Poincaré ring spectra

Viktor Burghardt

These notes formed out of a talk given at the IAS/Park City Mathematics Institute in July 2024 by the author. The purpose of these notes is to give a brief demonstration of the emerging theory of Poincaré ring spectra as developed in [BRY25]. We emphasize that the treatment here is far from exhaustive and is solely meant to serve as an appetizer. In particular, it does not treat schemes, so only talks about the affine version of the theory, does not establish a connection to the classical involutive Brauer group, and omits most proofs. We encourage the reader to dive deeper into the topic via [BRY25].

Outline 0.1. We will start by recalling the classical Brauer group of a field and then generalize it to $\mathbf{E}_\infty$-ring spectra via the Brauer space. Our path towards the Brauer space shall serve us as a guideline for the definition of the Poincaré Brauer space. To motivate the theory of Poincaré ring spectra, we pick up the question if 2-torsion in the classical Brauer group is solely due to involutions on Azumaya algebras. While this question has been answered in the affirmative by Albert for fields, and by Saltman for commutative rings, we outline how to answer it for $\mathbf{E}_\infty$-ring spectra using Poincaré ring spectra. Having defined Poincaré ring spectra, we will define the Poincaré Picard space and the Poincaré Brauer space and describe their relation. At the end, we will relate the Poincaré Brauer space to the Brauer space via the norm fiber sequence of Theorem 8.12, which will allow us to attack the above mentioned question about 2-torsion in the Brauer group of an $\mathbf{E}_\infty$-ring spectrum.

Resources 0.2. Throughout, we will be using the language of ∞-categories. As references we mention [Lur09] and [Lur17]. We will also make use of the language of Poincaré ∞-categories as developed in [Cal+23]. Lastly, the main source of the applications presented here will be [BRY25]. We will not attempt to be fully self-contained and refer the reader to these references for more background.

Acknowledgements 0.3. The author wishes to thank the Institute for Advanced Study and the organizers of the 2024 Park City Mathematics Institute on motivic homotopy theory. It was a wonderful time. We also want to thank the referee for helpful comments, and for pointing out Remark 8.15 to us.

1. Brauer group

Let $\mathbf{k}$ be a field. An important invariant of $\mathbf{k}$ is its Brauer group, which we will now define. To do so, we need to define the notion of an Azumaya algebra and the notion of Morita equivalence. Both of these definitions seem rather adhoc at first sight. In section 3 they will be reinterpreted in the context of module categories in which they appear naturally.

Notation 1.1. Throughout these notes, $\mathbf{k}$ will denote a field, S will denote a discrete commutative ring, and k will denote an $\mathbf{E}_\infty$-ring spectrum.

Definition 1.2. An associative algebra A over $\mathbf{k}$ is called *Azumaya* if it satisfies the following two conditions:

1. A is nonzero and finite dimensional as a $\mathbf{k}$-vector space.
2. The map

$$\begin{aligned} A \otimes_{\mathbf{k}} A^{\mathrm{op}} &\to \underline{\mathrm{Hom}}_{\mathbf{k}}(A, A) \\ a \otimes b &\mapsto [x \mapsto axb] \end{aligned}$$

   is an isomorphism of $\mathbf{k}$-algebras.

Definition 1.3. Let A, B be Azumaya algebras over $\mathbf{k}$. Then A, B are said to be *Morita equivalent* if there exists a finite dimensional $\mathbf{k}$-module P, such that

$$A \otimes_{\mathbf{k}} B^{\mathrm{op}} \simeq \underline{\mathrm{Hom}}_{\mathbf{k}}(P, P)$$

as $\mathbf{k}$-algebras.

This defines an equivalence relation. Let A be an Azumaya algebra over $\mathbf{k}$ and denote by $[A]$ its equivalence class with respect to Morita equivalence. The operation

$$[A] \cdot [B] := [A \otimes_{\mathbf{k}} B]$$

defines a group structure on the set of Morita equivalence classes with unit object $[k]$. The inverse of $[A]$ is given by $[A^{\mathrm{op}}]$:

$$[A] \cdot [A^{\mathrm{op}}] = [A \otimes_{\mathbf{k}} A^{\mathrm{op}}] = [\underline{\mathrm{Hom}}_{\mathbf{k}}(P, P)] = [k] = 0.$$

Definition 1.4. Let $\mathbf{k}$ be a field. We will denote the group of Morita equivalence classes of Azumaya algebras, with product induced by $- \otimes_{\mathbf{k}} -$, by $\mathrm{Br}(\mathbf{k})$ and refer to it by the *classical Brauer group* of $\mathbf{k}$.

Definition 1.5. Let A be an Azumaya algebra over a field $\mathbf{k}$. We call an isomorphism of $\mathbf{k}$-algebras $\sigma : A \xrightarrow{\sim} A^{\mathrm{op}}$ an *involution*, if it satisfies $\sigma^{\mathrm{op}} \circ \sigma = \mathrm{id}_A$.

Let A be an Azumaya algebra over a field $\mathbf{k}$ with an involution $\sigma : A \xrightarrow{\sim} A^{\mathrm{op}}$. Then,

$$[A] \cdot [A] = [A] \cdot [A^{\mathrm{op}}] = 0$$

in the classical Brauer group $\mathrm{Br}(\mathbf{k})$. In other words, the equivalence class $[A]$ of an Azumaya algebra A which admits an involution is 2-torsion in $\mathrm{Br}(\mathbf{k})$. Informally speaking, involutions lead to 2-torsion in the classical Brauer group. One may ask: are involutions the only cause for 2-torsion in the classical Brauer group?

Question 1.6. Does every 2-torsion class in $\mathrm{Br}(\mathbf{k})$ contain an Azumaya algebra which admits an involution?

This question has been answered by Albert.

Theorem 1.7. *Let $\mathbf{k}$ be a field. An equivalence class in the classical Brauer group of $\mathbf{k}$ is 2-torsion if and only if it contains an Azumaya algebra which admits an involution.*

We will raise a generalization of Question 1.6 in the context of $\mathbf{E}_\infty$-ring spectra and approach it using the theory of Poincaré ring spectra.

Remark 1.8. The classical Brauer group can be defined for commutative rings, and also schemes. Question 1.6 can be asked in these more general cases, and answering it becomes more difficult the more general objects you consider. In the case of commutative rings, this question has been answered by Saltman, see [Sal78].

Remark 1.9. Albert proves a stronger version of Theorem 1.7 in [Alb35, Thm. 12]. He shows that an equivalence class in the classical Brauer group of a field is 2-torsion if and only if *every* Azumaya algebra in that equivalence class admits an involution. We chose to state the weaker version, because it generalizes to commutative rings and $\mathbf{E}_\infty$-ring spectra.

2. Picard space

We will soon generalize the above definition of the classical Brauer group to $\mathbf{E}_\infty$-ring spectra. Before we do so, let us consider invertible objects in module categories and demonstrate how this leads to a space which is related to the Picard group.

Notation 2.1. We will denote the ∞-category of $\mathbf{E}_\infty$-ring spectra by CAlg. Given an algebra object A in a symmetric monoidal ∞-category $\mathcal{C}$, we will denote the ∞-category of right A-modules in $\mathcal{C}$ by Mod_A.

Let $\mathcal{C}$ be an ∞-category. Let $\mathcal{C}^{\simeq}$ be the core of $\mathcal{C}$, i.e. the subcategory spanned by all objects and only equivalences. If $\mathcal{C}$ is symmetric monoidal, then its core $\mathcal{C}^{\simeq}$ is an $\mathbf{E}_\infty$-space. In general, $\mathcal{C}^{\simeq}$ is not grouplike.

Example 2.2. Let k be an $\mathbf{E}_\infty$-ring spectrum. The ∞-category of right k-modules Mod_k is symmetric monoidal via the tensor product $- \otimes_k -$. If k is nonzero, then the zero object of Mod_k is not invertible. Thus, $\mathrm{Mod}_k^{\simeq}$ is an $\mathbf{E}_\infty$-space, which is not grouplike.

Definition 2.3. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. We will call the full subcategory of $\mathcal{C}^{\simeq}$ spanned by invertible objects, the *Picard space* of $\mathcal{C}$ and denote it by $\mathcal{P}ic(\mathcal{C})$. If $\mathcal{C} = \mathrm{Mod}_k$ for k an $\mathbf{E}_\infty$-ring spectrum, we will write $\mathcal{P}ic(k)$ for $\mathcal{P}ic(\mathcal{C})$ and call it the *Picard space* of k.

Remark 2.4. The Picard space of a symmetric monoidal ∞-category is tautologically a grouplike $\mathbf{E}_\infty$-space, since we only keep the objects which are invertible.

Remark 2.5. Let S be a discrete commutative ring. Recall that the Picard group of S, denoted $\mathrm{Pic}(S)$, is defined as the set of isomorphism classes of discrete modules over S with the group structure induced by the tensor product $- \otimes_S -$. Let $\mathcal{P}ic^{\heartsuit}(S)$ denote the full subcategory of $\mathcal{P}ic(S)$ spanned by discrete S-modules. Then we have an isomorphism $\pi_0\left(\mathcal{P}ic^{\heartsuit}(S)\right) \simeq \mathrm{Pic}(S)$.

The previous remark identifies the connected componenents of $\mathcal{P}ic^{\heartsuit}(S)$ for a discrete commutative ring S. The following shows that connected components of $\mathcal{P}ic(S)$ are given by shifts of connected components of $\mathcal{P}ic^{\heartsuit}(S)$.

Theorem 2.6 [Fau03][AG14]**.** *Let S be a discrete commutative ring. Then we have an isomorphism*

$$\pi_0(\mathcal{P}ic(S)) \simeq \mathrm{Pic}(S) \times \Gamma(S; \underline{\mathbf{Z}}),$$

where $\underline{\mathbf{Z}}$ is the constant sheaf on S taking the value $\mathbf{Z}$.

3. Brauer space

Note that we have obtained the Picard space by taking the module category and then invertible objects in it. We will apply the same strategy to obtain the Brauer space. Given an $\mathbf{E}_\infty$-ring spectrum k. The ∞-category Mod_k is an $\mathbf{E}_\infty$-algebra object in the ∞-category of stable presentable ∞-categories with exact functors. If we only consider compact modules, i.e. Mod_k^ω, then this is an $\mathbf{E}_\infty$-algebra in $\mathrm{Cat}_{\infty,\mathrm{idem}}^{\mathrm{ex}}$, the ∞-category of small idempotent complete stable ∞-categories and exact functors. In particular, we can consider modules over module categories.

Definition 3.1. Let k be an $\mathbf{E}_\infty$-ring spectrum. We will denote the ∞-category $\mathrm{Mod}_{\mathrm{Mod}_k^\omega}(\mathrm{Cat}_{\infty,\mathrm{idem}}^{\mathrm{ex}})$ by $\mathrm{Cat}_{k,\omega}$. We will write $\mathcal{B}r(k)$ for $\mathcal{P}ic(\mathrm{Cat}_{k,\omega})$ and call it the *Brauer space* of k.

Remark 3.2. Let k be an $\mathbf{E}_\infty$-ring spectrum, then $\mathrm{Cat}_{k,\omega}$ is a symmetric monoidal ∞-category. It follows that the Brauer space of an $\mathbf{E}_\infty$-ring spectrum is a grouplike $\mathbf{E}_\infty$-space.

Remark 3.3. Let k be an $\mathbf{E}_\infty$-ring spectrum. The unit object of $\mathrm{Cat}_{k,\omega}$ is Mod_k^ω. An endomorphism of Mod_k^ω in $\mathrm{Cat}_{k,\omega}$ is a functor $- \otimes_k M : \mathrm{Mod}_k^\omega \to \mathrm{Mod}_k^\omega$ for a compact k-module M. Such an endomorphism is an equivalence if and only if M is an invertible object in Mod_k^ω. We obtain an equivalence of spaces

$$\Omega\mathcal{B}r(k) \simeq \mathcal{P}ic(k).$$

We need to justify the name Brauer space. So let us establish a connection between the Brauer space of Definition 3.1 and the classical Brauer group of Definition 1.4.

Definition 3.4. Let k be an $\mathbf{E}_\infty$-ring spectrum and let A be an $\mathbf{E}_1$-algebra over k. We call A *Azumaya* if it satisfies the following two conditions:

1. A is a compact generator of Mod_k.

2. The natural k-algebra map

$$A \otimes_k A^{\mathrm{op}} \to \underline{\mathrm{Hom}}_k(A, A)$$

   is an equivalence of k-algebras.

Let A be an Azumaya algebra over an $\mathbf{E}_\infty$-ring spectrum k. Then the conditions of Definition 3.4 imply that the map

$$\mathrm{Mod}_A^\omega \otimes_{\mathrm{Mod}_k^\omega} \mathrm{Mod}_{A^{\mathrm{op}}}^\omega \simeq \mathrm{Mod}_{A\otimes_k A^{\mathrm{op}}}^\omega \to \mathrm{Mod}_k^\omega,$$

which sends $A \otimes_k A^{\mathrm{op}}$ to A, is essentially surjective and fully faithful. Thus, it is an equivalence, and Mod_A^ω is invertible in $\mathrm{Cat}_{k,\omega}$. The following theorem shows that all invertible objects in $\mathrm{Cat}_{k,\omega}$ are of this form.

Theorem 3.5 [AG14, Thm. 3.15]. *Let k be an $\mathbf{E}_\infty$-ring spectrum. Then any object of the Brauer space $\mathcal{B}r(k)$ is of the form Mod_A^ω for some Azumaya algebra A over k.*

This allows us to construct a map from the classical Brauer group of a field $\mathbf{k}$ of Definition 1.4 to the connected components of the Brauer space of $\mathbf{k}$ viewed as a discrete spectrum.

Theorem 3.6. *Let $\mathbf{k}$ be a field. The map*

$$\begin{aligned} \mathrm{Br}(\mathbf{k}) &\to \pi_0(\mathcal{B}r(\mathbf{k})) \\ [A] &\mapsto [\mathrm{Mod}_A^\omega] =: [A] \end{aligned}$$

is an isomorphism of groups.

Proof. By [AG14, Cor. 7.13 and Lem. 7.15], we have

$$\pi_0(\mathcal{B}r(\mathbf{k})) \simeq \mathrm{H}^1_{\text{ét}}(\mathrm{Spec}(\mathbf{k}), \mathbf{Z}) \times \mathrm{H}^2_{\text{ét}}(\mathrm{Spec}(\mathbf{k}), \mathbf{G}_m) \simeq \mathrm{H}^2_{\text{ét}}(\mathrm{Spec}(\mathbf{k}), \mathbf{G}_m).$$

Moreover, $\mathrm{H}^2_{\text{ét}}(\mathrm{Spec}(\mathbf{k}), \mathbf{G}_m)$ is torsion by Grothendieck [Gro68, Cor. 1.8]. The map in question now takes the form $\mathrm{Br}(\mathbf{k}) \to \mathrm{H}^2_{\text{ét}}(\mathrm{Spec}(\mathbf{k}), \mathbf{G}_m)_{\mathrm{tors}}$ and is an isomorphism by Gabber [Gab81, Ch. II 1st Remark]. ☐

Remark 3.7. By Theorem 3.6, two Azumaya algebras A, B over a field $\mathbf{k}$ are Morita equivalent if and only if Mod_A^ω and Mod_B^ω are equivalent as Mod_k^ω-modules.

Warning 3.8. An analogous statement of Theorem 3.6 does not hold for more general discrete commutative rings S. By [AG14, Cor. 7.13], we have $\pi_0(\mathcal{B}r(S)) \simeq \mathrm{H}^1_{\text{ét}}(\mathrm{Spec}(S), \mathbf{Z}) \times \mathrm{H}^2_{\text{ét}}(\mathrm{Spec}(S), \mathbf{G}_m)$. If S is a regular domain, then the right hand side coincides with the classical Brauer group, essentially by the proof of Theorem 3.6. In general, the classical Brauer group injects into the Brauer group, but they are generally not isomorphic. See [AG14, section 7.5] for a counterexample.

Given the identification in Theorem 3.6, we can extend our definition of the Brauer group of a field to $\mathbf{E}_\infty$-ring spectra as follows.

Definition 3.9. Let k be an $\mathbf{E}_\infty$-ring spectrum. We will call the group $\pi_0(\mathcal{B}r(k))$ the *Brauer group* of k and denote it by $\mathrm{Br}(k)$.

By a similar argument as in section 1, we see that if A is an Azumaya algebra over an $\mathbf{E}_\infty$-ring spectrum k which admits an involution $\sigma : A \xrightarrow{\sim} A^{\mathrm{op}}$, then $[A]$ is 2-torsion in $\mathrm{Br}(k)$. We can thus extend Question 1.6 to $\mathbf{E}_\infty$-ring spectra.

Question 3.10. Let k be an $\mathbf{E}_\infty$-ring spectrum. Does every 2-torsion class in $\mathrm{Br}(k)$ contain an Azumaya algebra which admits an involution?

Far below, we will answer this question in the case where 2 acts invertibly on k.

4. Hermitian Picard group

To answer Question 3.10, we will need to study rings with C_2-actions and invariants of them. We will do so via the notion of forms with respect to a given C_2-action. In this section we introduce these concepts on discrete commutative rings via the Hermitian Picard group. We will see later how it is an instance of an invariant naturally defined on Poincaré ring spectra.

Definition 4.1. Let S be a discrete commutative ring with a C_2-action $\lambda\colon S \to S$, and let M be an S-module. We define the λ-*dual* of M to be the S-module

$$\mathrm{D}_\lambda(M) := \underline{\mathrm{Hom}}_S\left(\lambda^* M, S\right).$$

In the case of the trivial action $\lambda = \mathrm{id}_S$ we will write D for $\mathrm{D}_{\mathrm{id}_S}$.

Remark 4.2. Let S be a discrete commutative ring with a C_2-action $\lambda\colon S \to S$ and denote by $\mathrm{Proj}(S)$ the category of finitely generated projective modules over S. Then the assignment of the λ-dual of Definition 4.1 can be extended to a duality functor $\mathrm{D}_\lambda : \mathrm{Proj}(S)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Proj}(S)$ which is symmetric monoidal and satisfies $\mathrm{D}_\lambda^{\mathrm{op}} \circ \mathrm{D}_\lambda \simeq \mathrm{id}$.

Definition 4.3. Let S be a discrete commutative ring with a C_2-action $\lambda\colon S \to S$. A *nondegenerate λ-Hermitian form* over S is a finitely generated projective module I over S together with an S-linear isomorphism $\varphi\colon I \xrightarrow{\sim} \mathrm{D}_\lambda(I)$ such that the following diagram commutes

$$\begin{tikzcd}
I \arrow[r, "\varphi"] \arrow[d, "\simeq"'] & \mathrm{D}_\lambda(I) \\
\mathrm{D}_\lambda^{\mathrm{op}} \circ \mathrm{D}_\lambda(I) \arrow[ur, "\mathrm{D}_\lambda(\varphi)"'] &
\end{tikzcd}.$$

We will denote λ-Hermitian forms over S by pairs (I, φ).

A *map of nondegenerate λ-Hermitian forms*, $f : (I, \varphi) \to (J, \psi)$, is a map of S-modules $f : I \to J$ such that the following diagram commutes

$$\begin{tikzcd}
I \arrow[r, "\varphi"] \arrow[d, "f"'] & \mathrm{D}_\lambda(I) \\
J \arrow[r, "\psi"'] & \mathrm{D}_\lambda(J). \arrow[u, "\mathrm{D}_\lambda(f)"']
\end{tikzcd}$$

We say f is an *isomorphism of nondegenerate λ-Hermitian forms* if its underlying S-module map is an isomorphism of S-modules.

Remark 4.4. Let S be a discrete commutative ring with a C_2-action $\lambda\colon S \to S$. Then nondegenerate λ-Hermitian forms on a finitely generated projective S-module I are in bijection with nondegenerate λ-symmetric bilinear forms on I via the $\otimes_S - \underline{\mathrm{Hom}}_S$ adjunction.

Remark 4.5. Let S be a commutative ring with a C_2-action $\lambda : S \to S$, and let I be an invertible S-module. Let $\varphi : I \to \mathrm{D}_\lambda(I)$ be an isomorphism of S-modules. Then φ corresponds to a nondegenerate pairing $< -, - >_\varphi\colon I \otimes_S \lambda^* I \to S$. Locally, at a prime $\mathfrak{p} \subset S$, this pairing is of the form

$$\begin{aligned} S_\mathfrak{p} \otimes_{S_\mathfrak{p}} \lambda^*(S_{\lambda(\mathfrak{p})}) &\to S_\mathfrak{p} \\ x \otimes y &\mapsto \lambda(y) a_\mathfrak{p} x \end{aligned}$$

for some $a_\mathfrak{p} \in S_\mathfrak{p}^\times$. A priori, the element $a_\mathfrak{p}$ depends on the isomorphism $(I \otimes_S \lambda^* I)_\mathfrak{p} \simeq S_\mathfrak{p} \otimes S_{\lambda(\mathfrak{p})}$. This ambiguity can be removed as follows. Given a trivialization $t_\mathfrak{p} : I_\mathfrak{p} \xrightarrow{\sim} S_\mathfrak{p}$ at $\mathfrak{p}$, we obtain a trivialization at $\lambda(\mathfrak{p})$ via the composition

$$t_{\lambda(\mathfrak{p})} : I_{\lambda(\mathfrak{p})} \xrightarrow{\varphi} \mathrm{D}_\lambda(I)_{\lambda(\mathfrak{p})} \simeq \mathrm{D}_\lambda(I_\mathfrak{p}) \xrightarrow{\circ {t_\mathfrak{p}}^{-1}} \mathrm{D}_\lambda(S_\mathfrak{p}) \simeq \mathrm{D}_\lambda(S)_{\lambda(\mathfrak{p})} \simeq S_{\lambda(\mathfrak{p})}.$$

If we trivialize $I \otimes_S \lambda^* I$ via $t_\mathfrak{p} \otimes t_{\lambda(\mathfrak{p})}$, then $a_\mathfrak{p} \in S_\mathfrak{p}^\times$ is independent of the choice of the initial trivializaiton $t_\mathfrak{p}$. Moreover, (I, φ) is a nondegenerate λ-Hermetian form if and only if $\lambda(a_\mathfrak{p}) = a_{\lambda(\mathfrak{p})}$ for all primes $\mathfrak{p} \subset S$.

The pairing

$$\begin{aligned}\langle -,-\rangle_\lambda :& S\times S\to S\\ &(r,s)\mapsto \lambda(s)r\end{aligned}$$

provides a canonical nondegenerate λ-Hermitian form $u_\lambda : S\to \mathrm{D}_\lambda(S)$, using Remark 4.4. Moreover, given two nondegenerate λ-Hermitian forms over S, (I,φ) and (J,ψ), tensoring provides a new nondegenerate λ-Hermitian form $(I\otimes_S J,\varphi\otimes_S\psi)$ with

$$\begin{aligned}\varphi\otimes_S\psi :& I\otimes_S J\to \mathrm{D}_\lambda(I\otimes_S J)\\ &i\otimes j\mapsto [x\otimes y\mapsto \varphi(i)(x)\cdot\psi(j)(y)]\,.\end{aligned}$$

Thus the set of nondegenerate λ-Hermitian forms over S forms a monoid under $\otimes_S$ with unit (S,u_λ).

Remark 4.6. Let S be a commutative ring with a C_2-action $\lambda : S\to S$. A nondegenerate λ-Hermetian form (I,φ) is invertible with respect to the monoid structure induced by $\otimes_S$ if and only if the underlying S-module I is invertible. In this case, the inverse of (I,φ) is given by $(\mathrm{D}(I),\varphi^{-1})$. More concretely, the isomorphism $(I\otimes_S \mathrm{D}(I),\varphi\otimes_S\varphi^{-1})\simeq(S,u_\lambda)$ is given by $\mathrm{eval} : I\otimes_S\mathrm{D}(I)\to S$. This can be checked locally, keeping in mind Remark 4.5.

Definition 4.7. Let S be a discrete commutative ring with a C_2-action $\lambda : S\to S$. The *Hermitian Picard group of* (S,λ) is the group of isomorphism classes of invertible λ-Hermitian forms over S under tensor product, with unit (S,u_λ). We will denote the Hermitian Picard group of (S,λ) by $\mathrm{Pic}^{\mathrm{h}}(S,\lambda)$. In case λ is the identity, we will also write $\mathrm{Pic}^{\mathrm{h}}(S)$.

Remark 4.8. Our definition of the Hermitian Picard group agrees with the absolute version of the Hermitian Picard group of Reyes Sanchez–Verhaeghe–Verschoren [RVV95]. In case of a discrete commutative ring S with trivial C_2-action, the Hermitian Picard group $\mathrm{Pic}^{\mathrm{h}}(S)$ is also the group of isometry classes of *discriminant bundles* [Knu91, p. 470].

The following follows from Remark 4.6.

Proposition 4.9. *Let S be a discrete commutative ring with trivial C_2-action. Then $\mathrm{Pic}^{\mathrm{h}}(S)$ is 2-torsion.*

Forgetting the Hermitian structure provides a map

$$\mathrm{Pic}^{\mathrm{h}}(S,\lambda)\to \mathrm{Pic}(S). \tag{4.10}$$

If the C_2-action on S is trivial, then the image of this map lies in the 2-torsion of $\mathrm{Pic}(S)$ by Proposition 4.9. The following result gives a means of computing the Hermitian Picard group in this case.

Proposition 4.11 [BRY25, Prop. 5.3.12]. *Let S be a discrete commutative ring. We have a split exact sequence*

$$0\to S^\times/(S^\times)^2\to \mathrm{Pic}^{\mathrm{h}}(S)\to \mathrm{Pic}(S)[2]\to 0.$$

Remark 4.12. Let S be a discrete commutative ring. A consequence of Proposition 4.11 is that the Hermitian Picard group is isomorphic to the first fppf cohomology group of R with μ_2-coefficients

$$\mathrm{Pic}^{\mathrm{h}}(S)\simeq \mathrm{H}^1_{\mathrm{fppf}}(S,\mu_2).$$

Since the λ-dual $\mathrm{D}_\lambda : \mathrm{Proj}(S)^{\mathrm{op}}\xrightarrow{\sim}\mathrm{Proj}(S)$ is symmetric monoidal by Remark 4.2, it induces a C_2-action on $\mathrm{Pic}(S)$, given by the assignment $[I]\mapsto[\mathrm{D}_\lambda(I)]=-[\lambda_*(I)]$, for I a module over S of rank 1. Note that modules underlying an invertible nondegenerate λ-Hermitian form are fixed points in the Picard group of S with respect to this action. Let us denote fixed points by this action by the superscript $(-)^{\lambda=-\mathrm{id}}$. Thus, the forgetful map (4.10) factors via

$$\mathrm{Pic}^{\mathrm{h}}(S,\lambda)\to \mathrm{Pic}(S)^{\lambda=-\mathrm{id}}.$$

The following result generalizes Proposition 4.11 to nontrivial C_2-actions on S with few zerodivisors. We say that a commutative ring S has *few zerodivisors*, if the subset of zerodivisors in S is a finite union of prime ideals.

Proposition 4.13 [BRY25, Prop. 5.3.17]. *Let S be a discrete commutative ring with few zerodivisors and a C_2-action $\lambda : S \to S$. Then we have the following split exact sequence*

$$0 \to (S^\times)^{C_2}/N_\lambda(S^\times) \to \mathrm{Pic}^{\mathrm{h}}(S,\lambda) \to \mathrm{Pic}(S)^{\lambda=-\mathrm{id}} \to 0,$$

where $N_\lambda : S^\times \to (S^\times)^{C_2}$ is the norm map given by $r \mapsto r \cdot \lambda(r)$.

Example 4.14. Let $S = \mathbf{Z}[\frac{1+\sqrt{-23}}{2}]$ with the action $\lambda : S \to S$ given by complex conjugation. The fixed points of S are given by integers, thus $(S^\times)^{C_2}/N_\lambda(S^\times) \simeq \mathbf{Z}/2$. The Picard group $\mathrm{Pic}(S) \simeq \mathbf{Z}/3$ of S is generated by the ideal $I = (2, \frac{1+\sqrt{-23}}{2})$. We have an isomorphism $\lambda^* I \xrightarrow{\sim} (2, \frac{1-\sqrt{-23}}{2})$, $x \mapsto \lambda(x)$. Since $(2, \frac{1+\sqrt{-23}}{2})(2, \frac{1-\sqrt{-23}}{2}) = (2)$ is a principal ideal, we know $[\lambda^* I] = -[I]$ in $\mathrm{Pic}(S)$. Thus, the C_2-action on the Picard group induced by the λ-dual is given by the identity. Therefore, $\mathrm{Pic}(S)^{\lambda=-\mathrm{id}} \simeq \mathrm{Pic}(S) \simeq \mathbf{Z}/3$. By Proposition 4.13, we have

$$\mathrm{Pic}^{\mathrm{h}}\left(\mathbf{Z}\left[\frac{1+\sqrt{-23}}{2}\right], \lambda\right) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/3.$$

This example shows that the forgetful map $\mathrm{Pic}^{\mathrm{h}}(S) \to \mathrm{Pic}(S)$ does generally not land in the 2-torsion part of the Picard group of S.

5. Poincaré ring spectra

In this section we will introduce the notion of a Poincaré ring spectrum. One can think of a Poincaré ring spectrum as an $\mathbf{E}_\infty$-ring spectrum equipped with a notion of forms with a compatible multiplicative structure. This will allow us to extend the Hermitian Picard group and build a suitable space, the Poincaré Picard space, that carries its information, much like the Picard space or Brauer space carry the information of the Picard group and the Brauer group. Later, we will also associate a suitable Brauer space to a Poincaré ring spectrum, which recovers the Poincaré Picard space by taking loops.

Notation 5.1. Let k be an $\mathbf{E}_\infty$-ring spectrum equipped with a C_2-action. We will denote by $k^{\mathrm{t}C_2}$ the Tate construction of k with respect to the equipped action. Since the Tate construction is lax symmetric monoidal, $k^{\mathrm{t}C_2}$ is naturally a k-algebra via the Tate-valued Frobenius.

Definition 5.2. Let k be an $\mathbf{E}_\infty$-ring spectrum. A *Poincaré structure* on k is the following data:

1. A C_2-action on k via maps of $\mathbf{E}_\infty$-ring spectra, i.e. a functor $\lambda : \mathrm{B}C_2 \to \mathrm{CAlg}$ with $\lambda(*) = k$.
2. A k-algebra $k \to C$.
3. A k-algebra map $C \to k^{\mathrm{t}C_2}$.

An $\mathbf{E}_\infty$-ring spectrum equipped with a Poincaré structure will be called a *Poincaré ring spectrum*.

Remark 5.3. Our notation for Poincaré ring spectra differs from that found in [Cal+23]. There the authors use the terminology $\mathbf{E}_\infty$*-ring spectrum with genuine involution*.

Informally, a Poincaré structure on k is a C_2-action on k together with a commutative diagram in CAlg of the form

$$\begin{tikzcd} C \arrow[rr] & & k^{\mathrm{t}C_2} \\ & k. \arrow[ul] \arrow[ur] & \end{tikzcd}$$

One can organize Poincaré ring spectra into a symmetric monoidal ∞-category $\mathrm{CAlg}^{\mathrm{p}}$.

Notation 5.4. Let R be a Poincaré ring spectrum given by an $\mathbf{E}_\infty$-ring spectrum k with C_2-action $\lambda : k \xrightarrow{\sim} k$ and a map of k-algebras $\alpha : C \to k^{\mathrm{t}C_2}$. We will denote the components of R as follows:

$$R^{\mathrm{e}} := k, \qquad \lambda_R := \lambda, \qquad R^{\varphi C_2} := C, \qquad \alpha_R := \alpha.$$

Remark 5.5. Let $\mathcal{C}$ be a stable ∞-category. A *Poincaré structure* on $\mathcal{C}$ is a reduced, 2-excisive functor $\Qoppa : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ with the following two properties:

1. The functor
$$\mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \qquad y \mapsto \mathrm{B}_{\Qoppa}(-, y),$$
where $\mathrm{B}_{\Qoppa}$ is the cross effect of $\Qoppa$, takes values in the essential image of the stable Yoneda embedding.
2. The resulting factorization of the functor in 1. as a *duality functor* $\mathrm{D}_{\Qoppa} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ is an equivalence.

A stable ∞-category equipped with a Poincaré structure is called a *Poincaré ∞-category*, and they can be organized into an ∞-category $\mathrm{Cat}^{\mathrm{p}}_{\infty}$. In Construction 6.1, we will construct a functor $\mathrm{CAlg}^{\mathrm{p}} \to \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{p}}_{\infty})$, which can be shown to be fully faithful with essential image those Poincaré ∞-categories whose underlying stable ∞-category is equivalent to $\mathrm{Mod}^{\omega}_{k}$ for some $\mathbf{E}_{\infty}$-ring spectrum k. This is the reason for the terminology *Poincaré ring spectrum*.

Example 5.6. Let k be an $\mathbf{E}_{\infty}$-ring spectrum with a C_2-action via maps of $\mathbf{E}_{\infty}$-ring spectra. Then the Tate-valued Frobenius is a map of k-algebras $k \to k^{\mathrm{t}C_2}$ and thus defines a Poincaré ring spectrum. We will call this structure the *Tate Poincaré ring spectrum* associated to R and denote it by k^{t}.

Example 5.7. Let $\mathbb{S}$ be the sphere spectrum equipped with the trivial C_2-action. We will denote the corresponding Tate Poincaré ring spectrum by $\mathbb{S}^{\mathrm{u}}$ and call it the *universal Poincaré* ring spectrum. This is the unit object in the symmetric monoidal ∞-category $\mathrm{CAlg}^{\mathrm{p}}$.

Example 5.8. Let k be an $\mathbf{E}_{\infty}$-ring spectrum with a C_2-action via maps of $\mathbf{E}_{\infty}$-ring spectra. The identity map $k^{\mathrm{t}C_2} \to k^{\mathrm{t}C_2}$ is a k-algebra map and thus defines a Poincaré ring spectrum. We will call this structure the *symmetric Poincaré ring spectrum* associated to k and denote it by k^{s}.

Example 5.9. Let k be a connective $\mathbf{E}_{\infty}$-ring spectrum with a C_2-action via maps of $\mathbf{E}_{\infty}$-ring spectra. The connective cover $\tau_{\geq 0}(k^{\mathrm{t}C_2}) \to k^{\mathrm{t}C_2}$ is a k-algebra map and thus defines a Poincaré ring spectrum. We will call this structure the *genuine symmetric Poincaré ring spectrum* associated to k and denote it by k^{gs}.

6. Space of Poincaré forms

We claimed in section 5 that a Poincaré structure on an $\mathbf{E}_{\infty}$-ring spectrum k essentially equips k with a notion of forms together with a compatible multiplicative structure. This is made precise by the following construction.

Construction 6.1. Let R be a Poincaré ring spectrum. Define a functor $\Qoppa_R : (\mathrm{Mod}^{\omega}_{R^{\mathrm{e}}})^{\mathrm{op}} \to \mathrm{Sp}$ via the following pullback square in $\mathrm{Fun}((\mathrm{Mod}^{\omega}_{R^{\mathrm{e}}})^{\mathrm{op}}, \mathrm{Sp})$:

$$\begin{tikzcd}
\Qoppa_R(-) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_{R^{\mathrm{e}}}(-, R^{\varphi C_2}) \arrow[d] \\
\underline{\mathrm{Hom}}_{R^{\mathrm{e}} \otimes R^{\mathrm{e}}}(- \otimes -, R^{\mathrm{e}})^{\mathrm{h}C_2} \arrow[r] & \underline{\mathrm{Hom}}_{R^{\mathrm{e}}}(-, R^{\mathrm{t}C_2}).
\end{tikzcd}$$

Here, the right vertical map is induced by the R^{e}-algebra map $R^{\varphi C_2} \to R^{\mathrm{t}C_2}$, and the bottom horizontal map is induced by the Tate-valued Frobenius $R^{\mathrm{e}} \to R^{\mathrm{t}C_2}$. This defines a functor $\Qoppa_{(-)} : \mathrm{CAlg}^{\mathrm{p}} \to \mathrm{CAlg}^{\mathrm{p}}(\mathrm{Cat}^{\mathrm{p}}_{\infty})$.

The functor $\Qoppa_R$ assigns to a finitely generated projective R^{e}-module M a spectrum $\Qoppa_R(M)$ which you can think of as the spectrum of *R-forms* on M. The multiplicative structure on $\Qoppa_R$ encodes how to multiply forms.

Example 6.2. Let k be an $\mathbf{E}_{\infty}$-ring spectrum with a C_2-action via maps of $\mathbf{E}_{\infty}$-ring spectra. Then we have for the symmetric Poincaré ring spectrum k^{s} from Example 5.8

$$\Qoppa_{k^{\mathrm{s}}}(M) \simeq \underline{\mathrm{Hom}}_{k \otimes k}(M \otimes M, k)^{\mathrm{h}C_2}$$

for every compact k-module M. The action here is given by swapping the domain and applying λ to the codomain. Let S be a discrete commutative ring with a C_2-action, and let M be a discrete finitely generated projective S-module. Then $\pi_0(\Qoppa_{S^{\mathrm{s}}}(M))$ is the group of λ-symmetric bilinear forms on M; see [Cal+23, Remark 4.2.21].

Example 6.3. Let R be a Poincaré ring spectrum. If $(R^{\mathrm{e}})^{\mathrm{t}C_2} \simeq 0$, then we have

$$\Qoppa_R(M) \simeq \Qoppa_{(R^{\mathrm{e}})^{\mathrm{s}}}(M) \oplus \underline{\mathrm{Hom}}_{R^{\mathrm{e}}}(M, R^{\varphi C_2}).$$

Note, if $R^{\varphi C_2} \simeq 0$, then R is given as the symmetric Poincaré ring spectrum associated to R^{e} with the C_2-action λ_R.

Using the functor $\Qoppa_R$, we can define a space of forms and nondegenerate forms over R.

Definition 6.4 [Cal+23, Def. 2.1.1, 2.1.3]. Let $\Qoppa : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ be a Poincaré ∞-category. Then the *space of Hermitian forms* of $\Qoppa$ is given as the core of the Grothendieck construction of $\Qoppa$:

$$\mathrm{Fm}(\Qoppa) = \left(\int_{\mathcal{C}} \Omega^\infty \Qoppa \right)^{\simeq}.$$

The *space of Poincaré objects* $\mathrm{Pn}(\Qoppa)$ is the full subgroupoid of $\mathrm{Fm}(\Qoppa)$ spanned by objects (x, q) such that the induced map $q_\# : x \to \mathrm{D}_\Qoppa(x)$ is an equivalence.

Definition 6.5. Let R be a Poincaré ring spectrum. We will denote the space $\mathrm{Pn}(\Qoppa_R)$ by $\mathcal{P}n(R)$ and call it the *space of Poincaré forms* over R.

Proposition 6.6 [Cal+23, Cor. 5.2.8]. *The functor* $\mathrm{Pn} : \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathcal{S}$ *is lax symmetric monoidal. In particular, if R is a Poincaré ring spectrum, then $\mathcal{P}n(R)$ is an $\mathbf{E}_\infty$-space.*

Remark 6.7. Let R be a Poincaré ring spectrum. Then $\mathcal{P}n(R)$ has two $\mathbf{E}_\infty$-structures given by taking tensor products of Poincaré objects and by taking direct sums of Poincaré objects, respectively. We will always view $\mathcal{P}n(R)$ as an $\mathbf{E}_\infty$-space via the former $\mathbf{E}_\infty$-structure.

7. Poincaré Picard space

Equipped with the space of Poincaré forms, we can now define a space that carries the information of the Hermitian Picard group.

Definition 7.1. Let R be a Poincaré ring spectrum. We will denote the full subcategory of $\mathcal{P}n(R)$ spanned by invertible objects by $\mathcal{P}ic^{\mathrm{p}}(R)$ and call it the *Poincaré Picard space of R*. Moreover, we will denote its group of connected components $\pi_0(\mathcal{P}ic^{\mathrm{p}}(R))$ by $\mathrm{Pic}^{\mathrm{p}}(R)$ and refer to it by the *Poincaré Picard group* of R.

Remark 7.2. By Proposition 6.6, the Poincaré Picard space $\mathcal{P}ic^{\mathrm{p}}(R)$ of a Poincaré ring spectrum R is a grouplike subalgebra of the $\mathbf{E}_\infty$-space $\mathcal{P}n(R)$. In particular, Pic^p is a grouplike $\mathbf{E}_\infty$-space.

Let S be a discrete commutative ring with a C_2-action, and let S^{s} be the corresponding symmetric Poincaré structure from Example 5.8. The Poincaré Picard group of S^{s} has a similar relationship with the Hermitian Picard group of S with its C_2-action as the Picard group with the Picard space.

Notation 7.3. Let R be a Poincaré ring spectrum. Objects in the Poincaré Picard space of R are Poincaré forms over R^{e}. If R^{e} is discrete, we will denote by $\mathcal{P}ic^{\mathrm{p},\heartsuit}(R)$ the full subcategory of $\mathcal{P}ic^{\mathrm{p}}(R)$ spanned by Poincaré forms on discrete R^{e}-modules.

Proposition 7.4 [BRY25, Prop. 5.4.6]. *Let S be a discrete commutative ring with a C_2-action $\lambda : S \xrightarrow{\simeq} S$, and let S^{s} be the corresponding symmetric Poincaré ring spectrum from Example 5.8. Then we have an isomorphism*

$$\pi_0(\mathcal{P}ic^{\mathrm{p},\heartsuit}(S^{\mathrm{s}})) \simeq \mathrm{Pic}^{\mathrm{h}}(S, \lambda).$$

Proof. After unravalling the notational differences, which are explained in [BRY25, Def. 5.4.1], this is a special case of [BRY25, Cor. 5.4.8, Thm. 5.4.9] for $X = \mathrm{Spec}(S)$. ☐

The following result is analogous to Theorem 2.6.

Theorem 7.5 [BRY25, Cor. 5.4.8, Thm. 5.4.9]. *Let S be a discrete commutative ring with a C_2-action $\lambda : S \xrightarrow{\simeq} S$. Let S^{a} be the corresponding symmetric or genuine symmetric Poincaré ring spectrum from Examples 5.8 and 5.9. Then we have an isomorphism*

$$\mathrm{Pic}^{\mathrm{p}}(S^{\mathrm{a}}) \simeq \mathrm{Pic}^{\mathrm{h}}(S, \lambda) \times \Gamma(S; \underline{\mathbf{Z}})^{\lambda = -\mathrm{id}},$$

where $(-)^{\lambda=-\mathrm{id}}$ denotes fixed points with respect to the action $n[C] \mapsto -n[\lambda(C)]$ for a connected component C of S.

Corollary 7.6. *Let S be a discrete commutative ring with a C_2-action $\lambda : S \xrightarrow{\simeq} S$. Then we have a canonical isomorphism*

$$\mathrm{Pic}^{\mathrm{p}}(S^{\mathrm{gs}}) \simeq \mathrm{Pic}^{\mathrm{p}}(S^{\mathrm{s}}),$$

To demonstrate computability of the Poincaré Picard group, we mention the following computation of the universal Poincaré ring spectrum $\mathbb{S}^{\mathrm{u}}$ from Example 5.7

Theorem 7.7 [BRY25, Cor. 5.1.17]. *Let $\mathbb{S}^{\mathrm{u}}$ be the universal Poincaré ring spectrum. Then we have an isomorphism*

$$\mathrm{Pic}^{\mathrm{p}}(\mathbb{S}^{\mathrm{u}}) \simeq \mathbf{Z}/2,$$

where a generator is given by the form $(-1, -1) \in \Qoppa_{\mathbb{S}^{\mathrm{u}}}(\mathbb{S}) \simeq \mathbb{S} \times_{\mathbb{S}^{tC_2}} \mathbb{S}$ over $\mathbb{S}$.

8. Poincaré Brauer space

To define a suitable Brauer space associated to a Poincaré ring spectrum, we will mimic the construction of the Brauer space from section 3. Given a Poincaré ring spectrum R, Construction 6.1 produces a commutative algebra object $\Qoppa_R$ in Poincaré ∞-categories $\mathrm{Cat}^{\mathrm{p}}_{\infty}$. The underlying ∞-category of $\Qoppa_R$ is $\mathrm{Mod}^{\omega}_{R^{\mathrm{e}}}$, which is idempotent complete. Therefore, $\Qoppa_R$ is a commutative algebra object in $\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}$, the full subcategory of $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ spanned by Poincaré ∞-categories whose underlying ∞-category is idempotent complete.

Definition 8.1. Let R be a Poincaré ring spectrum. We will denote the ∞-category $\mathrm{Mod}_{\Qoppa_R}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}})$ by $\mathrm{Cat}_{\Qoppa_R}$.

Definition 8.2. Let R be a Poincaré ring spectrum. We will denote the full subcategory of $\mathrm{Cat}^{\simeq}_{\Qoppa_R}$ spanned by invertible objects by $\mathcal{B}r^{\mathrm{p}}(R)$ and call it the *Poincaré Brauer space of R*. Moreover, we will denote its set of connected components $\pi_0(\mathcal{B}r^{\mathrm{p}}(R))$ by $\mathrm{Br}^{\mathrm{p}}(R)$ and refer to it by the *Poincaré Brauer group of R*.

Example 8.3 [BRY25, Example 6.2.4]. Let $\mathbb{S}^{\mathrm{u}}$ be the universal Poincaré ring spectrum from Example 5.7. Then we have an isomorphism

$$\mathrm{Br}^{\mathrm{p}}(\mathbb{S}^{\mathrm{u}}) \simeq \mathbf{Z},$$

where a generator is given by $\Sigma\mathbb{S}^{\mathrm{u}}$.

The Poincaré Brauer space bares a similar relation to the Poincaré Picard space as the Brauer space does to the Picard space.

Proposition 8.4 [BRY25, Prop. 6.2.2]. *Let R be a Poincaré ring spectrum. Then we have an equivalence of spaces*

$$\Omega\mathcal{B}r^{\mathrm{p}}(R) \simeq \mathcal{P}ic^{\mathrm{p}}(R).$$

We will now describe the objects of the Poincaré Brauer space $\mathcal{B}r^{\mathrm{p}}(R)$ more concretely.

Definition 8.5 [BRY25, Def. 6.3.5]. Let R be a Poincaré ring spectrum. An $\mathbf{E}_1$-*algebra with genuine involution* over R is the following data:

1. An $\mathbf{E}_1$-algebra B over R^{e} with a λ_R-involution $\sigma : B \xrightarrow{\sim} \lambda_R^* B^{\mathrm{op}}$, i.e. $\lambda^*(\sigma)^{\mathrm{op}} \circ \sigma \simeq \mathrm{id}_B$.
2. A $B \otimes_{R^{\mathrm{e}}} R^{\varphi C_2}$-module P together with a B-linear map $\alpha : P \to B^{\mathrm{t}C_2}$.

Notation 8.6. Let A be an $\mathbf{E}_1$-algebra with genuine involution over a Poincaré ring spectrum R, given by an $\mathbf{E}_1$-algeba B over R^e with λ_R-involution $\sigma : B \xrightarrow{\sim} \lambda_R(B)^{\mathrm{op}}$ and a B-linear map $\alpha : P \to B^{\mathrm{t}C_2}$. We will denote the components of A as follows:

$$A^{\mathrm{e}} := B, \qquad \sigma_A := \sigma, \qquad A^{\varphi C_2} := P, \qquad \alpha_A := \alpha.$$

We remark that Construction 6.1 can be extended to $\mathbf{E}_1$-algebras with genuine involution over Poincaré ring spectra, see [Cal+23, Const. 3.2.6]. By [BRY25, Prop. 4.1.9], an $\mathbf{E}_1$-algebra with genuine involution A over R then gives rise to a $\Qoppa_R$-module $\Qoppa_A$.

Definition 8.7. Let R be a Poincaré ring spectrum and let A be an $\mathbf{E}_1$-algebra with genuine involution over R. We will say that A is *Azumaya* if the following two conditions are satisfied:

1. The underlying $\mathbf{E}_1$-algebra A^{e} of A over R^{e} is an Azumaya algebra.
2. There exists an $\mathbf{E}_1$-algebra with genuine involution B over R such that $\Qoppa_A \otimes_{\Qoppa_R} \Qoppa_B \simeq \Qoppa_R$ as $\Qoppa_R$-modules.

Remark 8.8. We have a concrete description of the second condition in Definition 8.7 given in [BRY25, Def. 6.3.5]. For brevity, we will not expand it here.

Remark 8.9. Let R be a Poincaré ring spectrum such that $R^{\varphi C_2} \simeq 0$. This implies that $R^{\mathrm{t}C_2} \simeq 0$ as well and hence $\alpha_R \simeq \mathrm{id}_0$. In that case, any Azumaya algebra with a λ_R-involution over R^{e} produces an Azumaya algebra with genuine involution over R.

Theorem 8.10 [BRY25, Cor. 6.3.19]. *Let R be a Poincaré ring spectrum. Then any object of the Poincaré Brauer space $\mathcal{B}r^{\mathrm{p}}(R)$ is of the form $\Qoppa_A$ for A an Azumaya algebra with genuine involution over R.*

To say more about how the Poincaré Brauer space $\mathcal{B}r^{\mathrm{p}}(R)$ relates to the Brauer space $\mathcal{B}r(R^{\mathrm{e}})$ of the underlying $\mathbf{E}_\infty$-ring spectrum R^{e}, we need to introduce another variant of the Brauer space.

Definition 8.11. Let k be an $\mathbf{E}_\infty$-ring spectrum with a C_2-action via maps of $\mathbf{E}_\infty$-ring spectra. We will write $\mathcal{B}r^{C_2}(k)$ for $\mathcal{P}ic(\mathrm{Mod}_{(\mathrm{Mod}_k^\omega)^{\mathrm{h}C_2}}(\mathrm{Cat}^{\mathrm{ex}}_{\infty,\mathrm{idem}}))$.

Let k be an $\mathbf{E}_\infty$-ring spectrum with a C_2-action via maps of $\mathbf{E}_\infty$-ring spectra $\lambda : k \xrightarrow{\sim} k$. Then we have a map

$$\mathcal{B}r(k) \to \mathcal{B}r^{C_2}(k)$$

given by the assignment $\mathcal{C} \mapsto (\mathcal{C} \otimes_{\mathrm{Mod}_k^\omega} \lambda^*(\mathcal{C}))^{\mathrm{h}C_2}$; see [BRY25, Const. 7.1.3]. The following identifies its fiber as the Poincaré Brauer space and thus establishes the *norm fiber sequence*.

Theorem 8.12 [BRY25, Thm. 7.1.6]. *Let k be an $\mathbf{E}_\infty$-ring spectrum with a C_2-action via maps of $\mathbf{E}_\infty$-ring spectra. Then we have a fiber sequence of spaces*

$$\mathcal{B}r^{\mathrm{p}}(k^{\mathrm{s}}) \to \mathcal{B}r(k) \to \mathcal{B}r^{C_2}(k).$$

Finally, the norm fiber sequence from Theorem 8.12 allows us to find answers to Question 3.10.

Theorem 8.13 [BRY25, Thm. 7.3.2]. *Let k be an $\mathbf{E}_\infty$-ring spectrum such that $k^{\mathrm{t}C_2} \simeq 0$ for the trivial C_2-action on k. Then an equivalence class of the Brauer group* $\mathrm{Br}(k)$ *is 2-torsion if and only if it contains an Azumaya algebra which admits an involution.*

Proof. Let k^{s} be the symmetric Poincaré structure on k with the trivial C_2-action from Example 5.8. By Theorem 8.10, the objects of the Poincaré Brauer space $\mathcal{B}r^{\mathrm{p}}(k^{\mathrm{s}})$ are given by Azumaya algebras with genuine involution over k^{s}. Note that $\alpha_{k^{\mathrm{s}}} \simeq \mathrm{id}_0$ by assumption. In that case, Azumaya algebras with genuine involution over k^{s} are given by Azumaya algebras over k with an involution; see Remark 8.9. Objects of the

Brauer space $\mathcal{B}r(k)$ are given by module categories of Azumaya algebras over k, by Theorem 3.5. In the norm fiber sequence of Theorem 8.12

$$\mathcal{B}r^{\mathrm{p}}(k^{\mathrm{s}}) \to \mathcal{B}r(k) \to \mathcal{B}r^{C_2}(k)$$

the left map forgets the genuine involution. On connected components, the image of the left map is thus given by equivalence classes of Azumaya algebras over k which admit an involution. In particular, the image of the left map is contained in the subgroup of 2-torsion classes. On the other hand, the right map factors through multiplication by 2. Thus, the kernel of the right map is contained in and contains the subgroup of 2-torsion classes of $\mathrm{Br}(k)$. This proves the claim. □

Remark 8.14. The conclusion of Theorem 8.13 is an extension of Theorem 1.7 to $\mathbf{E}_\infty$-ring spectra. By Remark 8.15, the condition $k^{\mathrm{t}C_2} \simeq 0$ in Theorem 8.13 holds if and only if 2 is a unit in $\pi_0(k)$. However, Albert's and Saltman's theorems do not require the invertibility of 2. Presently, we have no evidence that this requirement is necessary for the conclusion of Theorem 8.13, and we hope to remove it in future work.

Remark 8.15. Let k be an $\mathbf{E}_\infty$-ring spectrum and let p be a prime. Then $k^{\mathrm{t}C_p} \simeq 0$ for the trivial action if and only if p acts invertibly on k. This can be seen as follows. Since p acts invertibly on k if and only if it does on $k_{(p)}$, we may assume that k is p-local. We will show that for a nonzero p-local $\mathbf{E}_\infty$-ring spectrum k, the following statements are equivalent:

1. The Tate construction for the trivial C_p-action on k vanishes.
2. The Tate construction for all C_p-actions on k vanishes.
3. The support of k is $\mathrm{supp}(k) := \{n \in \mathbf{Z}_{\geq 0} \cup \infty \mid k \otimes K(n) \not\simeq 0\} = \{0\}$.
4. The prime p acts invertibly on k.

The Tate construction is lax symmetric monoidal. The Tate construction for a C_p-action is thus an algebra over the Tate construction for the trivial C_p-action. This shows the equivalnece of (1) and (2). By Hahn's theorem [Cla+24, Thm. 4.6], (2) implies $\mathrm{supp}(k) \subset \{0, \infty\}$. The spectrum $k \otimes \mathbf{F}_p$ is free over $\mathbf{F}_p$, and thus nontrivial if and only if it contains $\mathbf{F}_p$ as a summand up to suspension. The Tate construction is exact and $\mathbf{F}_p^{\mathrm{t}C_p} \not\simeq 0$, hence $k \otimes \mathbf{F}_p$ vanishes if and only if $(k \otimes \mathbf{F}_p)^{\mathrm{t}C_p}$ vanishes. The latter vanishes as an algebra over $k^{\mathrm{t}C_p} \simeq 0$. Moreover, k is nonzero, so the nilpotence theorem [HS98, Thm. 3] implies that the support is nonempty. Thus (2) implies (3). Under the assumptions of (3), the support of k/p is empty. By the nilpotence theorem $k/p \simeq 0$, which is equivalent to (4). Finally, (4) implies (1), since $k^{\mathrm{t}C_p}$ is an algebra over $\mathbf{Q}^{\mathrm{t}C_p} \simeq 0$. We note that statements (1), (2), (4) are equivalent for any $\mathbf{E}_\infty$-ring spectrum without the requirements of being nonzero p-local.

References

[AG14] Benjamin Antieau and David Gepner. "Brauer groups and étale cohomology in derived algebraic geometry". In: *Geom. Topol.* 18.2 (2014), pp. 1149–1244. ISSN: 1465-3060,1364-0380. DOI: `10.2140/gt.2014.18.1149`. URL: `https://doi.org/10.2140/gt.2014.18.1149`.

[Alb35] A. Adrian Albert. "Involutorial simple algebras and real Riemann matrices". English. In: *Ann. Math. (2)* 36 (1935), pp. 886–964. ISSN: 0003-486X. DOI: `10.2307/1968595`.

[BRY25] Viktor Burghardt, Noah Riggenbach, and Lucy Yang. *Involutive Brauer groups and Poincaré rings*. 2025. arXiv: `2509.25737 [math.AG]`. URL: `https://arxiv.org/abs/2509.25737`.

[Cal+23] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. "Hermitian K-theory for stable ∞-categories I: Foundations". In: *Selecta Math. (N.S.)* 29.1 (2023), Paper No. 10, 269. ISSN: 1022-1824,1420-9020. DOI: `10.1007/s00029-022-00758-2`. URL: `https://doi.org/10.1007/s00029-022-00758-2`.

[Cla+24] Dustin Clausen, Akhil Mathew, Niko Naumann, and Justin Noel. "Descent and vanishing in chromatic algebraic K-theory via group actions". In: *Ann. Sci. Éc. Norm. Supér. (4)* 57.4 (2024), pp. 1135–1190. ISSN: 0012-9593,1873-2151.

[Fau03] H. Fausk. "Picard groups of derived categories". In: *J. Pure Appl. Algebra* 180.3 (2003), pp. 251–261. ISSN: 0022-4049,1873-1376. DOI: `10.1016/S0022-4049(02)00145-7`. URL: `https://doi.org/10.1016/S0022-4049(02)00145-7`.

[Gab81] Ofer Gabber. "Some theorems on Azumaya algebras". In: *The Brauer group (Sem., Les Plans-sur-Bex, 1980)*. Vol. 844. Lecture Notes in Math. Springer, Berlin, 1981, pp. 129–209. ISBN: 3-540-10562-X.

[Gro68] Alexander Grothendieck. "Le groupe de Brauer. II. Théorie cohomologique". In: *Dix exposés sur la cohomologie des schémas*. Vol. 3. Adv. Stud. Pure Math. North-Holland, Amsterdam, 1968, pp. 67–87.

[HS98] Michael J. Hopkins and Jeffrey H. Smith. "Nilpotence and stable homotopy theory. II". In: *Ann. of Math. (2)* 148.1 (1998), pp. 1–49. ISSN: 0003-486X,1939-8980. DOI: `10.2307/120991`. URL: `https://doi.org/10.2307/120991`.

[Knu91] Max-Albert Knus. *Quadratic and Hermitian forms over rings*. Vol. 294. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. With a foreword by I. Bertuccioni. Springer-Verlag, Berlin, 1991, pp. xii+524. ISBN: 3-540-52117-8. DOI: `10.1007/978-3-642-75401-2`. URL: `https://doi.org/10.1007/978-3-642-75401-2`.

[Lur09] Jacob Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. DOI: `10.1515/9781400830558`. URL: `https://doi.org/10.1515/9781400830558`.

[Lur17] Jacob Lurie. *Higher algebra*. Manuscript available at `https://www.math.ias.edu/~lurie/papers/HA.pdf`. 2017.

[RVV95] M. V. Reyes Sánchez, P. Verhaeghe, and A. Verschoren. "The relative Hermitian Picard group". In: *Comm. Algebra* 23.10 (1995), pp. 3915–3941. ISSN: 0092-7872,1532-4125. DOI: `10.1080/00927879508825439`. URL: `https://doi.org/10.1080/00927879508825439`.

[Sal78] David J. Saltman. "Azumaya algebras with involution". In: *Journal of Algebra* 52.2 (1978), pp. 526–539. ISSN: 0021-8693. DOI: `https://doi.org/10.1016/0021-8693(78)90253-3`. URL: `https://www.sciencedirect.com/science/article/pii/0021869378902533`.