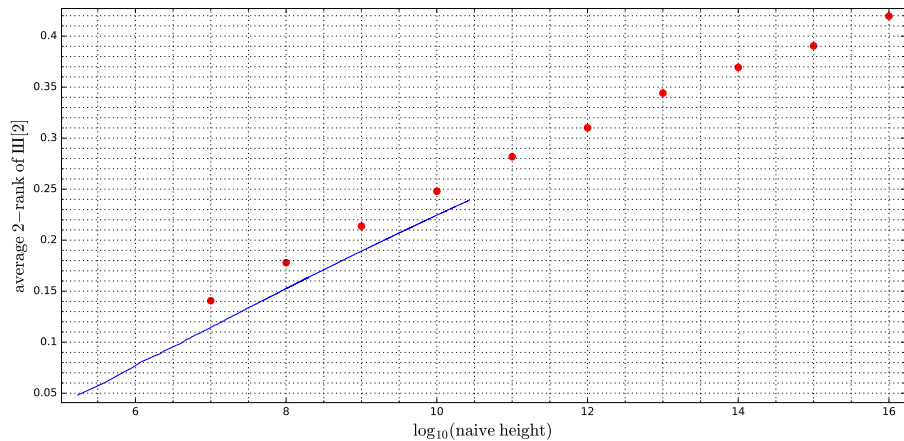
# Distribution of 2-Selmer groups: data versus heuristics

2-rank of $\text{Sel}_2$	predicted % Poonen-Rains	% of all curves $\text{ht} < 2.7 \cdot 10^{10}$	% for rank 0	% for rank 1
0	20.97	23.81	72.85	0
1	41.94	44.49	0.0865	93.83
2	27.96	25.78	26.93	0.098
3	7.989	5.511	0.0209	6.064
4	1.065	0.395	0.111	0.00351
5	0.0687	0.00643	$1.41 \cdot 10^{-5}$	0.00173

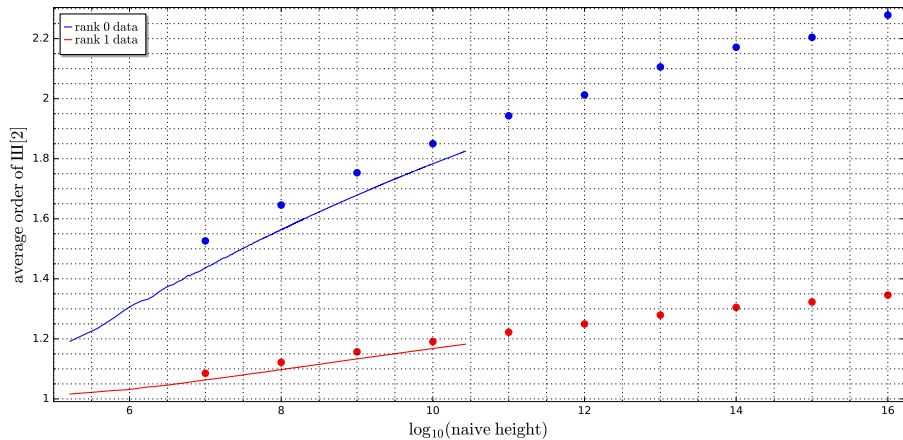
# Average order of the 2-Selmer group, by rank



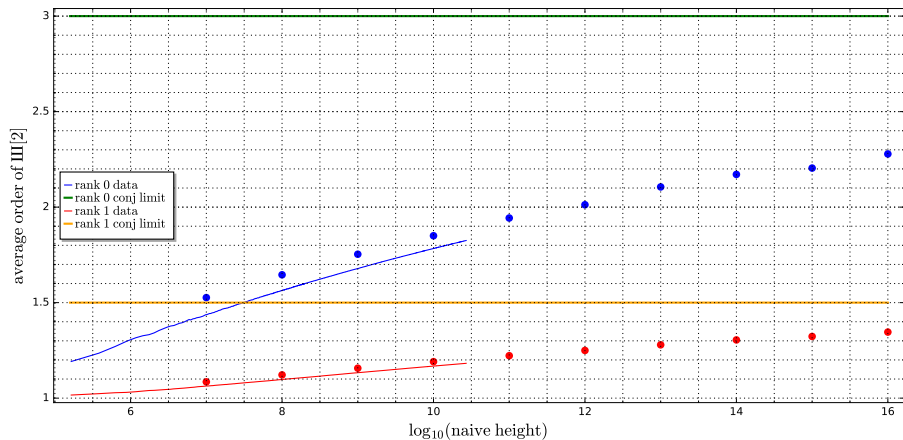
# Average 2-rank of $\text{III}[2]$



# Average order of $\text{III}[2]$ : rank 0 versus rank 1



# Average order of $\text{III}[2]$ : rank 0 versus rank 1

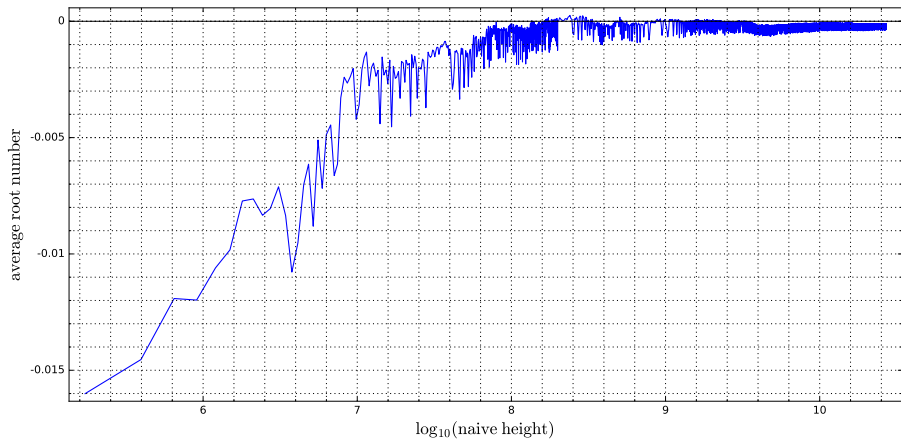


# Distribution of $\text{III}[2]$ : data versus heuristics

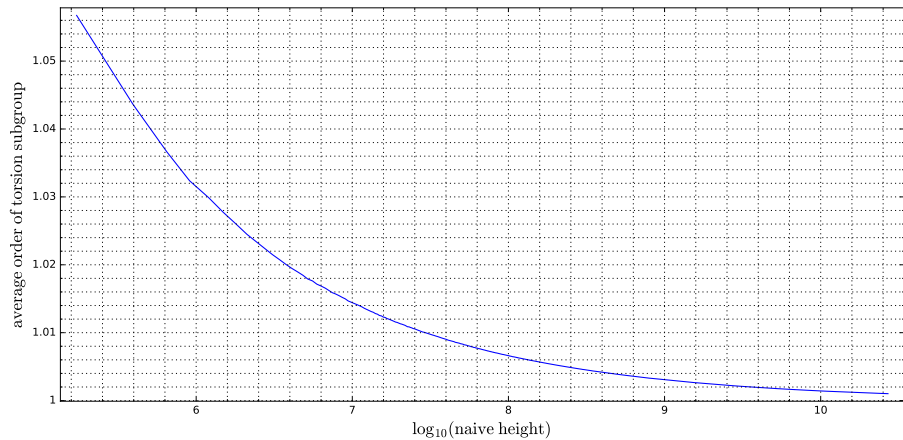
rank	2-rank of $\text{III}[2]$	predicted prop. Delaunay	proportion for all curves of height $< 2.7 \cdot 10^{10}$
0	0	0.4194	0.7294
0	2	0.5592	0.2695
0	4	0.0213	0.0011
1	0	0.8388	0.9393
1	2	0.1598	0.0607



# Average root number



# Average order of the torsion subgroup



# Distribution of torsion subgroups

Torsion group	# Curves	Rank 0	Rank 1	Rank 2	Rank $\geq 3$
trivial	238528817 (99.901%)	0.327	0.474	0.172	0.028
$\mathbb{Z}/2\mathbb{Z}$	233153	0.359	0.492	0.141	0.008
$\mathbb{Z}/3\mathbb{Z}$	1020	0.463	0.466	0.072	0
$\mathbb{Z}/4\mathbb{Z}$	257	0.521	0.463	0.016	0
$\mathbb{Z}/6\mathbb{Z}$	23	0.870	0.130	0	0
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	1035	0.453	0.496	0.051	0
$\mathbb{Z}/5\mathbb{Z}$	1	1	0	0	0
$\mathbb{Z}/7\mathbb{Z}$	1	1	0	0	0
$\mathbb{Z}/9\mathbb{Z}$	1	1	0	0	0
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	2	1	0	0	0

# Different properties

Property	# Curves	Rk 0	Rk 1	Rk 2	Rk $\geq 3$
height $\leq 2.7 \cdot 10^{10}$	238764310	0.327	0.474	0.172	0.028
conductor $\leq 10^8$	4908673	0.305	0.474	0.193	0.027

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$\text{rk}_2(\text{Sel}_2) = 6$ & $\text{rk} = 4$	3				

# Different properties

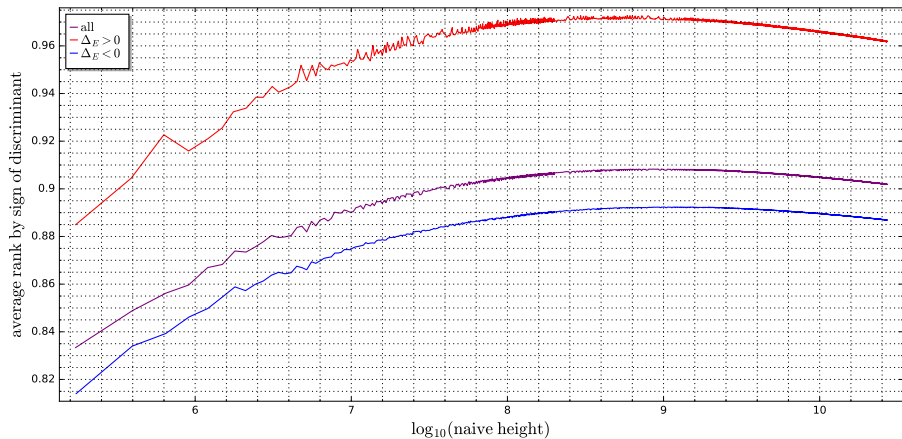
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$\text{rk}_2(\text{Sel}_2) = 6$ & $\text{rk} = 4$	3				
$E$ has CM	65732	0.328	0.474	0.170	0.028

# Different properties

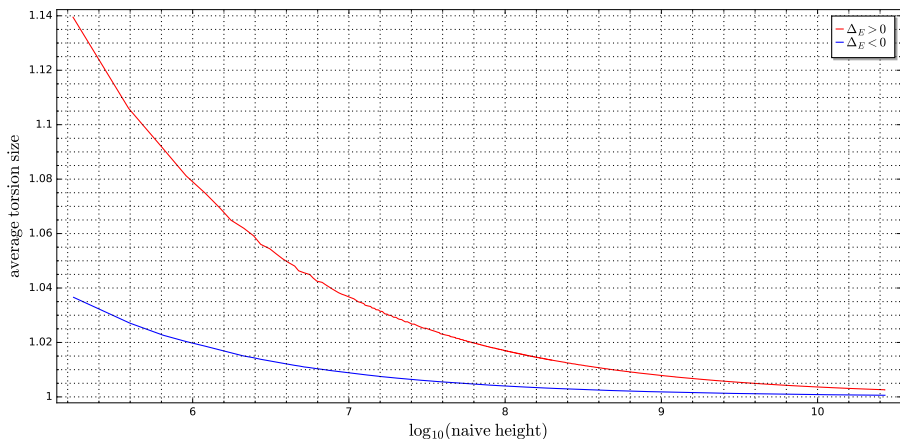
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$E$ has CM	65732	0.328	0.474	0.170	0.028
$\Delta_E > 0$	47738800 (19.994%)	0.305	0.466	0.192	0.036
$\Delta_E < 0$	191025510 (80.006%)	0.332	0.476	0.166	0.026



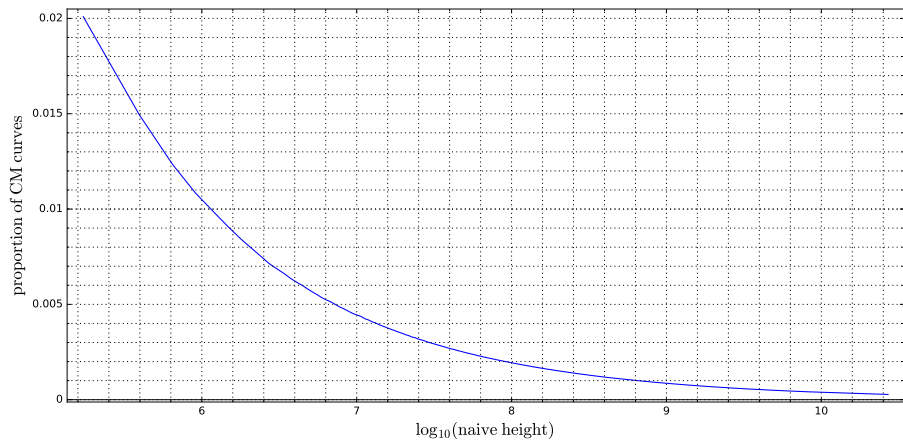
# Average rank: positive versus negative discriminant



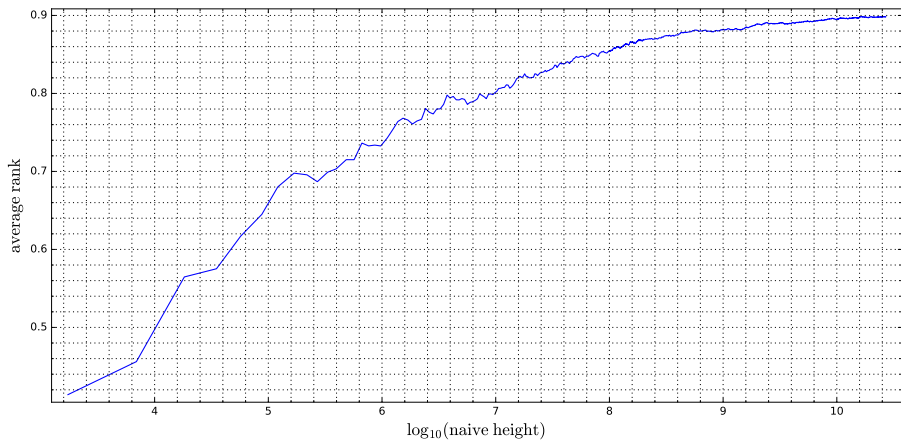
# Average order of the torsion subgroup: positive versus negative discriminant



# Proportion of curves with complex multiplication



# Average rank of curves with complex multiplication



# Family $\mathcal{F}_1$ with one marked point: height $< 10^8$

Rank	# Curves	% Curves	2-Selmer Rank	# Curves
0	15783	0.44%	0	364
1	1239600	34.48	1	1145633
2	1724209	47.96	2	1727290
3	564784	15.71	3	657323
4	49642	1.38	4	63235
5	872	0.024	5	1045
6	1	0.000028	6	1
total	3594891		total	3594891

average rank = 1.83185

average # Sel<sub>2</sub> = 4.31296

overlap with main db: 693601 curves (19.29% of these)