

STABLE COOPERATIONS OF WITT MOTIVIC COHOMOLOGY

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ABSTRACT. In this paper we begin the study of the (dual) Steenrod algebra of the Witt motivic cohomology spectrum $H_W\mathbb{Z}$ by determining the algebra structure of $H_W\mathbb{Z}_{**}H_W\mathbb{Z}$ over fields k of characteristic not 2 which are extensions of fields F with $K_2^M(F)/2 = 0$. For example, this includes all fields of odd characteristic, as well as fields that are extensions of quadratically closed fields of characteristic 0. After inverting η , this computes the $HW := H_W\mathbb{Z}[\eta^{-1}]$ -algebra $HW_{**}HW$. In particular, for the given base fields, this implies the HW -module structure of $HW \wedge HW$ which has been computed by Bachmann and Hopkins in [BH21][Bac21].

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1. INTRODUCTION

In classical homotopy theory, the determination of the structure of the Steenrod algebra $\mathcal{A}_p$ was key to the development of the field. It lead, among other results, to Adams' resolution of the Hopf invariant one problem [Ada60], Thom's classification of smooth manifolds up to cobordism [Tho54], accessibility of the Adams and Adams-Novikov spectral sequences [Ada58] [Nov67] and the many computations that come with it.

In motivic homotopy theory, Voevodsky computed the Steenrod algebra, and its dual, of motivic cohomology mod p [Voe03b]. For $p = 2$ the dual motivic Steenrod algebra is given by

$$H\mathbb{Z}/2_{**}H\mathbb{Z}/2 \cong H\mathbb{Z}/2_{**}[\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots]/(\tau_i^2 - \rho\tau_{i+1} - (\tau + \rho\tau_0)\xi_{i+1}),$$

where $\rho \in \pi_{-1, -1}(\mathbb{S})$ corresponds to $-1 \in \mathcal{O}^\times$. The computation of the Steenrod algebra was central to the establishment of the Milnor, and more generally Bloch-Kato, conjectures, now also known as the norm-residue isomorphism. The resolution of the Milnor conjecture [Voe03a] and the Beilinson-Lichtenbaum conjecture [SV00, Section 6] implies, see for example [KRØ20, 7.2], that the coefficient ring of $H\mathbb{Z}/2$ is given by a polynomial ring over Milnor K-theory mod 2 in one variable τ with degree $|\tau| = (0, -1)$:

$$H\mathbb{Z}/2_{**} \cong \underline{K}_{**}^M/2[\tau].$$

Let $\mathcal{SH}(k)$ be the stable motivic homotopy category of Morel and Voevodsky over a perfect field k as described in [Mor03]. Given a spectrum $E \in \mathcal{SH}(k)$, we denote by $\pi_i(E)_* := \pi_{i-*,-*}E$ the i -line of the stable homotopy groups of E . The 0-line of the stable homotopy groups of the motivic sphere spectrum is given by Milnor-Witt K-theory [Mor12] [Mor04a]

$$\pi_0(\mathbb{S})_* \cong K_*^{\text{MW}}(k).$$

As a consequence of the Milnor conjecture, it decomposes via the pullback square [Mor04b, Thm 5.3][Mor12, Chapter 5]

$$(1.1) \quad \begin{array}{ccc} K_*^{\text{MW}}(k) & \longrightarrow & K_*^{\text{M}}(k) \\ \downarrow & & \downarrow \\ K_*^{\text{W}}(k) & \longrightarrow & k_*^{\text{M}}(k), \end{array}$$

where K_*^{M} denotes the graded ring of Milnor K-theory, K_*^{W} denotes the graded ring of Witt K-theory and k_*^{M} denotes the graded ring of Milnor K-theory modulo 2. In degree $(0, 0)$ this is the pullback square

$$\begin{array}{ccc} \text{GW}(k) & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ \text{W}(k) & \longrightarrow & \mathbb{Z}/2. \end{array}$$

It relates the Grothendieck-Witt ring $\text{GW}(k)$ to the Witt ring of symmetric bilinear forms $\text{W}(k)$. These theories are very arithmetic in nature. Let us consider this at the level of motivic Eilenberg-MacLane spectra. The square (1.1) can be realized as homotopy groups of the pullback square of the corresponding homotopy modules in the heart of $\mathcal{SH}(k)$ with respect to Morel's homotopy t -structure [Mor03]. By taking effective covers one arrives at a homotopy pullback square

$$(1.2) \quad \begin{array}{ccc} \text{H}\tilde{\mathbb{Z}} & \longrightarrow & \text{H}\mathbb{Z} \\ \downarrow & & \downarrow \\ \text{H}_\text{W}\mathbb{Z} & \longrightarrow & \text{H}\mathbb{Z}/2. \end{array}$$

Motivic cohomology, represented by the motivic spectrum $\text{H}\mathbb{Z} \in \mathcal{SH}(k)$, is in various ways similar to singular cohomology. The spectrum $\text{H}\mathbb{Z}/2$ is the mod 2 version of $\text{H}\mathbb{Z}$. The other two spectra, Milnor-Witt motivic cohomology $\text{H}\tilde{\mathbb{Z}}$ and Witt motivic cohomology $\text{H}_\text{W}\mathbb{Z}$, are less classical and arise from new phenomena in $\mathcal{SH}(k)$. People have intensively studied η -inverted variants of $\text{H}_\text{W}\mathbb{Z}$, $\text{H}\text{W} := \text{H}_\text{W}\mathbb{Z}[\eta^{-1}]$, the structure of $\text{H}\text{W} \wedge \text{H}\text{W}$, as well as structures of η -periodic relatives such as Balmer-Witt K-theory, $\text{KW} = \text{KO}[\eta^{-1}]$ [Bac17][RSØ18][Ana17][ARØ20][KRØ20] [BH21][Bac21][FH20].

Our goal is to study the spectra on the left hand side of the square (1.2), specifically the algebras of self-cohomology (Steenrod algebra) and their self-homology (dual Steenrod algebra). We hope this will shed some light on the Witt part of $\mathcal{SH}(k)$. In this paper we start by determining the algebra structure of $\text{H}_\text{W}\mathbb{Z}_{**}\text{H}_\text{W}\mathbb{Z}$ for certain base fields. We plan this to be the first in a series of studies to determine the entire structure.

The coefficient ring $\text{H}_\text{W}\mathbb{Z}_{**}$ is known. Combining our knowledge of the coefficient ring $\text{H}\mathbb{Z}/2_{**}$ with the resolution of Morel's structure conjecture [Bac17, Thm. 17], we get

$$\text{H}_\text{W}\mathbb{Z}_{**} \cong \underline{K}_{**}^{\text{W}}[\tau]/(\tau\eta).$$

The element η is already present in $\pi_{1,1}\mathbb{S}$ and originates from the unstable Hopf map $\mathbb{A}^2 \setminus 0 \rightarrow \mathbb{P}^1$.

We now describe the main theorem of this paper. Let $I \subset \mathbb{Z}_{\geq 2}$ be a fixed finite set of integers greater or equal to 2, and define

$$\begin{aligned} \bar{\xi}_I &:= \prod_{i \in I} \bar{\xi}_i, & c(I) &:= \sum_{i \in I} \overline{\xi_{i-1}^2} \bar{\xi}_{I \setminus i}, & c_1(I) &:= \tau_0 \bar{\xi}_I + \tau_1 c(I), \\ t(I) &:= \tau \bar{\xi}_I, & t_1(I) &:= \tau \tau_1 \bar{\xi}_I, \end{aligned}$$

where a line above an element stands for a conjugated element. We will see that there exist canonical lifts of these elements under the canonical map $\text{H}_\text{W}\mathbb{Z}_{**}\text{H}_\text{W}\mathbb{Z} \rightarrow \text{H}\mathbb{Z}/2_{**}\text{H}\mathbb{Z}/2$. By abuse of notation we denote the lifted elements also as defined above. Furthermore, there exist lifts $t_j \in \text{H}_\text{W}\mathbb{Z}_{**}\text{H}_\text{W}\mathbb{Z}$ of $\tau_j \in \text{H}\mathbb{Z}/2_{**}\text{H}\mathbb{Z}/2$ for $j \geq 2$ and $s \in \text{H}_\text{W}\mathbb{Z}_{**}\text{H}_\text{W}\mathbb{Z}$ of $\tau_0^3 \tau_1 \in \text{H}\mathbb{Z}/2_{**}\text{H}\mathbb{Z}/2$. With this notation, the main theorem takes the following form.

Theorem 1.3 (see Thm. 5.4). *Let the base field k be an extension of a field F satisfying $k_2^M(F) = 0$. Then we have an isomorphism of right $H_W\mathbb{Z}_{**}$ -algebras*

$$H_W\mathbb{Z}_{**}H_W\mathbb{Z} \cong H_W\mathbb{Z}_{**}[s, t_2, t_3, \dots, c(I), t(I), c_1(I), t_1(I)]/(\dots),$$

with index $I \subset \mathbb{Z}_{\geq 2}$ running over all finite subsets of $\mathbb{Z}_{\geq 2}$. The ideal of η -torsion is generated by $c(I), c_1(I), t(I), t_1(I)$. The ideal $(\dots)$ is generated by the relations in Lemma 4.22, Lemma 5.2 and

$$\begin{aligned} \eta c(I) &= \eta c_1(I) = \eta t(I) = \eta t_1(I) = c(\emptyset) = 0, & \tau_0^4 &= c(2)\tau^2, & t_j^2 &= \rho t_{j+1} + \xi_{j+1}\tau, \\ t(I)s &= t_1(I)\tau_0^3, & t_1(I)s &= \tau_0^3 t(I)(\rho t_2 + \xi_2\tau), & s^2 &= \tau_0^2(\rho t_2 + \xi_2\tau)c(2)\tau^2, \\ sc(I) &= \tau_0^3 c_1(I) + \tau t(I)c(2), & sc_1(I) &= \tau_0^3(\rho t_2 + \xi_2\tau)c(I) + \tau t_1(I)c(2). \end{aligned}$$

The degrees here are

$$\begin{aligned} |t_1| &= (3, 0), & |s| &= (6, 1), & |t_i| &= (2^i - 1)(2, 1) + (1, 0), \\ |c(I)| &= ((\sum_{i \in I} 2^i - 1) - 1)(2, 1), & |c_1(I)| &= \sum_{i \in I} (2^i - 1)(2, 1) + (1, 0), \\ |t(I)| &= \sum_{i \in I} (2^i - 1)(2, 1) + (0, -1), & |t_1(I)| &= \sum_{i \in I} (2^i - 1)(2, 1) + (3, 0). \end{aligned}$$

The theorem shows that $H_W\mathbb{Z}_{**}H_W\mathbb{Z}$ has η -torsion. Thus, $H_W\mathbb{Z}_{**}H_W\mathbb{Z}$ is not free as an $H_W\mathbb{Z}$ -module. It will therefore not be the dual of $H_W\mathbb{Z}^{**}H_W\mathbb{Z}$ in the category of $H_W\mathbb{Z}^{**}$ -modules. The existence of η -torsion makes the computation much more difficult. One could compare the phenomena in this case to similar phenomena arising in the classical computation of $bo_{**}bo$ [Mah81]. Along the way we show that there is no higher η -torsion. A fact we believe will further become useful in the future analysis of $H_W\mathbb{Z}^{**}H_W\mathbb{Z}$.

After inverting η , Theorem 1.3 computes the $HW := H_W\mathbb{Z}[\eta^{-1}]$ -algebra $HW_{**}HW$.

Corollary 1.4 (see Cor. 5.8). *Let the base field k be an extension of a field F satisfying $k_2^M(F) = 0$. Then we have an isomorphism of $W(k)[\eta^{-1}]$ -algebras*

$$HW_{**}HW \cong W(k)[\eta^{\pm 1}][x_2, y, x_3, x_4, \dots]/(y^2, x_j^2 - 2x_{j+1}),$$

where $|x_j| = (2^j, 0)$ and $|y| = (5, 0)$.

In particular, over base fields with the given assumptions, this shows the result [BH21, Cor. 8.20] of Bachmann and Hopkins about the HW -module structure of $HW \wedge HW$.

1.1. Outline. To establish our result we will make heavy use of the homotopy pullback square of Proposition 4.4 in $\mathcal{SH}(k)$

$$\begin{array}{ccc} H_W\mathbb{Z} & \xrightarrow{r} & H\mathbb{Z}/2 \\ \downarrow \pi_W & & \downarrow \pi \\ \underline{K}^W & \xrightarrow{r} & \underline{k}^M. \end{array}$$

We smash with $H_W\mathbb{Z}$ and $H\mathbb{Z}/2$ to get two more pullback squares

$$\begin{array}{ccc} H_W\mathbb{Z} \wedge H_W\mathbb{Z} & \longrightarrow & H\mathbb{Z}/2 \wedge H_W\mathbb{Z} \\ \downarrow & & \downarrow \\ \underline{K}^W \wedge H_W\mathbb{Z} & \longrightarrow & \underline{k}^M \wedge H_W\mathbb{Z} \end{array} \quad \begin{array}{ccc} H\mathbb{Z}/2 \wedge H_W\mathbb{Z} & \longrightarrow & H\mathbb{Z}/2 \wedge H\mathbb{Z}/2 \\ \downarrow & & \downarrow \\ H\mathbb{Z}/2 \wedge \underline{K}^W & \longrightarrow & H\mathbb{Z}/2 \wedge \underline{k}^M. \end{array}$$

We want to get to the object at the top left corner of the left square. To do so we compute homotopy groups of all the others first. Clockwise we start at $H\mathbb{Z}/2 \wedge H_W\mathbb{Z}$ via the right square. There we know the top right and can easily determine the bottom right by setting $\tau \in H\mathbb{Z}/2_{**}$ to zero in $H\mathbb{Z}/2_{**}H\mathbb{Z}/2$. The bottom left corner of the right square can be computed using the well-known cofiber sequence [Mor04b][Mor03, Remark 3.12]

$$\Sigma^{1,1}\underline{K}^W \xrightarrow{\eta} \underline{K}^W \rightarrow \underline{k}^M.$$

After smashing with $H\mathbb{Z}/2$, this splits as spectra in $\mathcal{SH}(k)$ and the map that continues to the right will be determined by the reduced power operation $Sq^2 \in H\mathbb{Z}/2_{**}H\mathbb{Z}/2$, the dual element of $\xi_1 \in H\mathbb{Z}/2_{**}H\mathbb{Z}/2$. Thanks to the known structure of the motivic Steenrod algebra and its dual we can carry out this computation. A pullback square gives us $H\mathbb{Z}/2_{**}H_W\mathbb{Z}$. Again, modding out τ we can establish the bottom right corner of the left square. We attack the bottom left corner of the left square similarly via the above mentioned cofiber sequence. This time we smash with $H_W\mathbb{Z}$, but do not get a splitting.

However, it can be shown that there is no higher η -torsion in the picture. So the long exact sequence of homotopy groups provides enough information to conclude the result. To deal with η -completion, we use a result by Bachmann and Hopkins [BH21, Thm. 5.1] which relates 2 and η -completion. We then conclude by pulling back the information we gathered as shown in the left square.

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1.3. Conventions. We will generally assume that k is a field of characteristic not 2, unless stated otherwise. $\mathcal{SH}(k)$ will mean the motivic stable homotopy category of $\mathbb{P}^1$ -spectra as described in [Mor03]. Let $E, F, X \in \mathcal{SH}(k)$ be spectra. We write $E_{**}F = \pi_{**}(E \wedge F)$ and $F_{**}E = \pi_{**}(F \wedge E)$. Given a map $f : E \rightarrow F$, we denote by $f_* = f_*^R : X_{**}E \rightarrow X_{**}F$ the map $\pi_{**}(X \wedge f)$ and by $f_*^L : E_{**}X \rightarrow F_{**}X$ the map $\pi_{**}(f \wedge X)$. The homotopy groups $\pi_{i-j, -j}E$ will also be denoted by $\pi_i(E)_j$ to emphasize that they form the i -line of stable homotopy groups of E , i.e. the line with slope 1 that contains the point $(i, 0)$.

K_*^{MW}	graded ring of Milnor-Witt K-theory
K_*^W	graded ring of Witt K-theory
K_*^M	graded ring of Milnor K-theory
k_*^M	graded ring of Milnor K-theory mod 2
$\underline{K}^{MW}$	homotopy module of Milnor-Witt K-theory
$\underline{K}^W$	homotopy module of Witt K-theory
$\underline{K}^M$	homotopy module of Milnor K-theory
$\tilde{k}^M$	homotopy module of Milnor K-theory mod 2
$H\tilde{\mathbb{Z}}$	Milnor-Witt motivic cohomology ring spectrum
$H\mathbb{Z}$	motivic cohomology ring spectrum
$H_W\mathbb{Z}$	Witt motivic cohomology ring spectrum
$H\mathbb{Z}/2$	motivic cohomology mod 2
$HW = H_W\mathbb{Z}[\eta^{-1}]$	η -inverted Witt motivic cohomology ring spectrum
$E_x^\wedge$	x -completion of a module or a spectrum E
$\mathrm{vcd} \, 2(k)$	étale 2-cohomological dimension of the field $k[\sqrt{-1}]$
$\pi_i E_j$	bigraded stable homotopy groups of a spectrum E with $\pi_i E_j = \pi_{i-j, -j}E$
$\mathbb{N}$	natural numbers without zero
$\mathbb{Z}_{\geq i}$	Integers greater or equal to a fixed integer i .

2. PRELIMINARIES

This section recalls known results and defines some objects we will work with. It in particular recalls the definitions of the objects mentioned in the introduction.

2.1. The Homotopy t-Structure. Let k be a field of characteristic not 2. Morel has defined a t -structure on $\mathcal{SH}(k)$ [Mor03], the *homotopy t-structure*. The corresponding subcategories of n -connective spectra $\mathcal{SH}(k)_{\geq n}$ and n -coconnective spectra $\mathcal{SH}(k)_{\leq n}$ can be described by [Hoy15, Thm. 2.3]

$$\begin{aligned} E \in \mathcal{SH}(k)_{\geq n} &\iff \pi_{p,q}(E) = 0 \text{ for } p - q < n, \\ E \in \mathcal{SH}(k)_{\leq n} &\iff \pi_{p,q}(E) = 0 \text{ for } p - q > n. \end{aligned}$$

The objects $\pi_{p,q}(E)$ on the right hand side are the *bigraded homotopy sheaves* of E . They are given as the Nisnevich sheafification of the presheaves of smooth schemes over k defined by $X \mapsto [\Sigma^{p,q}\Sigma^\infty X_+, E]$. When the base k is a perfect field, Morel showed that homotopy sheaves detect equivalences on finitely generated field extensions $k \hookrightarrow L$, see [Hoy15, Thm. 2.7]. We will denote for a motivic spectrum $E \in \mathcal{SH}(k)$ the groups

$$\pi_{p,q}(E) = [\mathbb{S}^{p,q}, E],$$

where $\mathbb{S} = \Sigma^\infty \mathrm{Spec}(k)_+$ is the sphere spectrum. The heart of the homotopy t -structure is given by the abelian category $\mathcal{SH}(k)^\heartsuit$ of *homotopy modules*. By the above description of the t -structure, the only

non-vanishing homotopy sheaves of $F \in \mathcal{SH}(k)^\heartsuit$ are on the 0-line, i.e. $\pi_0(F)_* = \pi_{-*, -*}(F)$. Since the sphere spectrum is connective its coconnective cover $\mathbb{S}_{\leq 0}$ lies in $\mathcal{SH}(k)^\heartsuit$. By Morel's computation its bigraded homotopy groups are given by Milnor-Witt K-theory [Mor12] [Mor04a]

$$\pi_0(\mathbb{S})_* \cong K_*^{\text{MW}}(k).$$

It thus deserves the name $\underline{K}^{\text{MW}} := \mathbb{S}_{\leq 0}$, the *Milnor-Witt homotopy module*. Generally, underlines in the name of a spectrum will indicate that the spectrum lies in the heart $\mathcal{SH}(k)^\heartsuit$. One further defines, by taking cones in the abelian category $\mathcal{SH}(k)^\heartsuit$, the *Milnor*, *Witt* and *Milnor mod 2 homotopy modules* [Mor03, Example 3.33]

$$\begin{aligned} \underline{K}^{\text{M}} &:= \underline{K}^{\text{MW}}/\eta, \\ \underline{K}^{\text{W}} &:= \underline{K}^{\text{MW}}/h, \\ \underline{k}^{\text{M}} &:= \underline{K}^{\text{W}}/\eta. \end{aligned}$$

In the above $\eta : \mathbb{S}^{1,1} \rightarrow \mathbb{S}$ is a stable version of the unstable Hopf map $\mathbb{A}^2 \setminus 0 \rightarrow \mathbb{P}^1$, and $h : \mathbb{S} \rightarrow \mathbb{S}$ is defined as $h := 2 + \eta\rho$, with $\rho : \mathbb{S}^{-1,-1} \rightarrow \mathbb{S}$ being a stable version of the map $\text{Spec}(k) \rightarrow \mathbb{G}_m$ corresponding to $-1 \in k^\times$.

It follows from [Mor04b][Mor03, Remark 3.12] that we have a cofiber sequence of motivic spectra

$$(2.1) \quad 0 \rightarrow \Sigma^{1,1}\underline{K}^{\text{W}} \xrightarrow{\eta} \underline{K}^{\text{W}} \xrightarrow{h} \underline{k}^{\text{M}} \rightarrow 0.$$

The homotopy groups of these homotopy modules are given by

$$\pi_0(\underline{K}^{\text{MW}})_* \cong K_*^{\text{MW}}(k), \quad \pi_0(\underline{K}^{\text{M}})_* \cong K_*^{\text{M}}(k), \quad \pi_0(\underline{K}^{\text{W}})_* \cong K_*^{\text{W}}(k), \quad \pi_0(\underline{k}^{\text{M}})_* \cong k_*^{\text{M}}(k)$$

on the diagonal and by 0 otherwise. The homotopy groups of $\underline{K}^{\text{MW}}, \underline{K}^{\text{M}}, \underline{K}^{\text{W}}, \underline{k}^{\text{M}}$ will be denoted by $\underline{K}_{**}^{\text{MW}}, \underline{K}_{**}^{\text{M}}, \underline{K}_{**}^{\text{W}}$ and $\underline{k}_{**}^{\text{M}}$ respectively.

2.2. Effective and Very Effective Spectra. Voevodsky defined [Voe02] the subcategory of *effective spectra* $\mathcal{SH}(k)^{\text{eff}} \subset \mathcal{SH}(k)$ which can be given as the full subcategory generated by homotopy colimits and extensions by suspension spectra $\Sigma^{p,q}\Sigma^\infty X_+$ of smooth schemes $X \in \mathbf{Sm}_k$ and $p \in \mathbb{Z}, q \geq 0$. The inclusion $i_t : \Sigma^{0,t}\mathcal{SH}(k)^{\text{eff}} \hookrightarrow \mathcal{SH}(k)$ has a right adjoint $r_t : \mathcal{SH}(k) \rightarrow \Sigma^{0,t}\mathcal{SH}(k)^{\text{eff}}$. The composition $f_t := i_t \circ r_t$ is called the *t-th effective cover* functor. It preserves homotopy colimits and there are natural transformations $f_{t+1} \rightarrow f_t$. The *t-th slice* of a spectrum $E \in \mathcal{SH}(k)$ is defined by the cofiber sequence

$$f_{t+1}(E) \rightarrow f_t(E) \rightarrow s_t(E).$$

With the help of the effective cover we furthermore define

$$\begin{aligned} \text{H}\tilde{\mathbb{Z}} &:= f_0(\underline{K}^{\text{MW}}), \\ \text{H}\mathbb{Z} &:= f_0(\underline{K}^{\text{M}}), \\ \text{H}_\text{W}\mathbb{Z} &:= f_0(\underline{K}^{\text{W}}), \\ \text{H}\mathbb{Z}/2 &:= \text{cofib}(2 : \text{H}\mathbb{Z} \rightarrow \text{H}\mathbb{Z}). \end{aligned}$$

It was shown in [BF18] that $\text{H}\mathbb{Z}$ defined as above represents motivic cohomology. The coefficient ring $\text{H}\mathbb{Z}/2_{**}$ is known due to the resolution of the Milnor conjecture [Voe03a] and the Beilinson-Lichtenbaum conjecture [SV00, Section 6], see for example [KRØ20, 7.2]. It is given as a polynomial ring over Milnor K-theory mod 2 [Mil70] in one variable $\tau \in \text{H}\mathbb{Z}/2_{0,-1}$

$$\text{H}\mathbb{Z}/2_{**} \cong \underline{k}_{**}^{\text{M}}[\tau].$$

More precisely, $\text{H}\mathbb{Z}/2_{p,q} \cong k_{-p}^{\text{M}}(k)\tau^{p-q}$ for $0 \geq p \geq q$ and 0 otherwise. This implies that there is a cofiber sequence

$$(2.2) \quad \Sigma^{0,-1}\text{H}\mathbb{Z}/2 \xrightarrow{\tau} \text{H}\mathbb{Z}/2 \rightarrow \underline{k}^{\text{M}}.$$

Together with the resolution of Morel's structure conjecture [Bac17, Thm. 17] this computes $\text{H}_\text{W}\mathbb{Z}_{**}$ as

$$\text{H}_\text{W}\mathbb{Z}_{**} \cong \underline{K}_{**}^{\text{W}}[\tau]/(\tau\eta),$$

and we have a cofiber sequence

$$\Sigma^{0,-1}\text{H}\mathbb{Z}/2 \xrightarrow{\tilde{\tau}} \text{H}_\text{W}\mathbb{Z} \rightarrow \underline{K}^{\text{W}}.$$

Bachmann defined [Bac17] a t-structure on $\mathcal{SH}(k)^{\text{eff}}$ via

$$\begin{aligned} E \in \mathcal{SH}(k)_{\geq n}^{\text{eff}} &\longleftrightarrow \pi_p(E)_0 = 0 \text{ for } p < n, \\ E \in \mathcal{SH}(k)_{\leq n}^{\text{eff}} &\longleftrightarrow \pi_p(E)_0 = 0 \text{ for } p > n. \end{aligned}$$

Denote the non-negative part $\mathcal{SH}(k)_{\geq 0}^{\text{eff}}$ with respect of this t-structure by $\mathcal{SH}(k)^{\text{veff}}$; it was first defined in [SØ12]. This is a full subcategory of $\mathcal{SH}(k)$ stable under homotopy colimits and extensions and its objects are called *very effective* spectra. We say a spectrum E is *very n -effective*, if $\Sigma^{-n,0}E$ is very effective.

Proposition 2.3. *The spectra $\tilde{H}\mathbb{Z}, H\mathbb{Z}, H_W\mathbb{Z}, H\mathbb{Z}/2$ and $\Sigma^{1,1}\underline{K}^W$ as well as any combination of their smash products are very effective.*

Proof. Any effective spectrum that is connective with respect to the homotopy t-structure is clearly very effective. Furthermore, $\mathcal{SH}(k)^{\text{eff}}$ is stable under smash products [SØ12, Lemma 5.6]. It remains to show that $\Sigma^{1,1}\underline{K}^W$ is very effective. By taking effective covers of (2.1) we get a cofiber sequence

$$f_0(\Sigma^{1,1}\underline{K}^W) \xrightarrow{\tilde{\eta}} H_W\mathbb{Z} \xrightarrow{r} H\mathbb{Z}/2.$$

The resolution of Morel's structure conjecture [Bac17, Thm. 17] implies that $f_0(\Sigma^{1,1}\underline{K}^W) \simeq \Sigma^{1,1}\underline{K}^W$. The claim follows. $\square$

Remark 2.4 When the characteristic of the base field is not 2, It has been shown [Hoy15][BH21][Bac21] that the spectra $\underline{K}^W, \underline{k}^M$ and their effective covers $H_W\mathbb{Z}, H\mathbb{Z}/2$ are cellular. Therefore, for all our purposes it suffices to check equivalences on bigraded homotopy groups $\pi_{p,q}$ [DI05, Prop. 7.1].

2.3. Base Change. Let K be a field and F a field extension of K . The base change functor on smooth schemes $f^* : \mathbf{Sm}_K \rightarrow \mathbf{Sm}_F$ induces a symmetric monoidal base change functor $f^* : \mathcal{SH}(K) \rightarrow \mathcal{SH}(F)$. The spectra we have defined behave well with respect to base change. Denote by $\underline{K}_F^{\text{MW}}$ the Milnor-Witt homotopy module in $\mathcal{SH}(F)$ and similarly for the Milnor, Witt and Milnor mod 2 homotopy modules.

Proposition 2.5. *Let k be a perfect field and F a field extension. Then $f^* : \mathcal{SH}(k) \rightarrow \mathcal{SH}(F)$ preserves $\underline{K}^{\text{MW}}, \underline{K}^W, \underline{K}^M, \underline{k}^M$ and also their effective covers, i.e. we have a natural equivalence $f^*(\underline{K}_k^R) \simeq \underline{K}_F^R$.*

Proof. The base change functor f^* preserves homotopy colimits and truncations [Hoy15, Lemma 2.2]. Thus, it preserves the sphere spectrum and its 0-truncation $\underline{K}^{\text{MW}}$. Furthermore, base change f^* commutes with effective covers and slices [RØ16, Lemma 2.4]. Hence, it preserves $s_0(\mathbb{S}) \simeq H\mathbb{Z}$ [Lev08, Thm. 9.0.3 and 10.5.1][Hoy15, Rem. 4.20] and its 0-truncation $\underline{K}^M$. Base change also preserves $H\mathbb{Z}/2$ and by the cofiber sequence

$$\Sigma^{0,-1}H\mathbb{Z}/2 \xrightarrow{\tau} H\mathbb{Z}/2 \rightarrow \underline{k}^M$$

it further preserves $\underline{k}^M$. By [Mor04b, Thm. 5.3] we have a cofiber sequence

$$\underline{K}^{\text{MW}} \rightarrow \underline{K}^M \vee \underline{K}^W \rightarrow \underline{k}^M.$$

We have shown that f^* preserves all terms but $\underline{K}^W$; which it also must preserve by the cofiber sequence. $\square$

2.4. Completion. Let E be a ring spectrum, $x \in \pi_0(E)_*$ and M a left E -module spectrum. The x -completion $E_x^\wedge$ of E is defined by the homotopy limit

$$M_x^\wedge := \lim_i M/x^i,$$

where M/x^i is the cofiber of the map $x^i : \Sigma^{i|x}|M \rightarrow M$. M is called *x -complete* if the canonical map induces an equivalence $M \simeq M_x^\wedge$. A theorem by Lurie shows that completion behaves well with respect to taking homotopy groups.

Theorem 2.6. *Let $E \in \mathcal{SH}(k)$ be a connective E_2 -ring spectrum, M a left E -module and $x \in \pi_0(E)_*$. If M is x -complete, then $\pi_k(M)_*$ is x -complete as a $\pi_0(E)_*$ -module for every $k \in \mathbb{Z}$.*

Proof. This follows from [Lur04, XII, Prop. 4.2.16]. The arguments in loc. cit. analogously apply for $\mathcal{SH}(k)$ with the homotopy t -structure. $\square$

We will be most concerned with completing with respect to $\eta \in \pi_0(\mathbb{S})_{-1}$. In this case, Bachmann and Hopkins have established a connection to 2-completion for fields of finite virtual cohomological 2-dimension [BH21, Thm. 5.1]. It implies the following Proposition.

Proposition 2.7. *Let $E \in \mathcal{SH}(k)$ be an $H_W\mathbb{Z}$ -module, then η -completion factors through 2-completion, i.e. the canonical map is an equivalence: $E_\eta^\wedge \xrightarrow{\simeq} E_{\eta,2}^\wedge$. If $\text{vcd}_2(k) < \infty$, and E is very n -effective for some $n \in \mathbb{Z}$, then we have canonical equivalences*

$$E_\eta^\wedge \xrightarrow{\simeq} E_{\eta,2}^\wedge \xleftarrow{\simeq} E_2^\wedge.$$

Proof. Since the relation $\eta\rho = -2$ holds in $H_W\mathbb{Z}_{**}$, we have $(2) \subset (\eta) \subset H_W\mathbb{Z}_{**}$. It follows that η -completion factors through 2-completion; see for example [Lur04, XII, Prop. 4.2.11 and 4.2.12]. The second claim is [BH21, Thm. 5.1]. $\square$

This proposition reduces η -completion to 2-completion for $H_W\mathbb{Z}$ -modules. When -1 can be expressed as a sum of squares, this even frees us from the necessity of completing.

Corollary 2.8. *Let the base field k be nonreal of characteristic not 2 with $\mathrm{vcd}_2(k) < \infty$. Then every n -effective $H_W\mathbb{Z}$ -modules are η -complete.*

Proof. When k is nonreal, then $W(k) = \pi_0(H_W\mathbb{Z})_0$ is 2-primary torsion of finite exponent [EKM08, 31.4(6)]. Hence, every $H_W\mathbb{Z}$ -module is 2-complete. The claim follows by Proposition 2.7. $\square$

3. THE MOTIVIC STEENROD ALGEBRA AND ITS DUAL

The structure of $H_W\mathbb{Z}_{**}H_W\mathbb{Z}$ is deeply intertwined with the structure of the motivic Steenrod algebra mod 2 and its dual, $\mathcal{A}_2^{**} \cong H\mathbb{Z}/2^{**}H\mathbb{Z}/2$ and $\mathcal{A}_2^* \cong H\mathbb{Z}/2_{**}H\mathbb{Z}/2$ respectively. Since their properties will be used extensively in what follows, we decided to dedicate a new section to this. We collect some important facts about them here.

3.1. Self Homology and Cohomology of Spectra. Let E be a commutative ring spectrum in $\mathrm{SH}(k)$ with multiplication $\mu : E \wedge E \rightarrow E$ and unit $1 : \mathbb{S} \rightarrow E$. Then E_{**} is a ring via $x \otimes y \mapsto \mu^*(x \wedge y)$ and both $E_{**}E$ and $E^{**}E$ are E_{**} -algebras. Their ring structures are defined as follows. Let $\mu_{i,j} : E^{\wedge 4} \rightarrow E^{\wedge 3}$ be the multiplication map acting on the i -th and j -th coordinate and $\iota_{i,j} : E^{\wedge n} \rightarrow E^{\wedge n}$ the map swapping coordinates i and j . The multiplication on $E_{**}E$ is given by $x \otimes y \mapsto (\mu_{1,2} \wedge \mu_{3,4}) \circ \iota_{2,3} \circ (x \wedge y)$. It is graded commutative. The multiplication on $E^{**}E$ is given by composition $f \otimes g \mapsto f \circ g$. This is in general not commutative. We denote by $\iota : E \wedge E \rightarrow E \wedge E$ the conjugation map. $E^{**}E$ is an E_{**} -algebra, and hence a bimodule, via the ring map

$$E_{**} \rightarrow E^{**}E, \quad s \mapsto E \simeq \mathbb{S} \wedge E \xrightarrow{s \wedge E} E \wedge E \xrightarrow{\mu} E.$$

$E_{**}E$ gets its E_{**} -algebra structure via the ring map

$$E_{**} \xrightarrow{\eta_L} E_{**}E, \quad s \mapsto s \wedge 1.$$

There is a second ring map, η_R , which defines a right E_{**} -module structure on E_{**} given by $s \mapsto 1 \wedge s$.

Remark 3.1 The image of E_{**} does generally not lie in the center of $E^{**}E$. For example, when E is motivic cohomology mod 2, then $\mathrm{Sq}^1 \cdot \tau - \tau \cdot \mathrm{Sq}^1 = \rho \neq 0$. Classical examples of this phenomenon are also MU and BP.

Recall the exterior products for $X, Y \in \mathrm{SH}(k)$

$$\begin{aligned} E_{**}(X) \otimes E_{**}(Y) &\xrightarrow{\times} E_{**}(X \wedge Y), & x \otimes y &\mapsto (\mu_{1,3} \wedge X \wedge Y) \circ (x \wedge y), \\ E^{**}(X) \otimes E^{**}(Y) &\xrightarrow{\times} E^{**}(X \wedge Y), & f \otimes g &\mapsto \mu \circ (f \wedge g), \end{aligned}$$

as well as the Kronecker pairing

$$E^{-**}(X) \otimes E_{**}(X) \xrightarrow{\langle -, - \rangle} E_{**}, \quad f \otimes x \mapsto \langle f, x \rangle := \mu \circ (E \wedge f) \circ x.$$

Here the degrees are additive, i.e. $|\langle f, x \rangle| = |x| - |f|$. Given a map $\phi : X \rightarrow Y$, we have $\langle f, \phi_*(x) \rangle = \langle \phi^*(f), x \rangle$. The pairings are meant to be understood to be over X and Y respectively. Also note that in this language we have $x \cdot y = \mu_*(x \times y)$ for $x, y \in E_{**}E$.

Proposition 3.2. *Let $\alpha, f \in E^{**}E$, $x \in E_{**}E$, $s \in E_{**}$. Then we have*

- (i) α_* is dual to α^* : $\langle f, \alpha_*(x) \rangle = \langle \alpha^*(f), x \rangle = \langle f \cdot \alpha, x \rangle$,
- (ii) $\langle f, \alpha_*(x) \rangle = \langle \alpha, f_*(\iota(x)) \rangle$,
- (iii) the pairing is left E_{**} -linear in both variables: $s\langle f, x \rangle = \langle sf, x \rangle = \langle f, sx \rangle$,
- (iv) the pairing respects the right E_{**} -actions: $\langle fs, x \rangle = \langle f, xs \rangle$.

Proof. See [Sch12] or [Voe03b] specifically for the case of the motivic Steenrod algebra. $\square$

Remark 3.3 (i) The pairing does not necessarily commute with right actions. Let $E = H\mathbb{Z}/2$ be motivic cohomology mod 2 and let $\mathrm{Sq}^1 \in H\mathbb{Z}/2^{**}H\mathbb{Z}/2$ be the Bockstein. Using the description (3.4), we have $\langle \mathrm{Sq}^1, 1 \cdot \tau \rangle = \langle \mathrm{Sq}^1, \tau + \rho\tau_0 \rangle = \rho \neq \tau \langle \mathrm{Sq}^1, 1 \rangle = 0$, since Sq^1 is dual to τ_0 .

- (ii) c swaps the left and right action on E_{**} . When they disagree, then ι is not E_{**} -linear and hence cannot be dualized via the pairing, in case it is perfect. In other words, there is no map $\iota : E^{**}E \rightarrow E^{**}E$ satisfying $\langle f, \iota(x) \rangle = \langle \iota(f), x \rangle$. The existence of a dual conjugate forces the pairing to be right E_{**} -linear in both variables which on the other hand forces the left and right actions to be identical in the perfect case.

The multiplication map $\mu : E \wedge E \rightarrow E$ defines a diagonal $\mu^* : E^{**}E \rightarrow E^{**}(E \wedge E)$. For $e, h \in E^{**}(X \wedge Y)$, $a, b \in E_{**}(X \wedge Y)$ we have [Sch12, Prop. 6.17 (iii)]

$$\langle e \times h, a \times b \rangle = (-1)^{\text{tdim}(e) \cdot \text{tdim}(a)} \langle e, a \rangle \cdot \langle h, b \rangle.$$

If we have a decomposition $\mu^*(f) = \sum_i f'_i \times f''_i$, since the diagonal is dual to multiplication on $E_{**}E$, the above formula implies:

$$\langle f, x \cdot y \rangle = \sum_i (-1)^{\text{tdim}(f'_i) \text{tdim}(x)} \langle f'_i, x \rangle \cdot \langle f''_i, y \rangle,$$

where we define $\text{tdim}(a) = p$ for $a \in E_{p,q}(X \wedge Y)$ and $\text{tdim}(e) = i$ for $e \in E^{i,j}(X \wedge Y)$. Similarly, one defines a diagonal

$$\psi : E_{**}E \rightarrow E_{**}(E \wedge E), \quad x \mapsto \iota_{2,3} \circ (x \wedge 1),$$

with $\iota_{i,j} : E^{\wedge n} \rightarrow E^{\wedge n}$ swapping coordinates i and j .

3.2. The (Dual) Motivic Steenrod Algebra. We now specialize to $E = \mathbb{H}\mathbb{Z}/2$, i.e. motivic cohomology mod 2. In this case $\mathbb{H}\mathbb{Z}/2 \wedge \mathbb{H}\mathbb{Z}/2$ is a reflexive $\mathbb{H}\mathbb{Z}/2$ -module, i.e. the canonical map

$$\mathbb{H}\mathbb{Z}/2 \wedge \mathbb{H}\mathbb{Z}/2 \rightarrow \text{Hom}_{\mathbb{H}\mathbb{Z}/2}(\text{Hom}_{\mathbb{H}\mathbb{Z}/2}(\mathbb{H}\mathbb{Z}/2 \wedge \mathbb{H}\mathbb{Z}/2, \mathbb{H}\mathbb{Z}/2), \mathbb{H}\mathbb{Z}/2)$$

is an equivalence of $\mathbb{H}\mathbb{Z}/2$ -modules [HKØ17, Prop. 5.3][Hoy15, Prop. 5.5, Thm. 5.10]. Thus, the Kronecker pairing is perfect. The diagonal on $\mathbb{H}\mathbb{Z}/2^{**}\mathbb{H}\mathbb{Z}/2$ can now be identified with

$$\Delta : \mathbb{H}\mathbb{Z}/2^{**} \xrightarrow{\mu^*} \mathbb{H}\mathbb{Z}/2^{**}(\mathbb{H}\mathbb{Z}/2 \wedge \mathbb{H}\mathbb{Z}/2) \cong \mathbb{H}\mathbb{Z}/2^{**}\mathbb{H}\mathbb{Z}/2 \otimes_{\mathbb{H}\mathbb{Z}/2^{**}} \mathbb{H}\mathbb{Z}/2^{**}\mathbb{H}\mathbb{Z}/2.$$

The decomposition $\mu^*(f) = \sum_i f'_i \times f''_i = \sum_i f'_i \otimes f''_i$ now always holds, where we identified via above isomorphism. Hence, we always have

$$\langle f, x \cdot y \rangle = \sum_i \langle f'_i, x \rangle \cdot \langle f''_i, y \rangle.$$

Dually, we have a diagonal

$$\Delta : \mathbb{H}\mathbb{Z}/2_{**} \xrightarrow{\psi} \mathbb{H}\mathbb{Z}/2_{**}(\mathbb{H}\mathbb{Z}/2 \wedge \mathbb{H}\mathbb{Z}/2) \cong \mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2 \otimes_{\mathbb{H}\mathbb{Z}/2_{**}} \mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2$$

and for $\Delta(x) = \sum_i x'_i \otimes x''_i$ we get [Voe03b, 12.9][Boa82, Lemma 3.3]

$$\langle f \circ g, x \rangle = \sum_i \langle f \langle g, x''_i \rangle, x'_i \rangle,$$

where the scalar $\langle g, x''_i \rangle$ acts on the right on f , i.e. $f \langle g, x''_i \rangle = \eta_R(\langle g, x''_i \rangle) \cdot f$.

Recall from Section 2.2 that we know the coefficient ring $\mathbb{H}\mathbb{Z}/2_{**}$ which is given by

$$\mathbb{H}\mathbb{Z}/2_{**} \cong \underline{k}_{**}^M[\tau],$$

i.e. $\mathbb{H}\mathbb{Z}/2_{p,q} \cong \underline{k}_{-p}^M(k) \tau^{p-q}$ for $0 \geq p \geq q$ and 0 otherwise. Note that this is a commutative ring. The elements $\underline{k}_{**}^M(k)$ all come from the sphere spectrum while τ does not. Recall that we denote the element $\mathbb{S} \rightarrow \mathbb{G}_m \simeq \Sigma^{1,1}\mathbb{S}$ corresponding to $-1 \in \mathbb{G}_m(k)$ by the letter ρ .

The dual motivic Steenrod algebra is given as a commutative left $\mathbb{H}\mathbb{Z}/2_{**}$ -algebra by [Voe03b][Hoy15][HKØ17]

$$(3.4) \quad \mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2 \cong \mathbb{H}\mathbb{Z}/2_{**}[\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots] / (\tau_i^2 - \rho\tau_{i+1} - (\tau + \rho\tau_0)\xi_{i+1})$$

with degrees

$$|\xi_r| = (2^r - 1)(2, 1), \quad |\tau_i| = (2^i - 1)(2, 1) + (1, 0).$$

Remark 3.5 We would like to point out that there is no immediate connection between the element τ in the coefficient ring and the elements τ_i in the dual Steenrod algebra, contrary to what the notation might suggest.

We readily see that $\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2$ is free as a left $\mathbb{H}\mathbb{Z}/2_{**}$ -module. A basis can be described in terms of monomials. Let E and R be finite length sequences $E = (E_0, E_1, \dots)$, $R = (R_1, R_2, \dots)$ with $E_i \in \{0, 1\}$ and $R_i \in \mathbb{Z}_{\geq 0}$. Basis monomials can be given by

$$\tau_0^{E_0} \tau_1^{E_1} \dots \xi_1^{R_1} \xi_2^{R_2} \dots$$

Notation 3.6 We will view E as a subset of $\mathbb{Z}_{\geq 0}$ and define for a finite sequence of elements $(x_i)_{i \in \mathbb{Z}_{\geq 0}}$ in a ring A

$$x_E := \prod_{e \in E} x_e.$$

For a finite sequence $R = (R_1, R_2, \dots)$ with $R_i \in \mathbb{Z}_{\geq 1}$ we define

$$x^R := \prod_{i=1}^{\infty} x_i^{R_i}.$$

In those terms a basis of $\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2$ is given by monomials $\tau_E \xi^R$ for $E \subset \mathbb{Z}_{\geq 0}$ finite and $R = (R_1, R_2, \dots)$ a finite sequence in $\mathbb{Z}_{\geq 0}$.

Furthermore, for $I \subset \mathbb{Z}_{\geq 1}$ finite, we define e_I to be the sequence satisfying $e_I(j) = 1$ for $j \in I$ and $e_I(j) = 0$ for $j \notin I$.

The pair $(\mathbb{H}\mathbb{Z}/2_{**}, \mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2)$ is a Hopf algebroid. Its structure is described in [Voe03b, Sec. 12][HKØ17, Sec. 5.1][Hoy15, Sec. 5.3]. We recall

$$\begin{aligned} \eta_L(\tau) &= \tau, \\ \eta_R(\tau) &= \tau + \rho\tau_0, \\ \Delta(\tau) &= \tau \otimes 1, \\ \Delta(\tau_r) &= \tau_r \otimes 1 + 1 \otimes \tau_r + \sum_{i=0}^{r-1} \xi_{r-i}^{2^i} \otimes \tau_i, \\ \Delta(\xi_r) &= \xi_r \otimes 1 + 1 \otimes \xi_r + \sum_{i=1}^{r-1} \xi_{r-i}^{2^i} \otimes \xi_i, \\ \iota(\tau) &= \tau + \rho\tau_0, \\ \iota(\tau_r) &= \tau_r + \sum_{i=0}^{r-1} \xi_{r-i}^{2^i} \iota(\tau_i), \\ \iota(\xi_r) &= \xi_r + \sum_{i=1}^{r-1} \xi_{r-i}^{2^i} \iota(\xi_i). \end{aligned} \tag{3.7}$$

We note that η_L , η_R , Δ and ι are $k_*^M(k)$ -linear, since these elements all come from the sphere spectrum. Whenever we write $x \cdot s = xs$ for a scalar $s \in \mathbb{H}\mathbb{Z}/2_{**}$, we will mean $\eta_R(s)x$. In particular, $x\tau = \eta_R(\tau)x = (\tau + \rho\tau_0)x$. Moreover, we will denote the following conjugates with overlines $\iota(\tau_i) = \overline{\tau_i}$, $\iota(\xi_i) = \overline{\xi_i}$.

The motivic Steenrod algebra $\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2$ is generated as an algebra by $\mathbb{H}\mathbb{Z}/2_{**}$ and reduced power operations

$$\text{Sq}^{2i} \in \mathbb{H}\mathbb{Z}/2^{2i,i}\mathbb{H}\mathbb{Z}/2, \quad \text{Sq}^{2i+1} \in \mathbb{H}\mathbb{Z}/2^{2i+1,i}\mathbb{H}\mathbb{Z}/2,$$

where $i \geq 0$, $\text{Sq}^0 = id$ and Sq^1 is the Bockstein. Since the Kronecker pairing is perfect, we can dualize the basis $\tau_E \xi^R$. The dual of τ_0 is Sq^1 and the dual of ξ_1 is Sq^2 . Our computations will involve the maps

$$\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2 \xrightarrow{(\text{Sq}^2)_*^R} \mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2 \quad \text{and} \quad \mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2 \xrightarrow{(\text{Sq}^2)_*^L} \mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2$$

given by $\pi_{**}(\mathbb{H}\mathbb{Z}/2 \wedge \text{Sq}^2)$ and $\pi_{**}(\text{Sq}^2 \wedge \mathbb{H}\mathbb{Z}/2)$ respectively. They can be computed as follows.

Proposition 3.8. *Let $x \in \mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2$ and $\alpha \in \mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2$. Then*

$$\alpha_*^R(x) = \sum_i x'_i \langle \alpha, x''_i \rangle \quad \text{and} \quad \alpha_*^L(x) = \sum_i \langle \alpha, \iota(x'_i) \rangle x''_i.$$

Proof. This can be shown using formulas for the Kronecker pairing that we have established earlier. The second formula follows from Corollary 6.4 in [Boa82]. $\square$

Remark 3.9 Over a field with no square root of -1 , the conjugation in the second factor is necessary even for $\alpha = \text{Sq}^1$, the dual element of τ_0 :

$$\begin{aligned}\langle \text{Sq}^1, \tau \rangle &= 0, \\ \langle \text{Sq}^1, \iota(\tau) \rangle &= \langle \text{Sq}^1, (\tau + \rho\tau_0) \rangle = \rho \neq 0.\end{aligned}$$

Classically, we can dualize the conjugation and the conjugate of Sq^1 is Sq^1 itself. Hence, this does not occur classically and our reasoning suggests that this is related to Remark 3.3 (iii).

Remark 3.10 We also have $\iota \circ (\text{Sq}^2)_*^R = (\text{Sq}^2)_*^L \circ \iota$.

Their behavior on products is described by

Proposition 3.11. *We have for $x, y \in \text{HZ}/2_{**}\text{HZ}/2$*

$$\begin{aligned}(\text{Sq}^1)_*^R(x \cdot y) &= (\text{Sq}^1)_*^R(x) \cdot y + x \cdot (\text{Sq}^1)_*^R(y), \\ (\text{Sq}^1)_*^L(x \cdot y) &= (\text{Sq}^1)_*^L(x) \cdot y + x \cdot (\text{Sq}^1)_*^L(y), \\ (\text{Sq}^2)_*^R(x \cdot y) &= (\text{Sq}^2)_*^R(x) \cdot y + x \cdot (\text{Sq}^2)_*^R(y) + (\tau + \rho\tau_0) \cdot (\text{Sq}^1)_*^R(x) \cdot (\text{Sq}^1)_*^R(y), \\ (\text{Sq}^2)_*^L(x \cdot y) &= (\text{Sq}^2)_*^L(x) \cdot y + x \cdot (\text{Sq}^2)_*^L(y) + \tau \cdot (\text{Sq}^1)_*^L(x) \cdot (\text{Sq}^1)_*^L(y).\end{aligned}$$

Proof. By the Cartan formula [Voe03b, Prop. 9.7, Lemma 11.6]) and the fact that the Bockstein Sq^1 [Voe03b, (8.1)] is a derivation, we have

$$\begin{aligned}\Delta(\text{Sq}^1) &= \text{Sq}^1 \otimes 1 + 1 \otimes \text{Sq}^1, \\ \Delta(\text{Sq}^2) &= \text{Sq}^2 \otimes 1 + 1 \otimes \text{Sq}^2 + \tau \text{Sq}^1 \otimes \text{Sq}^1.\end{aligned}$$

The result now follows from duality of Δ and multiplication on $\text{HZ}/2_{**}\text{HZ}/2$ together with Proposition 3.8. $\square$

The left $\text{HZ}/2_{**}$ -subalgebra generated by ξ_i is preserved under conjugation. Hence

$$\begin{aligned}\text{HZ}/2_{**}\text{HZ}/2 &\cong \text{HZ}/2_{**}[\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots] / (\tau_i^2 - \rho\tau_{i+1} - \xi_{i+1}\tau) \\ (3.12) \quad &\cong \text{HZ}/2_{**}[\tau_0, \tau_1, \dots, \bar{\xi}_1, \bar{\xi}_2, \dots] / (\tau_i^2 - \rho\tau_{i+1} - \xi_{i+1}\tau).\end{aligned}$$

Let $K \subset \text{HZ}/2_{**}\text{HZ}/2$ be the left $\text{HZ}/2_{**}$ -module spanned by monomials $\tau_E \bar{\xi}^R$ with even powers of ξ_1 , i.e. with $E \subset \mathbb{Z}_{\geq 0}$ finite and $R = (2R_1, R_2, \dots)$ finite with $R_i \in \mathbb{Z}_{\geq 0}$. Then we have an obvious splitting

$$(3.13) \quad \text{HZ}/2_{**}\text{HZ}/2 \cong K \oplus \bar{\xi}_1 K.$$

We will use this splitting later to compute $\text{HZ}/2$ -homology of $H_W\mathbb{Z}$. The elements in K map to zero under $(\text{Sq}^2)_*^R$ modulo $(\tau + \rho\tau_0)$.

4. BUILDING BLOCKS OF $H_W\mathbb{Z} \wedge H_W\mathbb{Z}$

As mentioned in the introduction, $H_W\mathbb{Z}$ and $\text{HZ}/2$ share a relation with each other witnessed by the pullback square of rings [Mor04b, Thm 5.3][Mor12, Chapter 5]

$$(4.1) \quad \begin{array}{ccc} K_*^{MW}(k) & \longrightarrow & K_*^M(k) \\ \downarrow & & \downarrow \\ K_*^W(k) & \longrightarrow & k_*^M(k). \end{array}$$

We will make precise the details of their connection now. We then use this information to break down the task of computing $H_W\mathbb{Z}_{**}H_W\mathbb{Z}$ into pieces which we compute in later sections.

Theorem 17 of [Bac17] demonstrates that we have a diagram of cofiber sequences

$$(4.2) \quad \begin{array}{ccccc} * & \longrightarrow & \Sigma^{0,-1}\text{HZ}/2 & \xrightarrow{id} & \Sigma^{0,-1}\text{HZ}/2 \\ \downarrow & & \downarrow \tilde{\tau} & & \downarrow \tau \\ \Sigma^{1,1}\underline{K}^W & \xrightarrow{\tilde{\eta}} & H_W\mathbb{Z} & \xrightarrow{r} & \text{HZ}/2 \\ \downarrow id & & \downarrow \pi_W & & \downarrow \pi \\ \Sigma^{1,1}\underline{K}^W & \xrightarrow{\eta} & \underline{K}^W & \xrightarrow{r} & \underline{k}^M. \end{array}$$

We will denote the maps continuing to the right of r and $\underline{r}$ by ∂ and $\underline{\partial}$ respectively. Here the middle row is the effective cover f_0 of the bottom row which is the cofiber sequence (2.1). The two bottom vertical maps π_W and π are given by zero truncation with respect to the homotopy t-structure. That is, the columns are of the form $X_{\geq 1} \rightarrow X \rightarrow X_{\leq 0}$.

Notation 4.3 Underlined maps will usually mean 0-truncated maps with respect to the homotopy t-structure. In that case we have a relationship such as $\pi \circ r = \underline{r} \circ \pi_W$ in (4.2). A tilde map in most cases is a lifted version of a map. For example: $\eta = \pi_W \circ \tilde{\eta}$ in (4.2).

Diagram (4.2) immediately implies the following Proposition.

Proposition 4.4. *We have a homotopy pullback square*

$$\begin{array}{ccc} H_W \mathbb{Z} & \xrightarrow{r} & H\mathbb{Z}/2 \\ \downarrow \pi_W & & \downarrow \pi \\ \underline{K}^W & \xrightarrow{\underline{r}} & \underline{k}^M. \end{array}$$

We smash the pullback square in Proposition 4.4 with $H_W \mathbb{Z}$ on the right and get a pullback square

$$(4.5) \quad \begin{array}{ccc} H_W \mathbb{Z} \wedge H_W \mathbb{Z} & \xrightarrow{r \wedge H_W \mathbb{Z}} & H\mathbb{Z}/2 \wedge H_W \mathbb{Z} \\ \downarrow \pi_W \wedge H_W \mathbb{Z} & & \downarrow \pi \wedge H_W \mathbb{Z} \\ \underline{K}^W \wedge H_W \mathbb{Z} & \xrightarrow{\underline{r} \wedge H_W \mathbb{Z}} & \underline{k}^M \wedge H_W \mathbb{Z}. \end{array}$$

In the sections below we will determine the algebra structure of the homotopy groups of the pieces in this square surrounding $H_W \mathbb{Z} \wedge H_W \mathbb{Z}$. We proceed clockwise and begin with the top right corner, $H\mathbb{Z}/2_{**} H_W \mathbb{Z}$.

4.1. Computation of $H\mathbb{Z}/2 \wedge H_W \mathbb{Z}$. Smashing the pullback square in Proposition 4.4 with $H\mathbb{Z}/2$ on the left, we have a pullback square

$$\begin{array}{ccc} H\mathbb{Z}/2 \wedge H_W \mathbb{Z} & \xrightarrow{H\mathbb{Z}/2 \wedge r} & H\mathbb{Z}/2 \wedge H\mathbb{Z}/2 \\ \downarrow H\mathbb{Z}/2 \wedge \pi_W & & \downarrow H\mathbb{Z}/2 \wedge \pi \\ H\mathbb{Z}/2 \wedge \underline{K}^W & \xrightarrow{H\mathbb{Z}/2 \wedge \underline{r}} & H\mathbb{Z}/2 \wedge \underline{k}^M. \end{array}$$

We know the homotopy groups on the right. Hence, to compute $H\mathbb{Z}/2_{**} H_W \mathbb{Z}$ we are only missing the structure of $H\mathbb{Z}/2 \wedge \underline{K}^W$ and information on the maps involved.

The map $\eta : \mathbb{S}^{1,1} \rightarrow \mathbb{S}$ is a map between spheres and acts as 0 on $H\mathbb{Z}$, and thus also on $H\mathbb{Z}/2$. Therefore, when smashing the last row of grid (4.2) with $H\mathbb{Z}/2$ on the left all rows are split. That is, they are of the form $X \rightarrow X \vee F \rightarrow F$. Consequently, we have a split short exact sequence

$$(4.6) \quad 0 \rightarrow H\mathbb{Z}/2_{**} \underline{K}^W \xrightarrow{\underline{r}^R} H\mathbb{Z}/2_{**} \underline{k}^M \xrightarrow{\underline{\partial}^R} \Sigma^{2,1} H\mathbb{Z}/2_{**} \underline{K}^W \rightarrow 0.$$

Let $d := \underline{r} \circ \underline{\partial}$. The $H\mathbb{Z}/2_{**}$ -homology of $\underline{K}^W$ can now be computed as the kernel of d_*^R . The following Proposition enables us to do so by giving a description of d . This proposition can already be found in the proof of [BH21, Lemma 6.18] which we borrow below.

Proposition 4.7. *We have a commutative diagram*

$$\begin{array}{ccc} H\mathbb{Z}/2 & \xrightarrow{\text{Sq}^2} & \Sigma^{2,1} H\mathbb{Z}/2 \\ \downarrow \pi & & \downarrow \pi \\ \underline{k}^M & \xrightarrow{d} & \Sigma^{2,1} \underline{k}^M. \end{array}$$

Proof. We proceed as in the proof of [BH21, Lemma 6.18]. Taking $[H\mathbb{Z}/2, -]$ of the $\Sigma^{2,1}$ -shifted version of the cofiber sequence (2.2), we see that we have a surjection

$$\pi_* : H\mathbb{Z}/2^{2,1} H\mathbb{Z}/2 \twoheadrightarrow [H\mathbb{Z}/2, \Sigma^{2,1} \underline{k}^M], \quad f \mapsto \pi \circ f,$$

since τ is a non-zero divisor in the motivic Steenrod algebra. Hence, there exists $f \in \mathrm{HZ}/2^{2,1}\mathrm{HZ}/2$ such that $d \circ \pi = \pi \circ f$. Taking $\mathrm{HZ}/2$ -homology, we get a commutative diagram

$$\begin{array}{ccc} \mathrm{HZ}/2_{**}\mathrm{HZ}/2 & \xrightarrow{f_*^R} & \Sigma^{2,1}\mathrm{HZ}/2_{**}\mathrm{HZ}/2 \\ \downarrow \pi_*^R & & \downarrow \pi_*^R \\ \mathrm{HZ}/2_{**}\underline{k}^M & \xrightarrow{d_*^R} & \Sigma^{2,1}\mathrm{HZ}/2_{**}\underline{k}^M, \end{array}$$

where we used the rightmost column of (4.2) and the fact that τ is a non-zero divisor in the dual motivic Steenrod algebra to deduce that the vertical maps are surjective. By degree reasons we have $\mathrm{HZ}/2^{2,1}\mathrm{HZ}/2 \cong \mathbb{F}_2\{\mathrm{Sq}^2\} \oplus k_1^M(k)\{\mathrm{Sq}^1\}$. Using again the rightmost column of (4.2), we see $\mathrm{HZ}/2_{**}\underline{k}^M \cong (\mathrm{HZ}/2_{**}\mathrm{HZ}/2)/(\tau + \rho\tau_0)$. Since this is never trivial and the left and right side of (4.6) are shifts of each other, the splitting is never trivial. It follows that $d \neq 0$. After base change to a quadratically closed field we have $k_1^M(k) \cong 0$, hence f cannot be a scalar multiple of Sq^1 . We note that the short exact sequence (4.6) dictates $d_*^R \circ d_*^R = 0$. Using the formulas (3.7) for Δ we see that $((\mathrm{Sq}^2 + a\mathrm{Sq}^1)_*^R)^2(\tau_1) = (\mathrm{Sq}^2 + a\mathrm{Sq}^1)_*^R(\rho\xi_1) = \rho$, for $a \in k_1^M(k)$, is not zero under the map π_*^R . We must have $f = \mathrm{Sq}^2$. This shows the claim. $\square$

Notation 4.8 In concordance with our conventions in Notation 4.3, we will also refer to d by Sq^2 .

Before we begin our computation, let us remark how this computes the algebra $\mathrm{HZ}/2_{**}\mathrm{H}_W\mathbb{Z}$.

Lemma 4.9. *We have a pullback square of rings*

$$\begin{array}{ccc} \mathrm{HZ}/2_{**}\mathrm{H}_W\mathbb{Z} & \xleftarrow{r_*^R} & \mathrm{HZ}/2_{**}\mathrm{HZ}/2 \\ \downarrow (\pi_W)_*^R & & \downarrow \pi_*^R \\ \mathrm{HZ}/2_{**}\underline{K}^W & \xleftarrow{r_*^R} & \mathrm{HZ}/2_{**}\underline{k}^M. \end{array}$$

Proof. As (4.6) shows, r_*^R is an injection on $\mathrm{HZ}/2_{**}$ -homology, hence r_*^R injects $\mathrm{HZ}/2_{**}\mathrm{H}_W\mathbb{Z}$ into $\mathrm{HZ}/2_{**}\mathrm{HZ}/2$ by a simple diagram chase. The map π_*^R , which is just given by multiplication by τ on the right, is injective since τ , and therefore $\iota(\tau)$, is a non-zero divisor in $\mathrm{HZ}/2_{**}\mathrm{HZ}/2$. Hence, π_*^R and thus $(\pi_W)_*^R$ are surjective. The Mayer-Vietoris sequence shows that if one of the maps one pulls back along is surjective on homotopy groups, then one has a pullback square on homotopy groups. This and the fact that all involved maps are maps of ring spectra now imply the claim. $\square$

Note that, by 3.12 and the proof of Proposition 4.7, we have an isomorphism of left $\mathrm{HZ}/2_{**}$ -algebras

$$\mathrm{HZ}/2_{**}\underline{k}^M \cong \mathrm{HZ}/2_{**}[\tau_0, \tau_1, \dots, \overline{\xi_1}, \overline{\xi_2}, \dots]/(\tau + \rho\tau_0, \tau_i^2 - \rho\tau_{i+1}).$$

The kernel of d_*^R , i.e. $\mathrm{HZ}/2_{**}\underline{K}^W$, will be a subalgebra of the above. For its computation, we use the behavior of d_*^R on products. Recall Proposition 3.11:

$$(\mathrm{Sq}^2)_*^R(x \cdot y) = (\mathrm{Sq}^2)_*^R(x) \cdot y + x \cdot (\mathrm{Sq}^2)_*^R(y) + (\tau + \rho\tau_0) \cdot (\mathrm{Sq}^1)_*^R(x) \cdot (\mathrm{Sq}^1)_*^R(y).$$

This means d_*^R is a derivation, since it is obtained from $(\mathrm{Sq}^2)_*^R$ by modding out by $(\tau + \rho\tau_0)$ as can be seen by Corollary 4.7. The formulas (3.7), Proposition 3.8 and Proposition 3.11 imply inductively

$$(\mathrm{Sq}^2)_*^R(\tau_i) = 0, \quad (\mathrm{Sq}^2)_*^R(\overline{\xi_1}) = 1, \quad (\mathrm{Sq}^2)_*^R(\overline{\xi_j}) = 0, \quad \text{for } j \neq 1.$$

Proposition 4.10. *We have an isomorphism of left $\mathrm{HZ}/2_{**}$ -algebras*

$$\mathrm{HZ}/2_{**}\underline{K}^W \xrightarrow[\cong]{r_*^R} \mathrm{HZ}/2_{**}[\tau_0, \dots, \overline{\xi_1}^2, \overline{\xi_2}, \overline{\xi_3}, \dots]/(\tau + \rho\tau_0, \tau_i^2 - \rho\tau_{i+1}) \hookrightarrow \mathrm{HZ}/2_{**}\underline{k}^M.$$

Proof. Let $\underline{K}$ be as in (3.13). By what we have just computed, $\underline{K} := \pi_*^R(\underline{K})$ lies in the kernel of d_*^R . Furthermore, d_*^R acts on $\xi_1 \underline{K}$ via $\xi_1 k \mapsto k$. So $\mathrm{im}(r_*^R) = \underline{K}$ and since r_*^R is a ring map, $\underline{K}$ is a left $\mathrm{HZ}/2_{**}$ -algebra.

To deduce the relations, note that monomials $\tau_E \overline{\xi}^R$ with $E \subset \mathbb{Z}_{\geq 0}$ finite and $R = (R_1, R_2, \dots)$ a finite sequence in $\mathbb{Z}_{\geq 0}$, form a right $\underline{k}_{**}^M$ -basis for $\mathrm{HZ}/2_{**}\underline{k}^M$. In particular, such monomials with R_1 even are therefore k_*^M -linearly independent. Any other relations in $\mathrm{HZ}/2_{**}\underline{K}^W$ must therefore contain higher degrees of τ_i . Since the relations in the result obviously hold, we already have relations for τ_i^2 . Hence, we may reduce to lower degrees in them. The claim follows. $\square$

Remark 4.11 Since we had no assumptions on our base field, it follows that the right $k_*^M(k)$ -basis of $\mathbb{H}\mathbb{Z}/2_{**}\underline{K}^W$ mentioned in the proof describes $\mathbb{H}\mathbb{Z}/2 \wedge \underline{K}^W$ as a free right module over $\underline{k}^M$.

Proposition 4.12. *We have an isomorphism of left $\mathbb{H}\mathbb{Z}/2_{**}$ -algebras*

$$\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}_W\mathbb{Z} \xrightarrow[\cong]{r_*^R} \mathbb{H}\mathbb{Z}/2_{**}[\tau_0, \dots, \overline{\xi_1}\tau, \overline{\xi_1}^2, \overline{\xi_2}, \overline{\xi_3} \dots] / (\dots) \hookrightarrow \mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2,$$

where the ideal $(\dots)$ is generated by the relations

$$\tau_i^2 = \rho\tau_{i+1} + \xi_{i+1}\tau, \quad (\overline{\xi_1}\tau)^2 = \overline{\xi_1}^2\tau^2.$$

Proof. That we have the given generators follows from the pullback square of Lemma 4.9. The first column of the grid (4.2) gives a short exact sequence

$$0 \rightarrow \Sigma^{0,-1}\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2 \xrightarrow{\tilde{\tau}_*^R} \mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}_W\mathbb{Z} \xrightarrow{(\pi_W)_*^R} \mathbb{H}\mathbb{Z}/2_{**}\underline{K}^W \rightarrow 0$$

with $\tilde{\tau}_*^R(\xi_{i+1}) = \tau_i^2 - \rho\tau_{i+1}$ and $\tilde{\tau}_*^R(\tau_i) = (\tau + \rho\tau_0)\tau_i$. The formulas (3.7) for $\overline{\xi_i}$ demonstrate that it suffices to have $\overline{\xi_1}\tau$ and $\overline{\xi_j}$, with $j \geq 2$, to recreate $\overline{\xi_j}\tau$. For more details see Remark 4.16. The given set thus generates $\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}_W\mathbb{Z}$ as an algebra. The relations for τ_i^2 are from $\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2$. The second relation $(\overline{\xi_1}\tau)^2 = \overline{\xi_1}^2\tau^2$ obviously holds in $\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2$ and thus in $\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}_W\mathbb{Z}$. We have reduced all possible relations to lower powers of the generators. It remains to show that the monomials

$$\tau_0^{E_0}\tau_1^{E_1} \dots (\overline{\xi_1}\tau)^\epsilon \overline{\xi_1}^{2R_1} \overline{\xi_2}^{R_2} \overline{\xi_3}^{R_3} \dots,$$

with $\epsilon, E_i \in \{0, 1\}$ and $R_i \in \mathbb{Z}_{\geq 0}$, form a basis. In other words $\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}_W\mathbb{Z} \cong K \oplus K\overline{\xi_1}\tau$. To show that the above monomials are a basis it suffices to show that $K \oplus K\overline{\xi_1}\tau$ injects into $\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2$ via the natural map $(a, b) \mapsto a + b$. Given the splitting (3.13), it hence suffices to show that the map

$$(4.13) \quad K\overline{\xi_1}\tau \hookrightarrow \mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2 \cong K \oplus \overline{\xi_1}K \rightarrow \overline{\xi_1}K$$

is an injection. Multiplying a basis element b with $\overline{\xi_1}\tau = \tau\overline{\xi_1} + \rho\tau_0\overline{\xi_1}$ will produce a term of the form $\tau\overline{\xi_1}b + T$ with T some term with all summands being greater than $\overline{\xi_1}b$ in terms of a lexicographic order $\tau_i < \overline{\xi_i}$. Indeed, expanding τ_i^2 we get $\rho\tau_{i+1} + \tau\xi_{i+1} + \rho\tau_0\xi_{i+1}$, a term that only leads to greater value with respect to the lexicographic order by the conjugation formulas (3.7). Thus, the matrix of the map

$$(4.14) \quad K \hookrightarrow \mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2 \cong K \oplus \overline{\xi_1}K \xrightarrow{\cdot \overline{\xi_1}\tau} K \oplus \overline{\xi_1}K \rightarrow K\overline{\xi_1}\tau$$

is lower triangular with τ on a diagonal shifted by the value of $\overline{\xi_1}$ with respect to the chosen lexicographic order. Hence (4.14), and thus also (4.13), are injective. Therefore $\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}_W\mathbb{Z} \cong K \oplus K\overline{\xi_1}\tau$ as left $\mathbb{H}\mathbb{Z}/2_{**}$ -modules. This proves the claim. $\square$

Remark 4.15 This result shows $\mathbb{H}\mathbb{Z}/2 \wedge \mathbb{H}_W\mathbb{Z}$ is a free left module over $\mathbb{H}\mathbb{Z}/2$. It is also free as a right module over $\mathbb{H}\mathbb{Z}/2$. Indeed, this statement is equivalent to the map

$$\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}_W\mathbb{Z} \rightarrow \mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}_W\mathbb{Z}, \tau \mapsto \tau + \rho\tau_0,$$

being a $k_*^M(k)$ -isomorphism. To see that it is one checks what happens to basis elements, that is monomials

$$\tau_0^{E_0}\tau_1^{E_1} \dots (\overline{\xi_1}\tau)^\epsilon \overline{\xi_1}^{2R_1} \overline{\xi_2}^{R_2} \overline{\xi_3}^{R_3} \dots,$$

with $\epsilon, E_i \in \{0, 1\}$ and $R_i \in \mathbb{Z}_{\geq 0}$. We have $(\tau + \rho\tau_0)^k = \tau^k + T$ with T divisible by powers of $\rho\tau_0$. When multiplying this with a basis monomial, the equations for τ_i^2 tell us that after we expand these into basis monomials, all the summands will have increased index in E or R . Therefore they will be greater with respect to a lexicographic order $\tau_i < \overline{\xi_i}$. So the matrix of this map has 1s on the diagonal and is lower triangular, proving the claim. This is also true for the basis on $\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2$ by the same proof, see [BH21, Cor 6.4].

Remark 4.16 We would like to emphasize that $\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}_W\mathbb{Z}$ contains all of $\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2\tau$, hence the relations of Proposition 4.12 make sense. It also contains all of ξ_r^2 . More explicitly, by the conjugation formulas (3.7)

$$\xi_1^2 = \overline{\xi_1}^2, \quad \xi_r^2 = \overline{\xi_r}^2 + \sum_{i=1}^{r-1} \xi_{r-i}^{2^{i+1}} \overline{\xi_i}^2, \quad \xi_r\tau = \overline{\xi_r}\tau + \sum_{i=1}^{r-1} \xi_{r-i}^{2^i} \overline{\xi_i}\tau.$$

The first element is in $\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}_W\mathbb{Z}$ as a generator. By induction, the second formula shows that all ξ_i^2 are in $\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}_W\mathbb{Z}$. Similarly, the element $\xi_1\tau$ is in $\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}_W\mathbb{Z}$ as a generator, hence the last formula shows by induction that all $\xi_i\tau$ are in $\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}_W\mathbb{Z}$. Ideally we would write the relations of Proposition 4.12 using an explicit formula in the $\overline{\xi}_i, \overline{\xi}_1^2, \overline{\xi}_1\tau$, but this seems unfeasible. We must therefore track carefully what is in $\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}_W\mathbb{Z}$. It is in particular important to not get confused between multiplication with τ on the left and right. We mention as an example $\xi_2\tau = \overline{\xi}_2\tau + \overline{\xi}_1^2(\overline{\xi}_1\tau)$.

We will prefer to suppress the generator $\overline{\xi}_1\tau$ as $\tau_0^2 - \rho\tau_1$ and instead use the presentation

Corollary 4.17. *We have an isomorphism of left $\mathbb{H}\mathbb{Z}/2_{**}$ -algebras*

$$\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}_W\mathbb{Z} \xrightarrow[r^R]{\cong} \mathbb{H}\mathbb{Z}/2_{**}[\tau_0, \dots, \overline{\xi}_1^2, \overline{\xi}_2, \overline{\xi}_3 \dots]/(\dots) \hookrightarrow \mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2,$$

where the ideal $(\dots)$ is generated by the relations

$$\tau_0^4 = \rho^3\tau_2 + \rho^2(\overline{\xi}_1^2(\tau_0^2 - \rho\tau_1) + \overline{\xi}_2\tau) + \overline{\xi}_1^2\tau^2, \quad \tau_i^2 = \rho\tau_{i+1} + \xi_{i+1}\tau, \quad \text{for } i \geq 1.$$

4.2. Computation of $\underline{k}^M \wedge \mathbb{H}_W\mathbb{Z}$. By Proposition 3.8, the map $\tau_*^L : \mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2 \rightarrow \mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2$ is given by multiplication by τ on the left. The element τ is a non-zero divisor in $\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2$ and since $r_*^R : \mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}_W\mathbb{Z} \hookrightarrow \mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}\mathbb{Z}/2$ is a subalgebra, τ is also a non-zero divisor in $\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}_W\mathbb{Z}$. So by the long exact sequence of the second column of (4.2) we have the short exact sequence

$$(4.18) \quad 0 \rightarrow \Sigma^{0,-1}\mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}_W\mathbb{Z} \xrightarrow{\tau_*^L} \mathbb{H}\mathbb{Z}/2_{**}\mathbb{H}_W\mathbb{Z} \xrightarrow{\pi_*^L} \underline{k}_{**}^M\mathbb{H}_W\mathbb{Z} \rightarrow 0.$$

It follows immediately

Proposition 4.19. *We have as a left $\underline{k}_{**}^M$ -algebra*

$$\underline{k}_{**}^M\mathbb{H}_W\mathbb{Z} \cong \underline{k}_{**}^M[\tau_0, \dots, \overline{\xi}_1^2, \overline{\xi}_2, \overline{\xi}_3 \dots]/(\dots),$$

where the ideal $(\dots)$ is generated by the relations

$$\tau_0^4 = \rho^3(\tau_0\overline{\xi}_2 + \tau_1\overline{\xi}_1^2 + \tau_2), \quad \tau_r^2 = \rho\tau_{r+1} + \xi_r^2(\tau_0^2 - \rho\tau_1) + \rho\tau_0 \sum_{i=2}^{r+1} \xi_{r+1-i}^2 \overline{\xi}_i, \quad \text{for } r \geq 1.$$

Remark 4.20 The description shows that $\underline{k}^M \wedge \mathbb{H}_W\mathbb{Z}$ is a free left module over $\underline{k}^M$ with basis given by monomials

$$\tau_0^{E_0} \tau_1^{E_1} \dots \overline{\xi}_1^{-2R_1} \overline{\xi}_2^{-R_2} \overline{\xi}_3^{-R_3} \dots,$$

with $E_0 \in \{0, 1, 2, 3\}, E_i \in \{0, 1\}$ and $R_i \in \mathbb{Z}_{\geq 0}$. Alternatively, one can enumerate these monomials as $\tau_0^f \tau_F \overline{\xi}_I \overline{\xi}_S^{2S}$ with $f \in \{0, 1, 2, 3\}, F \subset \mathbb{Z}_{\geq 1}$ finite, $I \subset \mathbb{Z}_{\geq 2}$ finite and $S = (S_1, S_2, \dots)$ a finite sequence in $\mathbb{Z}_{\geq 0}$.

4.3. Computation of $\underline{K}^W \wedge \mathbb{H}_W\mathbb{Z}$. We now want to upgrade the left side of $\mathbb{H}\mathbb{Z}/2 \wedge \mathbb{H}_W\mathbb{Z}$ from $\mathbb{H}\mathbb{Z}/2$ to $\underline{K}^W$ through $\underline{k}^M$. We proceed similarly to our computation of $\mathbb{H}\mathbb{Z}/2 \wedge \underline{K}^W$, but now we apply homology on the other side.

Taking $(-)_**\mathbb{H}_W\mathbb{Z}$ of (2.1) we get a diagram with long exact row

$$(4.21) \quad \begin{array}{ccccccc} \dots & \xrightarrow{\eta} & \underline{K}_{**}^W\mathbb{H}_W\mathbb{Z} & \xrightarrow{r_*^L} & \underline{k}_{**}^M\mathbb{H}_W\mathbb{Z} & \xrightarrow{\partial_*^L} & \Sigma^{2,1}\underline{K}_{**}^W\mathbb{H}_W\mathbb{Z} \xrightarrow{\eta} \dots \\ & & & & \searrow d_*^L & & \downarrow r_*^L \\ & & & & & & \Sigma^{2,1}\underline{k}_{**}^M\mathbb{H}_W\mathbb{Z}. \end{array}$$

We have a precise description of d_*^L below, and we want to use it for this computation. We will show that $\ker(\partial_*^L) = \ker(d_*^L)$, and we will demonstrate this by showing that $\underline{K}_{**}^W\mathbb{H}_W\mathbb{Z}$ has no higher η -torsion. To do so, we will first compute $\ker(d_*^L)$ and the homology $H(d_*^L)$. Together with the nonexistence of higher η -torsion, this allows us to make deductions about relations in $\underline{K}_{**}^W\mathbb{H}_W\mathbb{Z}$ and explains what the kernel of r_*^L is. The homology of d_*^L is a shadow of the $\underline{K}_*^W$ -free part of $\underline{K}^W \wedge \mathbb{H}_W\mathbb{Z}$.

Similar to d_*^R , d_*^L is dictated by its lift $(\mathrm{Sq}^2)_*^L$ to $\mathrm{HZ}/2_{**}\mathrm{H}_W\mathbb{Z} \hookrightarrow \mathrm{HZ}/2_{**}\mathrm{HZ}/2$ since π_*^L is surjective by (4.18). We compute in $\mathrm{HZ}/2_{**}\mathrm{HZ}/2$ using (3.7), Proposition 3.8 and Proposition 3.11, and define for finite $I \subset \mathbb{Z}_{\geq 1}$

$$\begin{aligned} (\mathrm{Sq}^2)_*^L(\tau_1) &= \tau_0, & (\mathrm{Sq}^2)_*^L(\tau_i) &= 0, \quad \text{for } i \neq 1, \\ (\mathrm{Sq}^2)_*^L(\bar{\xi}_I) &= \sum_{i \in I} \bar{\xi}_{i-1}^2 \bar{\xi}_{I \setminus i} =: c(I), & (\mathrm{Sq}^2)_*^L(\tau_1 \bar{\xi}_I) &= \tau_0 \bar{\xi}_I + \tau_1 c(I) := c_1(I). \end{aligned}$$

Note $c(j+1) = \bar{\xi}_j^2$ and $c(1) = 1$ and $c(\emptyset) = 0$ and $c_1(\emptyset) = \tau_0$. Let us recall that the degrees are

$$\begin{aligned} |\bar{\xi}_i| &= (2^i - 1)(2, 1), & |\tau_i| &= (2^i - 1)(2, 1) + (1, 0), \\ |c(I)| &= ((\sum_{i \in I} 2^i - 1) - 1)(2, 1), & |c_1(I)| &= \sum_{i \in I} (2^i - 1)(2, 1) + (1, 0). \end{aligned}$$

Lemma 4.22. *Let $I, J \subset \mathbb{Z}_{\geq 2}$ be finite. In $\mathrm{HZ}/2_{**}\mathrm{H}_W\mathbb{Z} \subset \mathrm{HZ}/2_{**}\mathrm{HZ}/2$ we have the relations*

$$\begin{aligned} c_1(\emptyset)^4 &= \tau_0^4 = \rho^3(\tau_2 + c_1(2)) + \rho^2 \tau \bar{\xi}_2 + \tau^2 c(2), \\ c(I)c(J) &= \sum_{i \in I \cap J} c(i)c(I \Delta J \amalg i) \prod_{j \in I \cap J \setminus i} c(j+1) + \sum_{i \in I \setminus J} c(i)c(I \Delta J \setminus i) \prod_{j \in I \cap J} c(j+1), \\ c(I)c_1(J) &= \sum_{i \in I \cap J} c(i)c_1(I \Delta J \amalg i) \prod_{j \in I \cap J \setminus i} c(j+1) + \sum_{i \in I \setminus J} c(i)c_1(I \Delta J \setminus i) \prod_{j \in I \cap J} c(j+1), \\ c_1(I)c_1(J) &= \tau_0 c_1(I \Delta J) \prod_{r \in I \cap J} c(r+1) + (\rho \tau_2 + \xi_2 \tau) c(I)c(J). \end{aligned}$$

Proof. The formula for $c_1(\emptyset)^4$ is a restatement of the relation in Corollary 4.17. We prove the second formula. The others follow by similar arguments.

$$\begin{aligned} c(I)c(J) &= \sum_{i \in I, j \in J} \bar{\xi}_{i-1}^2 \bar{\xi}_{j-1}^2 \bar{\xi}_{I \setminus i} \bar{\xi}_{J \setminus j} = \sum_{i \in I \cap J} \left[\sum_{j \in J \setminus I \amalg i} + \sum_{j \in I \cap J \setminus i} \right] + \sum_{j \in I \cap J} \sum_{i \in I \setminus J} + \sum_{j \in J \setminus I} \sum_{i \in I \setminus J} \\ &= \sum_{i \in I \cap J} \sum_{j \in I \Delta J \amalg i} + \sum_{i \in I \setminus J} \sum_{j \in J \setminus I} + \sum_{i \in I \setminus J} \sum_{j \in I \setminus J \setminus i} = \sum_{i \in I \cap J} \sum_{j \in I \Delta J \amalg i} + \sum_{i \in I \setminus J} \sum_{j \in I \Delta J \setminus i} \\ &= \sum_{i \in I \cap J} \bar{\xi}_{i-1}^2 \sum_{j \in I \Delta J \amalg i} \bar{\xi}_{j-1}^2 \bar{\xi}_{I \Delta J \amalg i \setminus j} \bar{\xi}_{I \cap J \setminus i}^2 + \sum_{i \in I \setminus J} \bar{\xi}_{i-1}^2 \sum_{j \in I \Delta J \setminus i} \bar{\xi}_{j-1}^2 \bar{\xi}_{I \Delta J \setminus i \setminus j} \bar{\xi}_{I \cap J}^2. \end{aligned}$$

Here we have omitted the terms $\bar{\xi}_{i-1}^2 \bar{\xi}_{j-1}^2 \bar{\xi}_{I \setminus i} \bar{\xi}_{J \setminus j}$ under the sums after the second and third equal sign to improve readability. In the third equation we use symmetry to remove the double sum $0 = \sum_{i \in I \cap J} \sum_{j \in I \cap J \setminus i}$ as well as to add $0 = \sum_{i \in I \setminus J} \sum_{j \in I \setminus J \setminus i}$. $\square$

Remark 4.23 We will often use the notation $c_1(\emptyset)$ and τ_0 interchangeably. In the above we also used $\xi_2 \tau$. It can be expressed as $\xi_2 \tau = \tau \bar{\xi}_2 + \tau^2 c(2) + \rho c_1(2)$.

The map d_*^L is a derivation on $k_{**}^M \mathrm{H}_W\mathbb{Z}$, thanks to Proposition 3.11. Therefore, its kernel is a subring of $k_{**}^M \mathrm{H}_W\mathbb{Z}$ and its image is an ideal inside it. The elements we have just computed are not all elements d_*^L sends to zero. For example, let $r \in k_*^M(k)$ be ρ^3 -torsion, then $rd_*^L(\tau_0^3 \tau_1) = r\tau_0^4 = r\rho^3(c_1(1) + \tau_2) = 0$. Define $b := \tau_0^3 \tau_1$ if there exists ρ -torsion in $k_*^M(k)$ and let $b = 0$ if $k_*^M(k)$ is ρ -torsion free. The degree of b is $|b| = (6, 1)$.

Lemma 4.24. *Let the base field k satisfy $\rho^3 = 0$. Then*

$$\begin{aligned} \ker(d_*^L) &= k_{**}^M[b, \tau_2, \tau_3, \dots, c(I), c_1(I)]/(\dots) \hookrightarrow k_{**}^M \mathrm{H}_W\mathbb{Z}, \\ \mathrm{im}(d_*^L) &= (c(I), c_1(I)) \hookrightarrow \ker(d_*^L), \\ \mathrm{H}(d_*^L) &= k_{**}^M[\tau_2, b, \tau_3, \tau_4, \dots]/(b^2, \tau_j^2 - \rho \tau_{j+1}), \end{aligned}$$

with index $I \subset \mathbb{Z}_{\geq 2}$ running over all finite subsets of $\mathbb{Z}_{\geq 2}$. The ideal $(\dots)$ is generated by the relations in Lemma 4.22 mod τ as well as

$$\begin{aligned} c(\emptyset) &= 0, & c_1(\emptyset) &= \tau_0, & \tau_0^4 &= 0, & \tau_j^2 &= \rho \tau_{j+1} + \xi_{j+1} \tau, \quad \text{for } j \geq 2, \\ b^2 &= b\tau_0 = 0, & bc(I) &= \tau_0^3 c_1(I), & bc_1(I) &= \rho \tau_0^3 (c_1(2) + \tau_2) c(I), \end{aligned}$$

Proof. We will divide the proof into to the following steps:

- (1) Generators of the image of d_*^L .
- (2) Relations in $\ker(d_*^L)$.
- (3) Computation of the kernel of d_*^L .
 - (3.1) Elements of $V \cap \ker(d_*^L)$.
 - (3.2) General elements of $\ker(d_*^L)$.
- (4) Computation of $H(d_*^L)$.

Here V is a convenient submodule of $k_{**}^M H_W \mathbb{Z}$ defined in Step (3) below.

Step (1): Generators of the image of d_^L :*

We begin by finding generators of the image. We have seen above that τ_i lies in $\ker(d_*^L)$ when $i \neq 1$. Since d_*^L is a derivation, all squares lie in its kernel as well. Let $f \in \{0, 1, 2, 3\}$, $F \subset \mathbb{Z}_{\geq 1}$ finite, $I \subset \mathbb{Z}_{\geq 2}$ finite and $S = (S_1, S_2, \dots)$ a finite sequence in $\mathbb{Z}_{\geq 0}$. It follows that for a basis monomial we have

$$d_*^L(\tau_0^f \tau_F \bar{\xi}_I \bar{\xi}^{2S}) = \tau_0^f \tau_{F \setminus 1} \bar{\xi}^{2S} d_*^L(\tau_{F \cap 1} \bar{\xi}_I).$$

It thus suffices to show that we can write $d_*^L(\tau_{F \cap 1} \bar{\xi}_I)$ in the claimed form. This is clear since it is either $c(I)$ or $c_1(I)$. This completes Step (1).

Step (2): Relations in $\ker(d_^L)$:*

We now establish the relations. The relation for τ_0^4 follows because $\rho^3 = 0$. This immediately implies $b^2 = b\tau_0 = 0$, $bc(I) = \tau_0^3 c_1(I)$. Furthermore,

$$bc_1(I) = \tau_0^3(\rho\tau_2 + \xi_2\tau)c(I) = \tau_0^3(\rho\tau_2 + \rho c_1(2) + \tau_0^2 c(2))c(I) = \rho\tau_0^3(\tau_2 + c_1(2))c(I).$$

This completes Step (2).

Step (3): Computation of the kernel of d_^L :*

To see that the kernel is given as the claimed algebra we first note that all the generators are in the kernel of d_*^L . What remains to be shown to complete Step (3) is that any element in the kernel of d_*^L is a linear combination of the listed generators. We divide this task into two parts. Let us define the following submodule $V := \langle \tau_0^f \tau_{F \setminus 1} \bar{\xi}_I \bar{\xi}^{2S} \mid \bar{\xi}_I \bar{\xi}^{2S} \neq 1 \rangle$ of $k_{**}^M H_W \mathbb{Z}$. The first part (3.1) will be showing that any element of $V \cap \ker(d_*^L)$ is a linear combination of the given generators. The second part then reduces the general case to the case in (3.1).

Step (3.1): Elements of $V \cap \ker(d_^L)$:*

Now we consider linear combinations of basis elements $\tau_0^f \tau_{F \setminus 1} \bar{\xi}_I \bar{\xi}^{2S}$. We will show that any such linear combination in the kernel must already be a polynomial in $c(I)$, $\tau_0^f \tau_{F \setminus 1}$. Since the submodule $\langle \tau_0^f \tau_{F \setminus 1} \rangle$ is free and $d_*^L = \text{id} \otimes d_*^L$ on $V = \langle \tau_0^f \tau_{F \setminus 1} \bar{\xi}_I \bar{\xi}^{2S} \mid \bar{\xi}_I \bar{\xi}^{2S} \neq 1 \rangle \cong \langle \tau_0^f \tau_{F \setminus 1} \rangle \otimes \langle \bar{\xi}_I \bar{\xi}^{2S} \neq 1 \rangle$, we can reduce to linear combinations of $\bar{\xi}_I \bar{\xi}^{2S} \neq 1$. We can write any such linear combination as

$$T = \sum_{R \neq \emptyset} \sum_{J \subset \text{supp}(R)} a_{J,R} \bar{\xi}_{J+1} \bar{\xi}^{2(R-e_J)},$$

where $J \subset \mathbb{Z}_{\geq 1}$ finite, $R = (R_1, R_2, \dots)$ is a finite sequence in $\mathbb{Z}_{\geq 0}$ and $\text{supp}(R) = \{i \in \mathbb{Z}_{\geq 1} \mid R_i \neq 0\}$. Assume $T \in \ker(d_*^L)$, i.e.

$$0 = d_*^L(T) = \sum_{R \neq \emptyset} \sum_{J \subset \text{supp}(R)} a_{J,R} \sum_{j \in J} \bar{\xi}_{(J \setminus j)+1} \bar{\xi}^{2(R-e_{J \setminus j})}.$$

We have

$$\bar{\xi}_{(J_1 \setminus j_1)+1} \bar{\xi}^{2(R_1-e_{J_1 \setminus j_1})} = \bar{\xi}_{(J_2 \setminus j_2)+1} \bar{\xi}^{2(R_2-e_{J_2 \setminus j_2})}$$

if and only if $J_1 \setminus j_1 = J_2 \setminus j_2$ and $R_1 = R_2$. We therefore get for each $R \neq \emptyset$ the equations

$$\begin{aligned} 0 &= \sum_{J \subset \text{supp}(R)} a_{J,R} \sum_{j \in J} \bar{\xi}_{(J \setminus j)+1} \bar{\xi}^{2(R-e_{J \setminus j})} \\ &= \sum_{B \subset \text{supp}(R)} \bar{\xi}_{B+1} \bar{\xi}^{2(R-e_B)} \sum_{r \in \text{supp}(R) \setminus B} a_{B \amalg r, R}, \end{aligned}$$

implying

$$(4.25) \quad \sum_{r \in \text{supp}(R) \setminus B} a_{B \amalg r, R} = 0$$

for every set $B \subset \text{supp}(R)$. Define for a sequence R as above $t_R := \min(\text{supp}(R))$. We claim that the element

$$U := \sum_{R \neq \emptyset} \sum_{J \subset \text{supp}(R) \setminus t_R} a_{J, R} \bar{\xi}_{(J \amalg t_R) + 1} \bar{\xi}^{2(R - e_{J \amalg t_R})}$$

lies in the preimage of T , i.e. $d_*^L(U) = T$. Indeed, we have

$$\begin{aligned} d_*^L(U) &= \sum_{R \neq \emptyset} \sum_{J \subset \text{supp}(R) \setminus t_R} a_{J, R} \bar{\xi}_{J+1} \bar{\xi}^{2(R - e_J)} \\ &\quad + \sum_{R \neq \emptyset} \sum_{J \subset \text{supp}(R) \setminus t_R} a_{J, R} \sum_{j \in J} \bar{\xi}_{(J \amalg t_R \setminus j) + 1} \bar{\xi}^{2(R - e_{(J \amalg t_R) \setminus j})}. \end{aligned}$$

We can rewrite the terms in the second line using equation (4.25)

$$\begin{aligned} &\sum_{J \subset \text{supp}(R) \setminus t_R} a_{J, R} \sum_{j \in J} \bar{\xi}_{(J \amalg t_R \setminus j) + 1} \bar{\xi}^{2(R - e_{(J \amalg t_R) \setminus j})} \\ &= \sum_{B \subset \text{supp}(R) \setminus t_R} \bar{\xi}_{(B \amalg t_R) + 1} \bar{\xi}^{2(R - e_{B \amalg t_R})} \sum_{r \in \text{supp}(R) \setminus (B \amalg t_R)} a_{B \amalg r, R} \\ &= \sum_{B \subset \text{supp}(R) \setminus t_R} \bar{\xi}_{(B \amalg t_R) + 1} \bar{\xi}^{2(R - e_{B \amalg t_R})} a_{B \amalg t_R, R} \\ &= \sum_{J \amalg t_R \subset \text{supp}(R)} a_{J \amalg t_R, R} \bar{\xi}_{(J \amalg t_R) + 1} \bar{\xi}^{2(R - e_{J \amalg t_R})}. \end{aligned}$$

Thus $d_*^L(U) = T$ and therefore $\ker(d_{*|_V}^L) = \text{im}(d_{*|_V}^L)$. This completes Step (3.1).

Step (3.2): General elements of $\ker(d_^L)$:*

We now turn to a general element in the kernel of d_*^L . In the following all sums go over f, F, I, S as in Remark 4.20 and we suppress these index sets in the coefficients of the basis elements

$$\begin{aligned} \sum_{f, F, I, S} a_{f, F, I, S} d_*^L(\tau_0^f \tau_F \bar{\xi}_I \bar{\xi}^{2S}) &= \sum_{f, F, I, S} a_{f, F, I, S} d_*^L(\tau_{F \cap I}^f \bar{\xi}_I) \tau_0^f \tau_{F \setminus I} \bar{\xi}^{2S} \\ &= \sum_{f=0} [a_{f=0, 1 \notin F} c(I) + a_{f=0, 1 \in F} c_1(I)] \tau_{F \setminus 1} \bar{\xi}^{2S} \\ &\quad + \sum_{f=1} [a_{f=1, 1 \notin F} \tau_0 c(I) + a_{f=1, 1 \in F} \tau_0 c_1(I)] \tau_{F \setminus 1} \bar{\xi}^{2S} \\ &\quad + \sum_{f=2} [a_{f=2, 1 \notin F} \tau_0^2 c(I) + a_{f=2, 1 \in F} \tau_0^2 c_1(I)] \tau_{F \setminus 1} \bar{\xi}^{2S} \\ &\quad + \sum_{f=3} [a_{f=3, 1 \notin F} \tau_0^3 c(I) + a_{f=3, 1 \in F} \tau_0^3 \tau_1 c(I)] \tau_{F \setminus 1} \bar{\xi}^{2S}. \end{aligned}$$

We assume that the above sum is zero. In the last row we used the relation $\tau_0^3 c_1(I) = bc(I)$. Note that the very first sum $\sum_{f=0} a_{f=0, 1 \notin F} c(I)$ contains the only elements not divisible by τ_0 or τ_1 . It is thus independent of all the others. We have already shown that any sum of that form comes from an element in the image of d_*^L . We may thus assume that $a_{f=0, 1 \notin F} = 0$. Similarly, the very last sum contains the only elements divisible by $\tau_0^3 \tau_1$. By the same logic we may assume $a_{f=3, 1 \in F} = 0$. The elements $\tau_0^2 c_1(I) = \tau_0^3 \bar{\xi}_I + \tau_0^2 \tau_1 c(I)$ are the only ones that contain summands divisible by $\tau_0^2 \tau_1$. With the same argument we deduce $a_{f=2, 1 \in F} = 0$ which then leads to $a_{f=3, 1 \notin F} = 0$. We repeat the argument with elements divisible by $\tau_0 \tau_1$ to deduce $a_{f=1, 1 \in F} = 0$. This leads to $a_{f=2, 1 \notin F} = 0$. The last two vanish by the same reason. This shows that the kernel is generated by elements as claimed. The relations we have established reduce every possible relation to terms of low powers in the generators. One can check that the remaining monomials are linearly independent in $k_{**}^M H_W \mathbb{Z}$ and thus generate the ideal (...).

Step (4): Computation of $H(d_^L)$:*

Taking quotients we see immediately that $H(d_*^L)$ is generated as claimed. We have $\xi_1 \xi_i \tau \in H\mathbb{Z}/2_{**} H_W \mathbb{Z}$

and thus also in $\underline{k}_{**}^M H_W \mathbb{Z}$. Therefore, $\xi_j \tau = d_*^L(\xi_1 \xi_j \tau)$, for $j \geq 2$, is in the image of d_*^L . This shows the relation $\tau_j^2 - \rho \tau_{j+1} = 0$ in $H(d_*^L)$. The relation $b^2 = 0$ already holds before taking quotients. The last relation $\rho^3 \tau_2$ follows from the relation of τ_0^4 . Since all monomials with no higher powers in b and τ_j , for $j \geq 2$, are linearly independent in $\underline{k}_{**}^M H_W \mathbb{Z}$, these are all relations in $H(d_*^L)$. $\square$

Remark 4.26 The condition $\rho^3 = 0$ holds for any field of odd characteristic. It further holds for fields that are extensions of totally imaginary number fields, since k_3^M vanishes for these fields and field extensions induce a ring map on Milnor K-theory which preserves ρ . We note that all fields with $\rho^3 = 0$ are nonreal [Lam05, Cor. 6.20].

In the next two Propositions our base field will be assumed to satisfy $\rho^3 = 0$ and $\text{vcd}_2(k) < \infty$. In particular k is nonreal. By Corollary 2.8 and Proposition 2.3, all combinations of smash products of the spectra $H_W \mathbb{Z}, H\mathbb{Z}/2, \underline{K}^W, \underline{k}^M$ are η -complete under this assumption.

Proposition 4.27. *Let the base field k satisfy $\text{vcd}_2(k) < \infty$, $\rho^3 = 0$ and $k_4^M(k) = 0$. Then we have $\text{im}(\underline{r}_*^L) = \ker(d_*^L)$ and $\underline{K}_{**}^W H_W \mathbb{Z}$ has no higher η -torsion.*

Proof. Recall (4.21)

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\eta} & \underline{K}_{**}^W H_W \mathbb{Z} & \xrightarrow{\underline{r}_*^L} & \underline{k}_{**}^M H_W \mathbb{Z} & \xrightarrow{\underline{\partial}_*^L} & \Sigma^{2,1} \underline{K}_{**}^W H_W \mathbb{Z} \xrightarrow{\eta} \cdots \\ & & & & & \searrow d_*^L & \downarrow \underline{r}_*^L \\ & & & & & & \Sigma^{2,1} \underline{k}_{**}^M H_W \mathbb{Z}. \end{array}$$

Let $x \in \underline{K}_{**}^W H_W \mathbb{Z}$ be higher η -torsion, i.e. $x = \underline{\partial}_*^L(y) = a\eta^k$ with $k \geq 1$. Since x is divisible by η iff it lies in the kernel of $\underline{r}_*^L$, we know that $y \in \ker(d_*^L)$. Adding elements in $\text{im}(d_*^L) \subset \ker(\underline{\partial}_*^L)$ to y does not change their image under $\underline{\partial}_*^L$. Therefore, without loss of generality, y is a polynomial in b and τ_j for $j \geq 2$ by Lemma 4.24. Since $\underline{r}_*^L$ is a ring map and we want to show that $x = 0$, i.e. y is in the image of $\underline{r}_*$, it suffices to show that $b, \tau_j \in \text{im}(\underline{r}_*^L)$ for $j \geq 2$. Indeed, if they all are in the image, then so is any polynomial in them.

Let us first assume $y = \tau_j$ with $j \geq 2$. Since $\underline{K}^W$ is a connective E_∞ -ring spectrum and $\underline{K}^W \wedge H_W$ is η -complete, each i -line $\pi_i(\underline{K}^W \wedge H_W)_*$ is η -complete as a $\underline{K}_*^W$ -module by Theorem 2.6. In particular, every i -line $\pi_i(\underline{K}^W \wedge H_W)_*$ is separated with respect to (η) . Thus, we can pick $k \geq 1$ maximal with $x = \underline{\partial}_*^L(\tau_j) = a\eta^k$, i.e. s.t. $z := \underline{r}_*^L(a) \neq 0$. Clearly, $z \in \ker(d_*^L)$. τ_j lives on the 2^j -line, b lives on the $2^2 + 1 = 5$ -line and z on the $2^j - 1$ -line. By degree reasons, z maps to zero in $H(d_*^L)$, i.e. it lies in the image of d_*^L . Therefore, there are $u \in \underline{k}_{**}^M H_W \mathbb{Z}$ and $v \in \pi_{**} L$ such that

$$z = \underline{r}_*^L(a) = \underline{r}_*^L(\underline{\partial}_*^L(u)) \rightsquigarrow a - \underline{\partial}_*^L(u) \in \ker(\underline{r}_*^L) \longleftrightarrow a - \underline{\partial}_*^L(u) = \eta^m v.$$

Here we again choose $m \geq 1$ maximal. Since the image of $\underline{\partial}_*^L$ is η -torsion, this implies:

$$x = \eta^k a = \eta^k (\eta^m v + \underline{\partial}_*^L(u)) = \eta^{k+m} v$$

contradicting maximality of k . We conclude $x = 0$ and thus $\tau_j \in \text{im}(\underline{r}_*^L)$.

Let now $y = b = \tau_0^3 \tau_1$. In this case x lives on the 4-line, just like τ_2 . The only possibility is thus $z = s\tau_2$ for $s \in k_n^M(k)$. We have $|x| = (4, 0)$ and $|\tau_2| = (7, 3)$ and thus $n \geq 4$. By our assumptions on the base field, we must have $s = 0$. We reduce $x = 0$ and thus $b \in \text{im}(\underline{r}_*^L)$ by the non-existence of infinitely η -divisible elements, as before. $\square$

We thus see that lifts of monomials in τ_j , for $j \geq 2$, correspond to an η -torsion free part and η -torsion corresponds to lifts of $\text{im}(d_*^L)$. We can lift η -torsion canonically and $\underline{r}_*^L$ maps it isomorphically to $\text{im}(d_*^L)$. We will thus abuse notation and denote η -torsion in $\underline{K}_{**}^W H_W \mathbb{Z}$ by its image under $\underline{r}_*^L$. More precisely, we will denote

$$\underline{\partial}_*^L(\tau_1) = \tau_0 = c_1(\emptyset), \quad \underline{\partial}_*^L(\bar{\xi}_I) = c(I), \quad \underline{\partial}_*^L(\tau_1 \bar{\xi}_I) = c_1(I).$$

Moreover, we choose $\underline{r}_*^L$ lifts t_j of τ_j for $j \geq 2$ and s of b . They exist by the previous Proposition. Since $\underline{r}_*^L$ is a left $\underline{k}_*^M$ -algebra map, it is clear that the product of two lifts is a lift of the product. Using this property we have lifts of all elements in $\text{im}(\underline{r}_*^L)$. Note that our choice of lifts t_j and s is in general not canonical and only up to the ideal generated by η .

Proposition 4.28. *Assumptions as in Proposition 4.27, we have an isomorphism of left $\underline{K}_{**}^W$ -algebras*

$$\underline{K}_{**}^W H_W \mathbb{Z} \cong \underline{K}_{**}^W[s, t_2, t_3, \dots, c(I), c_1(I)]/(\dots),$$

with index $I \subset \mathbb{Z}_{\geq 2}$ running over all finite subsets of $\mathbb{Z}_{\geq 2}$. The ideal of η -torsion is generated by $c(I), c_1(I)$. The ideal $(\dots)$ is generated by the relations in Lemma 4.22 mod τ as well as the following

$$\begin{aligned} \eta c(I) = \eta c_1(I) = c(\emptyset) = 0, & \quad c_1(\emptyset) = \tau_0, & \quad \tau_0^4 = 0, & \quad t_j^2 = \rho t_{j+1} + \xi_{j+1} \tau + O(\eta), \\ s^2 = s\tau_0 = 0 & \quad sc(I) = \tau_0^3 c_1(I), & \quad sc_1(I) = \rho \tau_0^3 (c_1(2) + \tau_2) c(I). \end{aligned}$$

Proof. Since $\underline{K}^W$ is a connective E_∞ -ring spectrum and $\underline{K}^W \wedge H_W \mathbb{Z}$ is η -complete, each i -line $\pi_i(\underline{K}^W \wedge H_W \mathbb{Z})_*$ is η -complete as a $\underline{K}_{**}^W$ -module by Theorem 2.6 and $\underline{K}_{**}^W$ is η -complete. By Proposition 4.27, $r_*^L(\pi_i(\underline{K}^W \wedge H_W \mathbb{Z})_*)$ is finitely generated over $\underline{k}_{**}^M$ by certain monomials in $b, \tau_j, c(I), c_1(I)$ as described in Lemma 4.24. By Nakayama's Lemma [Mat89, Thm. 8.4], their lifts generate $\pi_i(\underline{K}^W \wedge H_W \mathbb{Z})_*$ as a left $\underline{K}_{**}^W$ -module. Hence, $\underline{K}_{**}^W H_W \mathbb{Z}$ is generated as left $\underline{K}_{**}^W$ -algebra by the lifts $s, t_j, c(I), c_1(I)$.

By Lemma 4.27 there is no higher η -torsion, hence η -torsion is generated as claimed. The map r_*^L maps η -torsion isomorphically to $\text{im}(d_*^L)$. Any relations in Lemma 4.24 involving only η -torsion thus hold verbatim in $\underline{K}_{**}^W H_W$. Relations that involve elements that are not η -torsion only hold up to the ideal generated by η . These relations thus have η -residue $O(\eta)$. More precisely, η -residue is present in the relations involving $t_j^2 - \rho t_{j+1}$. $\square$

Example 4.29 Let k be a field of characteristic not 2 with $\text{vcd}_2(k) < \infty$ and $k_2^M(k) = 0$, then $K_2^W(k) = 0$ by [Lam05, Cor. 6.19]. Therefore, in this case $(\eta)_1 = 0$ in $K_*^W(k)$. It follows that $\rho \tau_j$ have a unique lift over these fields. Examples of such fields are finite fields $\mathbb{F}_q$ with q odd as well as quadratically closed fields.

Remark 4.30 By the same arguments, one can prove η -completed versions of Proposition 4.27 and Proposition 4.28 for base fields k without the assumption on finite virtual cohomological 2-dimension. To do so one additionally uses the fact that η -completion of connective spectra is given by $\mathbb{H}\mathbb{Z}$ -localization [Man18, 5.2], which preserves cofiber sequences and ring structures. Hence, $(\underline{K}^W \wedge H_W \mathbb{Z})_\eta^\wedge$ is still a ring spectrum and one still has a diagram (4.21) after η -completion. In this case one gets an isomorphism of left $(\underline{K}_{**}^W)_\eta^\wedge$ -algebras

$$\pi_{**}((\underline{K}^W \wedge H_W \mathbb{Z})_\eta^\wedge) \cong (\underline{K}_{**}^W)_\eta^\wedge[s, t_2, t_3, \dots, c(I), c_1(I)]/(\dots),$$

where η -torsion and the ideal $(\dots)$ have the same generators and relations as in Proposition 4.28.

Remark 4.31 We hope that a choice of orientation $\text{MSL} \rightarrow H_W \mathbb{Z}$ will lead to a consistent choice of lifts and remove η -residues over general fields, similarly to how it was done for kw in [BH21] or [Ana17].

5. $H_W \mathbb{Z} \wedge H_W \mathbb{Z}$

We now come to $H_W \mathbb{Z} \wedge H_W \mathbb{Z}$. From the pullback square (4.5) we obtain, by similar arguments as in the proof of Lemma 4.9, a pullback square

$$(5.1) \quad \begin{array}{ccc} H_W \mathbb{Z}_{**} H_W \mathbb{Z} & \xrightarrow{r_*^L} & \mathbb{H}\mathbb{Z}/2_{**} H_W \mathbb{Z} \\ \downarrow (\pi_W)_*^L & & \downarrow \pi_*^L \\ \underline{K}_{**}^W H_W \mathbb{Z} & \xrightarrow{r_*^L} & \underline{k}_{**}^M H_W \mathbb{Z}. \end{array}$$

By this pullback square we will get generators for $\tau_0, s, t_j, c(I), c_1(I)$ as well as $\tau \bar{\xi}_1$ and $\tau \tau_1 \bar{\xi}_1$ for all $I \subset \mathbb{Z}_{\geq 2}$ finite. These lie in the kernel of $(\pi_W)_*^L$ and come from $\tau_*^L : \mathbb{H}\mathbb{Z}/2_{**} H_W \mathbb{Z} \rightarrow H_W \mathbb{Z}_{**} H_W \mathbb{Z}$. We denote

$$t(I) := \tau_*^L(\bar{\xi}_1), \quad t_1 := t_1(\emptyset) := \tau \tau_1, \quad t_1(I) := \tau_*^L(\tau_1 \bar{\xi}_1),$$

Note that the image of τ_*^L is η -torsion, since it comes from elements in $\mathbb{H}\mathbb{Z}/2_{**} H_W \mathbb{Z} \hookrightarrow \mathbb{H}\mathbb{Z}/2_{**} \mathbb{H}\mathbb{Z}/2$. Remember that we have a diagram

$$\begin{array}{ccc} \Sigma^{0,-1} \mathbb{H}\mathbb{Z}/2_{**} H_W \mathbb{Z} & \xrightarrow{id} & \Sigma^{0,-1} \mathbb{H}\mathbb{Z}/2_{**} H_W \mathbb{Z} \\ \downarrow \tau_*^L & & \downarrow \tau_*^L \\ H_W \mathbb{Z}_{**} H_W \mathbb{Z} & \xrightarrow{r_*^L} & \mathbb{H}\mathbb{Z}/2_{**} H_W \mathbb{Z} \end{array}$$

which we get from the top right square of (4.2). This means that we can determine relations of η -torsion inside $H_W \mathbb{Z}_{**} H_W \mathbb{Z}$ simply in $H\mathbb{Z}/2_{**} H_W \mathbb{Z}$ instead.

Lemma 5.2. *Let $I, J \subset \mathbb{Z}_{\geq 2}$ be finite. In $H_W \mathbb{Z}_{**} H_W \mathbb{Z}$ we have the relations*

$$\begin{aligned}
t(I)t(J) &= \tau t(I\Delta J) \prod_{i \in I \cap J} c(i+1), \\
t(I)t_1(J) &= \tau t_1(I\Delta J) \prod_{i \in I \cap J} c(i+1), \\
t_1(I)t_1(J) &= \tau(\rho\tau_2 + \xi_2\tau) t(I\Delta J) \prod_{i \in I \cap J} c(i+1), \\
c(I)t(J) &= \sum_{i \in I \cap J} c(i)t(I\Delta J \amalg i) \prod_{j \in I \cap J \setminus i} c(j+1) + \sum_{i \in I \setminus J} c(i)t(I\Delta J \setminus i) \prod_{j \in I \cap J} c(j+1), \\
c(I)t_1(J) &= \sum_{i \in I \cap J} c(i)t_1(I\Delta J \amalg i) \prod_{j \in I \cap J \setminus i} c(j+1) + \sum_{i \in I \setminus J} c(i)t_1(I\Delta J \setminus i) \prod_{j \in I \cap J} c(j+1), \\
c_1(I)t(J) &= \tau_0 t(I\Delta J) \prod_{i \in I \cap J} c(i+1) + \sum_{i \in I \cap J} c(i)t_1(I\Delta J \amalg i) \prod_{j \in I \cap J \setminus i} c(j+1) \\
&\quad + \sum_{i \in I \setminus J} c(i)t_1(I\Delta J \setminus i) \prod_{j \in I \cap J} c(j+1), \\
c_1(I)t_1(J) &= \tau_0 t_1(I\Delta J) \prod_{i \in I \cap J} c(i+1) + (\rho\tau_2 + \xi_2\tau) \left[\sum_{i \in I \cap J} c(i)t(I\Delta J \amalg i) \prod_{j \in I \cap J \setminus i} c(j+1) \right. \\
&\quad \left. + \sum_{i \in I \setminus J} c(i)t(I\Delta J \setminus i) \prod_{j \in I \cap J} c(j+1) \right].
\end{aligned}$$

Proof. The formulas follow from straightforward calculations similar to Lemma 4.22. $\square$

Remark 5.3 In the above Lemma we used $\xi_2\tau$. It can be written as $\xi_2\tau = t(e_2) + \tau_0^2 c(2) + \rho c_1(2)$. One can similarly express all $\xi_1\tau$ in terms of $t(I), c(I), c_1(I)$ using the formulas in Remark 4.16. This can be seen from the fact that in $k_{**}^M H_W \mathbb{Z}$ $\xi_1\tau$ lies in the image of d_*^L via $\xi_1\tau = d_*^L(\xi_1(\xi_1\tau))$, since $H\mathbb{Z}/2_{**} H\mathbb{Z}/2\tau \subset H\mathbb{Z}/2_{**} H_W \mathbb{Z}$. Hence, by Lemma 4.24 it can be written as claimed.

Theorem 5.4. *Let the base field k be an extension of a field F satisfying $k_2^M(F) = 0$. Then we have an isomorphism of right $H_W \mathbb{Z}_{**}$ -algebras*

$$H_W \mathbb{Z}_{**} H_W \mathbb{Z} \cong H_W \mathbb{Z}_{**} [s, t_2, t_3, \dots, c(I), t(I), c_1(I), t_1(I)] / (\dots),$$

with index $I \subset \mathbb{Z}_{\geq 2}$ running over all finite subsets of $\mathbb{Z}_{\geq 2}$. The ideal of η -torsion is generated by $c(I), c_1(I), t(I), t_1(I)$. The ideal $(\dots)$ is generated by the relations in Lemma 4.22, Lemma 5.2 and

$$\begin{aligned}
\eta c(I) &= \eta c_1(I) = \eta t(I) = \eta t_1(I) = c(\emptyset) = 0, & \tau_0^4 &= c(2)\tau^2, & t_j^2 &= \rho t_{j+1} + \xi_{j+1}\tau, \\
t(I)s &= t_1(I)\tau_0^3, & t_1(I)s &= \tau_0^3 t(I)(\rho t_2 + \xi_2\tau), & s^2 &= \tau_0^2(\rho t_2 + \xi_2\tau)c(2)\tau^2, \\
sc(I) &= \tau_0^3 c_1(I) + \tau t(I)c(2), & sc_1(I) &= \tau_0^3(\rho t_2 + \xi_2\tau)c(I) + \tau t_1(I)c(2).
\end{aligned}$$

The degrees here are

$$\begin{aligned}
|t_1| &= (3, 0), & |s| &= (6, 1), & |t_i| &= (2^i - 1)(2, 1) + (1, 0), \\
|c(I)| &= ((\sum_{i \in I} 2^i - 1) - 1)(2, 1), & |c_1(I)| &= \sum_{i \in I} (2^i - 1)(2, 1) + (1, 0), \\
|t(I)| &= \sum_{i \in I} (2^i - 1)(2, 1) + (0, -1), & |t_1(I)| &= \sum_{i \in I} (2^i - 1)(2, 1) + (3, 0).
\end{aligned}$$

Proof. By Remark 2.4 we may check this on homotopy groups $\pi_{p,q}$. By Proposition 2.5 we can reduce to F being our base field. If the characteristic of F is positive, then F is an extension of $\mathbb{F}_q$ with q odd and we may further reduce to $\mathbb{F}_q$. Assume the characteristic of F is 0. Since $\rho^2 = 0$ in $k_2^M(F)$, we have $-1 = a^2 + b^2$ in F [Lam05, Cor. 6.20]. Hence, F is a field extension of $\mathbb{Q}(a, b)$. Since this is a field extension of transcendence degree at most 1, we have $\text{cd}_2(\mathbb{Q}(a, b)) \leq 3$ [Ser02, II.4.4, Prop. 11 and 13]. Thus, by the resolution of the Milnor conjecture, $\mathbb{Q}(a, b)$ satisfies the assumptions of Proposition 4.28 and we may reduce to that situation.

At the beginning of this section we saw, with the help of the pullback square (5.1), that the given generators generate. The relations that are left are straightforward and can be imported from $H\mathbb{Z}/2_{**}H_W\mathbb{Z}$ and $\underline{K}^W \wedge H_W\mathbb{Z}$, keeping in mind $\rho^2 = 0$. By Example 4.29 we have unique lifts for $\rho\tau_j$ and thus τ_j^2 . Together with the fact that we have no higher η -torsion, we conclude that there are no η -residue terms in the relations. Linear independence of remaining monomials can now simply be checked in $H\mathbb{Z}/2_{**}H_W\mathbb{Z}$. $\square$

Remark 5.5 We can conjugate all the elements to get an isomorphism of left $H_W\mathbb{Z}_{**}$ -algebras.

Remark 5.6 The assumptions in Theorem 5.4 include fields k of odd characteristic as well as fields that are extensions of fields F with $\text{cd}_2(F) = 1$. For example, this includes all extensions of quadratically closed fields, in particular the complex numbers $\mathbb{C}$, and all extensions of $\mathbb{F}_p$ with $p \neq 2$.

Remark 5.7 Let F be a field of characteristic not 2 with assumptions as in Proposition 4.27, i.e. $\text{vcd}_2(F) < \infty$, $\rho^3 = 0$ and $k_4^M(k) = 0$. Let the base field k be a field extension of F . Then the arguments of the proof of Theorem 5.4 prove a version with the only difference being η -residues in the relations for t_j^2 . If one further drops the assumption $\text{vcd}_2(F) < \infty$, then one can prove an η -completed version as in Remark 4.30.

Let $HW = H_W\mathbb{Z}[\eta^{-1}]$. When we invert η , the coefficient ring becomes

$$H_W\mathbb{Z}_{**}[\eta^{-1}] \cong \underline{K}_{**}^W[\eta^{-1}] \cong W(k)[\eta^{\pm 1}]$$

by [Mor12, Cor. 3.11, Remark 3.12]. We can normalize the generators t_j and s to live in motivic weight 0 via $x_j := \eta^{1-2^j}t_j$ and $y := \eta^{-1}s$.

Corollary 5.8. *Let the base field k be an extension of a field F satisfying $k_2^M(F) = 0$. Then we have an isomorphism of $W(k)[\eta^{-1}]$ -algebras*

$$HW_{**}HW \cong W(k)[\eta^{\pm 1}][x_2, y, x_3, x_4, \dots]/(y^2, x_j^2 - 2x_{j+1}),$$

where $|x_j| = (2^j, 0)$ and $|y| = (5, 0)$.

Over fields with the given assumptions, this implies the result [BH21, Cor. 8.20] by Bachmann and Hopkins.

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