

LOYALTY OF THE AFFINE LINE IN MOTIVIC HOMOTOPY THEORY

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We show that integers act invertibly on a derived scheme S if and only if they act invertibly on the reduced motivic suspension spectrum of the affine line in MS_S . As a consequence, over regular locally noetherian schemes and derived schemes on which some nonzero integer acts as zero, the motivic sphere spectrum is $\mathbf{A}^1$ -invariant after inverting primes that are not invertible in the base scheme.

Outline 0.0.1. We start by briefly recalling some of the necessary notions and establishing the setting we are in. We then show that $\mathbf{A}^1$ is essentially blind to differences between the base S and finite étale covers of S . This will be our main tool and it will carry us all the way to the finish line via repeated use: the affine line in motivic homotopy theory respects invertibility of integers in the base. As an immediate application, away from noninvertible integers in the base, we establish $\mathbf{A}^1$ -invariance of the motivic sphere spectrum over locally noetherian regular schemes, as well as over derived schemes on which some nonzero integer acts as zero.

Resources 0.0.2. Our main sources will be [AHI24a], [AHI24b] and [Hoy24]. We will not attempt to be fully self-contained and refer the reader to these references for more background.

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1. set and setting

Convention 1.0.1. Unless otherwise specified, the letter S will stand for a derived scheme. We denote the ∞ -category of nisnevich local motivic spectra over S by MS_S . We use the notation $\mathbf{A}^1 = \Sigma^\infty(\mathbf{A}^1, 0) \in \mathrm{MS}_S$. Given an object E in a stable ∞ -category, we will denote its rationalization by $E_{\mathbf{Q}}$.

1.1. $\mathbf{A}^1$ -invariance We briefly recall the notion of $\mathbf{A}^1$ -invariance in motivic homotopy theory and demonstrate how it relates to torsion and rational parts.

Definition 1.1.1. We say that $E \in \mathrm{MS}_S$ is $\mathbf{A}^1$ -invariant if the internal mapping object $\underline{\mathrm{Hom}}(\mathbf{A}^1, E)$ vanishes. We will denote the full subcategory of $\mathbf{A}^1$ -invariant motivic spectra of MS_S by $\mathrm{Sp}(S)$.

One way to break this condition down into smaller pieces is as follows.

Proposition 1.1.2. *A motivic spectrum $E \in \mathrm{MS}_S$ is $\mathbf{A}^1$ -invariant if and only if $E_{\mathbf{Q}}$ and E/p are $\mathbf{A}^1$ -invariant for all primes $p \in \mathbf{Z}$.*

Proof. The full subcategory $\mathrm{Sp}(S) \subset \mathrm{MS}_S$ is stable and closed under colimits by [AHI24b, 6.1]. If $E \in \mathrm{Sp}(S)$, then E/p and $E_{\mathbf{Q}}$ belong to $\mathrm{Sp}(S)$ since they are colimits in $\mathrm{Sp}(S)$. On the other hand, if $E/p \in \mathrm{Sp}(S)$ for all primes p , then $E/n \in \mathrm{Sp}(S)$ for all integers n via cofiber sequences $E/n \rightarrow E/nm \rightarrow E/m$. Moreover, if $E_{\mathbf{Q}} \in \mathrm{Sp}(S)$, then E belongs to $\mathrm{Sp}(S)$ via the cofiber sequence $\mathrm{colim}_n \Sigma^{-1} E/n! \rightarrow E \rightarrow E_{\mathbf{Q}}$. $\square$

The following is an immediate consequence of the long exact sequence of homotopy groups.

Lemma 1.1.3. *Let $\mathcal{C}$ be a stable ∞ -category, $E \in \mathcal{C}$ and $l \in \mathbf{Z}$. Then $\pi_0 \mathrm{End}_{\mathcal{C}}(E/l)$ is a $\mathbf{Z}/l^2$ -module.*

Lemma 1.1.4. *Let $\mathcal{C}$ be a stable ∞ -category, let $A \in \mathcal{C}$ and $l \in \mathbf{Z}$. Then l acts invertibly on A if and only if the mapping spectrum $\mathrm{map}_{\mathcal{C}}(A, E/l)$ vanishes for all $E \in \mathcal{C}$.*

Proof. The integer l acts invertibly on $\mathrm{map}_{\mathcal{C}}(A, E/l)$ via A . At the same time, l^2 acts as zero on it via E/l by Lemma 1.1.3. Thus, we have $\mathrm{map}_{\mathcal{C}}(A, E/l) \simeq 0$. This proves one direction. The other direction follows by taking $E = A$. $\square$

Remark 1.1.5. We specialize Lemma 1.1.4 to $\mathcal{C} = \mathrm{MS}_S$: l acts invertibly on $\mathbf{A}^1$ in MS_S if and only if, for all motivic spectra $E \in \mathrm{MS}_S$, E/l is $\mathbf{A}^1$ -invariant.

The following says that $\mathbf{A}^1$ -invariance in MS_S is zariski local.

Lemma 1.1.6. *Let $E \in \mathrm{MS}_S$. Then $\mathbf{A}^1$ -invariance of E can be checked on a zariski cover of S .*

Proof. Let $j : U \hookrightarrow S$ be open. By the projection formula we have $j^*(\underline{\mathrm{Hom}}(\mathbf{A}_S^1, E)) \simeq \underline{\mathrm{Hom}}(\mathbf{A}_U^1, j^*(E))$. The claim now follows from zariski of MS . $\square$

2. taming the affine line

2.1. transport of vanishing In this section we show that the affine line cannot tell the difference between S and finite étale covers of S that embed into $\mathbf{G}_m$. This is our main tool. It will allow us to investigate how functions act on $\mathbf{A}^1$ in MS_S .

Theorem 2.1.1. *Let S be a qcqs derived scheme. Let $Z \subset \mathbf{G}_m$ be closed finite étale with surjective structure map $g : Z \subset \mathbf{G}_m \rightarrow S$. Then the unit $\mathbb{S} \rightarrow \mathbf{A}^1$ factors through the full subcategory $g_\#(\mathrm{MS}_Z^\omega)$.*

Proof. Let $K = \mathbf{P}_S^1 - Z$. Since $Z \rightarrow S$ is étale and separated, we may decompose $Z \times_S Z = \Delta(Z) \sqcup C$. Set $V = \mathbf{P}_Z^1 - C$. Then $V \rightarrow \mathbf{P}_S^1$ is étale, $V \times_{\mathbf{P}_S^1} Z = \Delta(Z) \rightarrow Z$ is an isomorphism, and $V \times_{\mathbf{P}_S^1} K = V - \Delta(Z)$. Thus,

$$\begin{array}{ccc} V - \Delta(Z) & \longrightarrow & V \\ \downarrow & & \downarrow \\ K & \hookrightarrow & \mathbf{P}_S^1 \end{array}$$

is a nisnevich square. Its top edge consists of open subschemes of $\mathbf{P}_Z^1$, hence of smooth Z -schemes. By nisnevich excision and exactness of $g_\#$, its fiber in MS_S is of the form $g_\#(W)$ for some $W \in \mathrm{MS}_Z^\omega$.

The automorphism $[x : y] \mapsto [x : y - x]$ of $\mathbf{P}^1$ sends 0 to 0 and 1 to ∞ . Thus, 0 and ∞ are $\mathbf{P}^1$ -homotopic, hence homotopic as maps $\mathbb{S} \rightarrow \mathbf{P}^1$ [AHI24a, 4.4]. Therefore, the map $\mathbb{S} \xrightarrow{\infty-0} K$ becomes nullhomotopic in $\mathbf{P}^1$ and factors through $g_\#(W)$.

It now suffices to construct a map $K \rightarrow \mathbf{A}^1$ realizing the claimed factorization. Without loss of generality, we may assume that g has constant rank d . Let $q \in \mathcal{O}(S)[s]$ be the characteristic polynomial of the coordinate on Z ; it is monic of positive degree d , $q(0)$ is a unit and $V(q) = Z$. Define $\varphi : K \rightarrow \mathbf{A}^1$ by $\varphi(x) = 1 - q^{(0)}/q(x)$. The map φ is regular on K , with $\varphi(0) = 0$ and $\varphi(\infty) = 1$. It follows that the composition $\varphi \circ (\infty - 0)$ is the unit of $\mathbf{A}^1$. This proves the claim. $\square$

Remark 2.1.2. When S is not qcqs, then the proof of Theorem 2.1.1 applies verbatim; only compactness of W is lost. For our use, compactness of W will not be needed.

Remark 2.1.3. A consequence of Theorem 2.1.1 is that we can transport vanishing of maps from MS_Z to MS_S . More precisely, let $f : E \rightarrow F$ be a map in MS_S . Then $g^*(f) \simeq 0$ implies $f \otimes \mathrm{id}_{\mathbf{A}^1} \simeq 0$. Indeed, due to the $\mathbf{E}_\infty$ -structure of $\mathbf{A}^1$ and by the projection formula, the map $f \otimes \mathrm{id}_{\mathbf{A}^1}$ has the factor $f \otimes \mathrm{id}_{g_\#(W)} \simeq g_\#(g^*(f) \otimes \mathrm{id}_W) \simeq 0$.

Remark 2.1.4. Let $q \in \mathcal{O}(S)[s]$ be monic of positive degree such that $q(0)$ and $\mathrm{disc}(q)$ are units. Then $Z = V(q) \subset \mathbf{G}_m$ is closed finite étale and surjective over S , and Theorem 2.1.1 applies.

The partial blindness of $\mathbf{A}^1$ established in Theorem 2.1.1 implies the a priori surprising statement that $\mathbf{A}^1$ -invariance comes for free in MS_S^- .

Proposition 2.1.5. *We have $\mathbf{A}^{1,-} \simeq 0$. In particular, $\mathrm{MS}_S^- = \mathrm{Sp}(S)^-$.*

Proof. If $2 \in \mathcal{O}(S)^\times$, then Remark 2.1.4 applied with $q(x) = x^2 + 1$ shows that $\langle -1 \rangle$ acts as the identity on $\mathbf{A}^1$, since -1 is a square on $V(x^2 + 1)$ and $\langle a^2 \rangle = 1$ for every unit a [Hoy24, 5.2]. If $3 \in \mathcal{O}(S)^\times$, then Remark 2.1.4 applied with $q(x) = x^2 + x + 1$ shows that $2 \langle -1 \rangle$ acts as 2 on $\mathbf{A}^1$. Indeed, on $V(x^2 + x + 1)$ we have $-1 = x + x^2$ with $x = (x^2)^2$, so that $2 = \langle x \rangle + \langle x^2 \rangle = 2 \langle -1 \rangle$ by [Hoy24, 5.3]. In either case, this implies $\mathbf{A}^{1,-} \simeq 0$. In general, we cover S by $S[1/2]$ and $S[1/3]$. The pullback to those opens will preserve $\mathbf{A}^{1,-}$; we know $\mathbf{A}^{1,-}$ vanishes over $S[1/2]$ and $S[1/3]$ by the above cases. The claim now follows from zariski descent of MS . $\square$

2.2. loyalty of the affine line The following definition captures the ability of objects to inherit invertibility in the action of primes on S , stably under base change.

Definition 2.2.1. For a prime $p \in \mathbf{Z}$, let $j_p : S[1/p] \hookrightarrow S$ be the open subscheme on which p is invertible. We say $A \in \mathrm{MS}_S$ is *loyal* if p acts invertibly on $j_p^*(A)$ for every prime p . We say A is *fully loyal* if moreover $x^*(A/p) \neq 0$ for every prime $p \in \mathbf{Z}$ and every point $x : \mathrm{Spec}(k) \rightarrow S$ of characteristic p .

Remark 2.2.2. Note that A is loyal if and only if, for every $f : S' \rightarrow S$ and every prime $p \in \mathbf{Z}$ invertible on S' , p acts invertibly on $f^*(A)$. Similarly, A is fully loyal if and only if, for every $f : S' \rightarrow S$ and every prime $p \in \mathbf{Z}$, p acts invertibly on $f^*(A)$ exactly when p is invertible on S' .

Theorem 2.2.3. *Let l be an integer that acts invertibly on S . Then l acts invertibly on $\mathbf{A}^1$ in MS_S .*

Proof. Let $p : \mathbf{G}_m \rightarrow S$ be the projection with coordinate x , and set $M = \mathbf{A}^1/l$, $M_{\mathbf{G}_m} = p^*(M) = \mathbf{A}_{\mathbf{G}_m}^1/l$. Denote $[x] : \mathbb{S} \xrightarrow{x} \mathbf{G}_{m+} \xrightarrow{r} \Sigma^{-1}\mathbf{P}^1$ and $[x]^{\natural} : \mathbf{G}_{m+} \xrightarrow{r} \Sigma^{-1}\mathbf{P}^1$. Moreover, define $q(s) = s^{l^2} - x \in \mathcal{O}(\mathbf{G}_m)[s]$, $Z = V(q)$, $g : Z \subset \mathbf{A}_{\mathbf{G}_m}^1 \rightarrow \mathbf{G}_m$ with coordinate y , and $M_Z = g^*(M_{\mathbf{G}_m})$. We divide the proof into the following steps.

- (1) $\langle -1 \rangle$ acts as the identity on $M_{\mathbf{G}_m}$: by Proposition 2.1.5 for l odd, and by its proof for l even.
- (2) $[x] \otimes \mathrm{id}_{M_{\mathbf{G}_m}} \simeq 0$: use Remark 2.1.4 over the base $\mathbf{G}_m$; here $q(0) = -x$ and $\mathrm{disc}(q) = \pm l^{2l^2} x^{l^2-1}$ are units because l is invertible. On Z we have $y^{l^2} = x$. The relations $[ab] = [a] + \langle a \rangle [b]$ for units a, b [Hoy24, 3.2 and 3.4] give $[y^{l^2}] = \sum_{i=0}^{l^2-1} \langle y \rangle^i [y]$. We claim that $\langle y \rangle$ acts as the identity on M_Z . If l is odd, then $\langle y \rangle + \langle -1 \rangle \langle y \rangle = 1 + \langle -1 \rangle$ [Hoy24, 3.5], so $2(\langle y \rangle - 1)$ acts as zero on M_Z by (1), and 2 acts invertibly on M_Z by Lemma 1.1.3. If l is even, then $2 \in \mathcal{O}(S)^\times$, and Remark 2.1.4 over the base Z , applied with $q(t) = t^2 - y$, together with $\langle t^2 \rangle = 1$ [Hoy24, 5.2] and Remark 2.1.3, shows that $\langle y \rangle - 1$ acts as zero on $\mathbf{A}_Z^1$, hence on M_Z . Therefore $g^*([x] \otimes \mathrm{id}_{M_{\mathbf{G}_m}}) = [y^{l^2}] \otimes \mathrm{id}_{M_Z} = [y] \otimes l^2 \mathrm{id}_{M_Z} \simeq 0$ by Lemma 1.1.3. Thus, $[x] \otimes \mathrm{id}_{M_{\mathbf{G}_m}} \otimes \mathrm{id}_{\mathbf{A}_{\mathbf{G}_m}^1} \simeq 0$ by Theorem 2.1.1. Finally, $M_{\mathbf{G}_m}$ is a retract of $M_{\mathbf{G}_m} \otimes \mathbf{A}_{\mathbf{G}_m}^1$ through its module structure.
- (3) $[x]^{\natural} \otimes \mathrm{id}_M \simeq 0$: $[x] \otimes \mathrm{id}_{M_{\mathbf{G}_m}}$ corresponds to $[x]^{\natural} \otimes \mathrm{id}_M$ under the adjunction $p_{\#} \dashv p^*$ and $p_{\#} p^*(M) \simeq \mathbf{G}_{m+} \otimes M$. Conclude with (2).
- (4) $M \simeq 0$: the map $[x]^{\natural} \otimes \mathrm{id}_M$ has a section by [Hoy24, 2.2]. Hence, $\mathrm{id}_{\Sigma^{-1}\mathbf{P}^1 \otimes M} \simeq 0$. The claim follows by invertibility of $\mathbf{P}^1$. $\square$

Remark 2.2.4. Let l be a prime that acts noninvertibly on S . Then l does not act invertibly on $\mathbf{A}^1$ in MS_S . Base change to residue fields is symmetric monoidal, exact, and preserves the affine line. Thus, we may assume that S is the spectrum of a characteristic l field. The Hochschild homology spectrum HH [AHI24b, 6.4] now satisfies $\mathrm{HH}_0(S) = \mathcal{O}(S)$. Thus, l acts as zero on HH . If l acted invertibly on $\mathbf{A}^1$ in MS_S , then $\mathrm{HH}/l \simeq \mathrm{HH} \oplus \Sigma \mathrm{HH}$ would be $\mathbf{A}^1$ -invariant by Remark 1.1.5. Since HH is clearly not $\mathbf{A}^1$ -invariant, l cannot act invertibly on $\mathbf{A}^1$.

Remark 2.2.5. As a consequence of Remark 2.2.4, the statement of Theorem 2.2.3 can be upgraded to an if and only if statement. In other words, the affine line $\mathbf{A}^1 \in \mathrm{MS}_S$ is fully loyal.

3. one consequence

In this section we establish one of the many applications of loyalty of the affine line in motivic homotopy theory. Further applications, such as étale descent for the kernel of $\mathbf{A}^1$ -localization, as well as implications for t-structures, effectivity and slices in MS_S , will be treated elsewhere.

3.1. $\mathbf{A}^1$ -invariance of the sphere

Theorem 3.1.1. *Let S be a regular locally noetherian scheme and let $P \subset \mathbf{Z}$ be the set of primes acting noninvertibly on S . Then, the P -inverted motivic sphere spectrum $\mathbb{S}[P^{-1}] \in \mathrm{MS}_S$ is $\mathbf{A}^1$ -invariant.*

Proof. By Proposition 1.1.2 it suffices to show that $\mathbb{S}[P^{-1}]$ is $\mathbf{A}^1$ -invariant rationally and mod p , for all primes $p \in \mathbf{Z}$. Rationally, we have $\mathbb{S}_{\mathbf{Q}} \simeq \mathrm{H}\mathbf{Q} \oplus \mathbb{S}_{\mathbf{Q}}^-$, by [Hoy24, 6.3]. By the assumptions on our base and Lemma 1.1.6, $\mathrm{H}\mathbf{Q}$ is $\mathbf{A}^1$ -invariant by [AHI24b, 9.12]; so is $\mathbb{S}_{\mathbf{Q}}^-$ by Proposition 2.1.5. Loyalty of the affine line together with Remark 1.1.5, implies that $\mathbb{S}[P^{-1}]/p$ is $\mathbf{A}^1$ -invariant. The claim follows. $\square$

Remark 3.1.2. The set P is empty for schemes over the rational numbers $\mathbf{Q}$. In the case of regular noetherian schemes over $\mathbf{Q}$, this answers affirmatively the question, asked in [AHI24b, 1.6], whether the motivic sphere over regular noetherian schemes is $\mathbf{A}^1$ -invariant.

Remark 3.1.3. The generality of Theorem 3.1.1 can be improved in various directions. For example, for S a derived scheme, we have: if there exists a nonzero integer that acts as zero on S , then $\mathbb{S}[P^{-1}] \in \mathrm{MS}_S$ is $\mathbf{A}^1$ -invariant. In particular, S does not need to be regular in positive characteristic. Loyalty of the affine line reduces this claim to $\mathbf{A}^1$ -invariance of $\mathrm{H}\mathbf{Q}$ and thus $\mathrm{KGL}_{\mathbf{Q}}$, which is $\mathbf{A}^1$ -invariant by results of Weibel [Wei81] and Goodwillie [Goo86].

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