CHAIN-LEVEL MODELS OF EQUIVARIANT TOPOLOGICAL HOCHSCHILD HOMOLOGY OF QUASIREGULAR SEMIPERFECT AND SMOOTH ALGEBRAS OVER FINITE FIELDS

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Abstract. We find an explicit model for equivariant topological Hochschild homology of semiperfect and smooth algebras over a finite field, as modules over the constant Green functor over a primary cyclic group.

1. Introduction

In [3], Bhatt, Morrow, and Scholze used the fixed point spectrum of equivariant topological Hochschild homology to define the motivic filtration on the De Rham-Witt complex of a smooth $\mathbb{F}_p$-algebra (first considered by Nygaard [23]). A key new idea in their approach was faithfully flat descent to quasiregular semiperfect $\mathbb{F}_p$-algebras. On the one hand, this was an application of topological Hochschild homology to obtaining a new technique in p-adic Hodge theory, on the other hand, it gave spectral sequences linking crystalline cohomology to different variants of topological cyclic homology.

An analogous story is also developed in [3] in mixed characteristic, where one can similarly treat quasisyntomic rings using descent to quasiregular semiperfectoid $\mathbb{Z}_p$-algebras. The role of crystalline cohomology is then replaced by the new concept of prismatic cohomology. In addition to further advancing the methods of p-adic Hodge theory, the spectral sequences based on prismatic cohomology led to substantial

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progress in the calculation of algebraic K-theory (see Antieau, Krause, and Nikolaus [2], also Mathew [21]).

It is a natural question to ask what role topology intrinsically plays in these new constructions. The purpose of this note is to address this question in the case of pure characteristic p, i.e. the case of smooth $\mathbb{F}_p$-algebras R. In this case, it turns out that topology is not involved in the sense that the genuine $\mathbb{Z}/p^r$-equivariant spectrum $THH_{\mathbb{Z}/p^r}(R)$ and its fixed points

$$TR^{r+1}(R) = THH^{\mathbb{Z}/p^r}(R) \tag{1}$$

can be explicitly modeled by purely algebraic, chain-level objects, i.e. in terms of homological algebra. The main result of our article is to construct and describe such a model. From one point of view, this can be considered a refinement of the spectral sequence method, which only gives associated graded information.

A remark may perhaps be in order at this point on the group over which we are doing equivariant topology: While $THH(R)$ is usually considered as an S^1-equivariant spectrum, its S^1-geometric fixed points (in the sense of equivariant stable homotopy theory) have no geometric meaning, and in fact are an artifact of the model chosen. This is one reason why we restrict the S^1-equivariant structure to a finite subgroup $\mathbb{Z}/p^r \subset S^1$ in the present paper. The other reason is that S^1 is a compact Lie group, and a finite subgroup is more suitable for discussing a purely algebraic model.

Before describing this in more detail, let us briefly reflect on the category in which we will be doing homological algebra. This is completely answered by genuine equivariant stable homotopy theory, the concepts of which remain in the background of topological Hochschild homology, even though Nikolaus and Scholze [22] used the "Tate square" to largely eliminate genuine equivariant spectra from the discussion. In equivariant stable homotopy theory, generalized Eilenberg-Mac Lane spectra can be represented by chain complexes of Mackey functors. In fact, the (unbounded) derived categories of generalized equivariant Eilenberg-Mac Lane spectra and Mackey functors are equivalent by the work of Greenlees and Shipley [11].

However, in our case, the ground category can be further refined. The reason we are dealing with equivariant generalized Eilenberg-Mac Lane spectra is the calculation of Hesselholt and Madsen [15] proving that the $\mathbb{Z}/p^r$-equivariant spectrum $TR(\mathbb{F}_p)$ (given as the homotopy limit of the $\mathbb{Z}/p^r$-equivariant spectra $(THH_{\mathbb{Z}/p^{n+r}}(\mathbb{F}_p))^{\mathbb{Z}/p^n}$ with respect to restriction) is the Eilenberg-Mac Lane spectrum $H\underline{\mathbb{Z}}_p$ of the constant Mackey functor $\underline{\mathbb{Z}}_p$. This suggests the right category to be the derived category of E_∞-$H\underline{\mathbb{Z}}_p$-modules, which is equivalent to the (unbounded) derived category of $\mathbb{Z}/p^r$-Mackey modules over the constant Green functor $\underline{\mathbb{Z}}_p$ (see [11]).

It is this category in which we give our chain-level model. Using faithfully flat descent, it suffices to describe the model for quasiregular semiperfect $\mathbb{F}_p$-algebras. In this case, the model turns out to be quite simple, (thus conforming with the pattern of [3]). The key ingredient is the "geometric fixed point functor" $\Phi^{\mathbb{Z}/p^i}$ from the derived category of $\mathbb{Z}/p^s$-$\underline{\mathbb{Z}}_p$-modules to the derived category of $\mathbb{Z}/p^r$-$\underline{\mathbb{Z}}_p$-modules for $i \leq s$, $r < s \leq r+i$. In terms of these building blocks, we can state our main result in broad terms as follows:

1. **Theorem.** *There is an explicit functorial model of $THH_{\mathbb{Z}/p^r}(R)$ for a quasiregular semiperfect $\mathbb{F}_p$-algebra R in the derived category of $\mathbb{Z}/p^r$-$\underline{\mathbb{Z}}_p$-modules which is a direct sum of even suspensions of chain complexes of the form $\Phi^{\mathbb{Z}/p^i}(\underline{\mathbb{Z}}_p)_{\mathbb{Z}/p^s}$.*

A more precise version of this statement is given in Theorem 6 below.

It should be noted that from the point of view of homotopy theory, the cases of positive and mixed characteristic are quite different. While, as already remarked, for $\mathbb{F}_p$-algebras R, (1) is an E_∞-module spectrum over the E_∞-ring spectrum $H\underline{\mathbb{Z}}_p$, which enables a purely chain-level model in the derived category of Mackey modules over $\underline{\mathbb{Z}}_p$, for $\mathbb{Z}_p$-algebras R, this approach only takes us to E_∞-modules over the $\mathbb{Z}/p^r$-equivariant spectrum $TR(\mathbb{Z}_p)$, which is known by Bökstedt and Madsen [5] to be a certain form of connective K-theory. Accordingly, the Bott periodicity element occurs in the calculations (see e.g. [2, 5, 21]). Thus, modules over connective K-theory would have to be used to discuss the mixed characteristic case. There is therefore no hope to model (1) in the mixed characteristic case in the derived category of an abelian category. One could hope for a concrete description in terms of some form of connective equivariant (topological) K-theory. This will be left for future work.

For this reason, we stick to the case of pure characteristic p in the present paper. In this case, remarkably enough, with the exception of quasiregular semiperfect descent, we are able to use, more or less, the classical method of Hesselholt-Madsen [15] who calculated the answer for a perfect commutative $\mathbb{F}_p$-algebra. As already noted, we observe that in this case, (1) is essentially given by the "geometric fixed points" of the constant $H\underline{\mathbb{Z}}_p$-module over a primary cyclic group $\mathbb{Z}/p^s$ with respect to the subgroup $\mathbb{Z}/p^{s-r}$ (Theorem 3 below). A key observation is that this does not depend on the choice of $s > r$. Explicit chain-level $\underline{\mathbb{Z}}_p$-Mackey module models are immediately visible. Additionally, geometric fixed points of (1) are also obtained as geometric fixed points of $H\underline{\mathbb{Z}}_p$ with respect to different subgroups.

The other key point is that this result also extends to the case of (1) for quasiregular semiperfect $\mathbb{F}_p$-algebras R, i.e. that for such R, (1) is described as a direct sum of the same building blocks (i.e. geometric fixed points of the $\mathbb{Z}/p^s$-equivariant $H\underline{\mathbb{Z}}_p$ with respect to different subgroups). This can also be proved using the methods of [15], along with the Quillen spectral sequence [24], Theorem 5.1 to start the induction. However, to use the results for the program we set out, it is important to discuss the precise functoriality for the case of quasiregular semiperfect $\mathbb{F}_p$-algebras. For this purpose, we use another device, namely the module-valued Witt vectors defined by Dotto, Krause, Nikolaus, and Patchkoria [6, 7]. Applying that construction to powers of the augmentation ideal J of the limit perfection S_R of a quasiregular semiperfect $\mathbb{F}_p$-algebra R, we get subobjects of $W(S_R)$, which can be used to describe our answer functorially (Theorem 6). Additionally, to show that our functoriality is the right one (thereby describing the correct presheaf on the crystalline site), we use a certain rigidity of the geometric fixed points of $H\underline{\mathbb{Z}}_p$ (Lemma 4).

It is worth noting that other approaches to the motivic filtration are possible, see e.g. Hahn, Raksit, and Wilson [12]. This is related to Hesselholt's description of topological cyclic homology of an associative $\mathbb{F}_p$-algebra as a derived functor [13]. The latter result is based on a description of TR for a unital tensor $\mathbb{F}_p$-algebra as a non-commutative analogue of the De Rham-Witt complex. This is then concentrated in homological degrees $0, 1$. However, it is worth noting that (1) for a unital tensor $\mathbb{F}_p$-algebra (or a smooth commutative $\mathbb{F}_p$-algebra) is actually not a direct sum of our building blocks given by geometric fixed points of homology with constant coefficients. Rather, finite $RO(\mathbb{Z}/p^r)$-graded suspensions are involved, which gives non-trivial extensions of the building blocks. This could make discussion of functoriality using this approach substantially more difficult. Therefore, our discussion could be considered as another application of the Bhatt-Morrow-Scholze approach, beyond the motivic filtration.

The present paper is organized as follows: In Section 2, we discuss some basic notation, and recall the theory of Mackey functors over a primary cyclic group, and Mackey modules over the constant Green functor. In Section 3, we introduce our building blocks, and discuss the chain-level model of (1) for a perfect $\mathbb{F}_p$-algebra R. In Section 4, we discuss the chain-level model for quasiregular semiperfect $\mathbb{F}_p$-algebras, and the functoriality.

2. Preliminaries

Equivariant stable homotopy theory for a finite group G is the basic background of our discussion. In particular, we are interested in genuine equivariant spectra, which represent $RO(G)$-graded equivariant homology and cohomology theories. The basic background reference for this topic is Lewis, May, Steinberger [19]. A G-equivariant spectrum is often decorated by a subscript $E = E_G$, while E^G denotes the corresponding fixed point spectrum. For a genuine equivariant spectrum E and a fixed $n \in \mathbb{Z}$, the system of groups $(\pi_n(E^H))_{H \subseteq G}$ over subgroups $H \subseteq G$ forms an additive functor from the stable orbit category to abelian groups. Such a functor is known as a *Mackey functor*, which can be described purely algebraically (see [8]). Additionally, there exist *equivariant Eilenberg-Mac Lane spectra* HM for a given Mackey functor M, where $(\pi_n(HM^H))_{H \subseteq G}$ is equal to M for $n = 0$, and equal to 0 else ([20]).

There is a natural tensor product $\square$ in the category of Mackey functors (see [8, 18]). In extreme brevity, one considers the category of *spans* whose objects are finite G-sets and morphisms are isomorphism classes (with the first and last term fixed) of diagrams of finite G-sets of the form $S \leftarrow T \rightarrow U$; composition is by pullback of the middle terms. Morphisms have a natural operation of addition and multiplication, and taking the Grothendieck group with respect to addition gives an abelian category called the *Burnside category* $\mathcal{B}$, which comes with a natural pairing $\pi : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$. Now one shows (or defines) Mackey functors to be equivalent to additive functors from $\mathcal{B}$ to the category *Ab* of abelian groups. The $\square$-product is then defined as the left Kan extension of the external tensor product over π.

One can further consider commutative ring objects in the category of Mackey functors, which are called *Green functors*. (While terminology may vary, non-commutative ring objects are not considered in the present paper.) One defines Mackey modules over a Green functor A in the obvious way, i.e. as Mackey functors M with a morphism $A \square M \to M$ satisfying the associativity and unit axioms. For a Green functor A, on the other hand, HA is an E_∞-ring spectrum, and by the work of Greenlees and Shipley ([11], Section 5), the derived category of E_∞- HA-modules (see [9] for the relevant background) is equivalent to the unbounded derived category of A-Mackey modules.

Recall that for a finite group G and a normal subgroup H, the *geometric H-fixed points* $\Phi^H E$ of a G-spectrum E is the G/H-equivariant spectrum

$$(E \wedge \widetilde{E\mathcal{F}[H]})^H$$

where $\mathcal{F}[H]$ is the family of subgroups of G not containing H, $E\mathcal{F}$ is the classifying space of a family $\mathcal{F}$ (i.e. a G-CW-complex whose K-fixed points are empty if $K \notin \mathcal{F}$ and contractible if $K \in \mathcal{F}$). The symbol $\widetilde{X}$ denotes the unreduced suspension of a G-space X.

From the point of view of representation theory, for a given finite-dimensional real G-representation V, letting $S^{\infty V}$ denote the colimit of the 1-point compactifications S^{nV} of nV with respect to the inclusion $S^0 \to S^V$, one has

$$S^{\infty V} = \widetilde{E\mathcal{F}[V]}$$

where $\mathcal{F}[V]$ is the family of all subgroups $K \subseteq G$ where

$$V^K \neq 0.$$

The essential point is that, in the full generality, for any A_∞-ring spectrum R, (1) is a genuine $\mathbb{Z}/p^r$-equivariant spectrum. This was first proved by Bökstedt, Hsiang, and Madsen [4]. More modern approaches now exist, see for example [1, 17]. Therefore, it is of interest to us to study the above concepts explicitly for the groups $G = \mathbb{Z}/p^r$.

Specifically, let us recall that a $\mathbb{Z}/p^r$-Mackey functor M can be understood as a sequence of abelian groups

$$M_i, \ i = 0, \dots r,$$

where M_i is an $\frac{\mathbb{Z}/p^r}{\mathbb{Z}/p^i}$-module (the value of the functor M at isotropy $\mathbb{Z}/p^i$). Further, we are given $\mathbb{Z}/p^r$-equivariant abelian group homomorphisms

$$r = r_i : M_i \to M_{i-1}, \ c = c_i : M_i \to M_{i+1}$$

(the restriction and corestriction) such that, denoting by γ the chosen generator of the ambient group $\mathbb{Z}/p^r$, we have

$$r_{i+1} \circ c_i = 1 + \gamma^{p^{r-i-1}} + \dots + \gamma^{p^{r-i-1}(p-1)}. \tag{2}$$

The *constant Green functor* $\underline{\mathbb{Z}_p}$ is defined to have the value $\mathbb{Z}_p$ in every isotropy where the restrictions are identities and corestrictions are multiplication by the index of the subgroups concerned. (One can define, analogously, a constant $\mathbb{Z}/p^r$-Mackey functor

$$\underline{A} = \underline{A}_{\mathbb{Z}/p^r} \tag{3}$$

for any abelian group A, which is a Green functor when A has a structure of a commutative ring.) Modules over the $\mathbb{Z}/p^r$-equivariant Green functor $\underline{\mathbb{Z}_p}$ (i.e. $\underline{\mathbb{Z}_p}$-modules in the category of $\mathbb{Z}/p^r$-Mackey functors) are sequences of $\mathbb{Z}_p$-modules M_i, $i = 0, \dots, r$ which satisfy the above axioms and the additional property

$$c_i \circ r_{i+1} = p \tag{4}$$

(see e.g. [18]). Morphisms are tuples of homomorphisms of $\mathbb{Z}_p$-modules which commute with restrictions and corestrictions.

The principal projectives in the abelian category of $\underline{\mathbb{Z}_p}$-modules in the category of $\mathbb{Z}/p^r$-Mackey functors are modules denoted by

$$\underline{M}[\mathbb{Z}/p^i] \tag{5}$$

where $0 \leq i \leq r$, M is a projective $\mathbb{Z}_p$-module and (5) is defined as follows:

$$\underline{M}[\mathbb{Z}/p^i]_j = \begin{cases} M[\dfrac{\mathbb{Z}/p^r}{\mathbb{Z}/p^j}] & \text{for } r-j \leq i \\ \\ M[\dfrac{\mathbb{Z}/p^r}{\mathbb{Z}/p^{r-i}}] & \text{for } r-j > i. \end{cases}$$

The restrictions and corestrictions are given by

$$r_j(1) = 1 \text{ for } j \leq r-i$$

and

$$c_j(1) = 1 \text{ for } j \geq r-i.$$

(The remaining corestrictions and restrictions are determined.) One can think of (5) as the free $\mathbb{Z}/p^r$-equivariant $\underline{\mathbb{Z}_p}$-module freely generated by M in isotropy $\mathbb{Z}/p^{r-i}$.

3. The building blocks and the case of perfect $\mathbb{F}_p$-algebras

Let α_s be the irreducible complex representation of $\mathbb{Z}/p^s$ given by sending the generator γ to $\zeta_{p^s} = e^{2\pi i/p^s}$. Let $\alpha_{s,i} = \alpha_s^{p^i}$ for $i = 0, \dots, s$ be the p^i-th tensor power of α_s. Let, in the unbounded derived category

$$D(\underline{\mathbb{Z}}_{\mathbb{Z}/p^r}\text{-}Mod)$$

of modules over the constant $\mathbb{Z}/p^r$-Green functor $\underline{\mathbb{Z}}_{\mathbb{Z}/p^r}$ (see (3) for the notation),

$$(6) \qquad \mathcal{W}_{r,j} = (\widetilde{C}_*(S^{\infty\alpha_{s,i}}))^{\mathbb{Z}/p^{s-r}},\ i \geq 0,\ i = j - r + s - 1,$$

for $j = 0, \dots, r$, $s > r$. (The definition appears to depend on s, but we will show independence in a suitable sense in the next paragraph.) Here $\widetilde{C}_*$ denotes the ordinary reduced (say, singular) chain complex, with G-action induced by the action on the space. For every based G-space X,

$$(\widetilde{C}_*(X^H))_{H\subseteq G}$$

for a finite group G always is a chain complex of $\underline{\mathbb{Z}}$-modules in the category of G-Mackey functors, since it automatically satisfies (4) (for a more complete discussion, see [18]).

The definition (6) appears to depend on the choice of $s > r$, but by construction, $\mathcal{W}_{r,j}$ is an E_∞-algebra over $\underline{\mathbb{Z}}_{\mathbb{Z}/p^r}$, since for any finite-dimensional G-representation β, (G a finite group), $S^{\infty\beta}$ always has an obvious action of the linear isometries operad, with respect to the smash product.

We will now show that the homotopy type of the E_∞-algebra $\mathcal{W}_{r,j}$ only depends on r and j.

Let us recall that (6) are chain-level models of constructions familiar in equivariant stable homotopy theory. In our present situation, we have

$$(\alpha_{s,i})^{\mathbb{Z}/p^j} = 0 \text{ if and only if } j > i,$$

so

$$S^{\infty\alpha_{s,i}} = E\widetilde{\mathcal{F}[\mathbb{Z}/p^{i+1}]}.$$

Now we have

$$\begin{array}{l}(H\underline{\mathbb{Z}}_{\mathbb{Z}/p^s} \wedge E\widetilde{\mathcal{F}[\mathbb{Z}/p^{i+1}]})^{\mathbb{Z}/p^{s-r}} = \\ ((H\underline{\mathbb{Z}}_{\mathbb{Z}/p^s} \wedge E\widetilde{\mathcal{F}[\mathbb{Z}/p^{i+1}]})^{\mathbb{Z}/p^{s-r-1}})^{\mathbb{Z}/p} = \\ (H\underline{\mathbb{Z}}_{\mathbb{Z}/p^{r+1}} \wedge E\widetilde{\mathcal{F}[\mathbb{Z}/p^{i+1}]})^{\mathbb{Z}/p}.\end{array}$$

Evidently, the same holds with $\underline{\mathbb{Z}}$ replaced by $\underline{\mathbb{Z}}_p$.

Now recall the equivalence of categories due to Greenlees and Shipley [11] between the unbounded derived category of G-Mackey functors and E_∞-modules over the G-equivariant E_∞-ring Eilenberg-Mac Lane spectrum $H\mathcal{A}_G$ where $\mathcal{A}_G$ is the Burnside Mackey functor. In terms of this comparison, taking fixed points on the level of chain complexes and spectra is equivalent.

The reason these complexes show up, and the fact that the definition (6) does not depend on s follows from the cyclotomic structure of THH, and from the fact that

$$TR(\mathbb{F}_p) = H\underline{\mathbb{Z}}_p,$$

which is uniquely determined.

In more detail, TR is the homotopy inverse limit of $THH^{\mathbb{Z}/p^r}$ with respect to the restriction maps (see [4, 15]). As already remarked in the Introduction, this also automatically becomes a genuine $\mathbb{Z}/p^r$-equivariant spectrum by the isomorphism

$$S^1/(\mathbb{Z}/p^r) \cong S^1.$$

Further, in this sense,

$$TR(\mathbb{F}_p) = H\underline{\mathbb{Z}}_p$$

(which follows from the same calculation as the computation of

$$THH^{\mathbb{Z}/p^r}(\mathbb{F}_p),$$

see [15]). We then obtain a map from the geometric fixed points of $TR_{\mathbb{Z}/p^s}$ to the geometric fixed points of $THH_{\mathbb{Z}/p^s}$, which is an equivalence by a direct computation.

In more detail, it induces an isomorphism in homotopy groups in degree 0 by naturality and the fact that the groups on both sides are isomorphic and cyclic. Now the homotopy groups on both sides are polynomial algebras on one generator in degree 2, which, by passing to Tate cohomology, inverts the generator. Since the comparison map is by definition multiplicative, it has to multiply the generator by a unit.

Further, the geometric fixed points of THH is THH again by the cyclotomic structure. We shall return to this point in Theorem 3.

By these considerations, we also have a canonical E_∞-map

$$\pi : \mathcal{W}_{r,j} \to \mathcal{W}_{r,j'},\ 0 \leq j \leq j' \leq r.$$

Comment: One can show that these algebras cannot in general be modeled by strictly (anti-)commutative DGA's. This is because the canonical morphism to the corresponding Tate complex induces an isomorphism on homology groups in non-negative degrees. On the other hand, the Tate complex inherits Steenrod operations from the Borel cohomology complex, which computes the cohomology of a lens space. Therefore, the Tate complex has non-trivial Steenrod operations which restrict to non-trivial Dyer-Lashof operations on the geometric fixed point complex (6).

If we relax our requirements of E_∞-coherence of the multiplicative structure, we have the following explicit chain models, where γ denotes a fixed generator of $\mathbb{Z}/p^r$:

$$\mathcal{W}_{r,0} : \underline{\mathbb{Z}} \xleftarrow{p\epsilon} \underline{\mathbb{Z}}[\mathbb{Z}/p^r] \xleftarrow{1-\gamma} \underline{\mathbb{Z}}[\mathbb{Z}/p^r] \xleftarrow{pN} \underline{\mathbb{Z}}[\mathbb{Z}/p^r] \dots \tag{7}$$

where ϵ is the augmentation and

$$N = 1 + \gamma + \dots + \gamma^{p^r-1},$$

(further to the right in (7), $1-\gamma$ and pN take turns periodically), while (8)

$$\mathcal{W}_{r,j} : \underline{\mathbb{Z}} \xleftarrow{\epsilon} \underline{\mathbb{Z}}[\mathbb{Z}/p^{r-j+1}] \xleftarrow{1-\gamma} \underline{\mathbb{Z}}[\mathbb{Z}/p^{r-j+1}] \xleftarrow{N_{r-j+1}} \underline{\mathbb{Z}}[\mathbb{Z}/p^{r-j+1}] \dots$$

where

$$N_i = 1 + \gamma + \dots + \gamma^{p^i-1},$$

(and again, $1-\gamma$ and N_{r-j+1} take turns periodically). These models follow simply from the periodic equivariant CW-module of the representation sphere of an infinite sum of the same representation of a finite cyclic group.

For any $\mathbb{F}_p$-algebra A, we have a $(\underline{\mathbb{Z}}_p)_{\mathbb{Z}/p^r}$-Mackey module $\underline{W}_{r+1}(A)$ which is truncated Witt complex $W_{i+1}(A)$ at isotropy $\mathbb{Z}/p^i$ and restrictions resp. corestrictions are given by the Frobenius resp. Verschiebung. (For an introduction to these concepts, see [15].)

More generally, we also have, for $0 \leq j \leq r$, a $(\underline{\mathbb{Z}}_p)_{\mathbb{Z}/p^r}$-Mackey module $\Phi_j \underline{W}_{r+1-j}(A)$ which is $W_{i+1-j}(A)$ at isotropy $\mathbb{Z}/p^i$ for $i \geq j$ and 0 for lower isotropy, and restrictions resp. corestrictions are given by the Frobenius resp. Verschiebung.

2. **Lemma.** *The $\mathbb{Z}/p^r$-Mackey functor valued homology of the Mackey functor chain complex $\mathcal{W}_{r,j}$ is given by*

$$H_s(\mathcal{W}_{r,j}) = \begin{cases} \Phi_j \underline{W}_{r+1-j}(\mathbb{F}_p) & \text{for } s \geq 0 \text{ even} \\ 0 & \text{else.} \end{cases} \tag{9}$$

Proof. Use (7), (8). (Recall that $W_j(\mathbb{F}_p) = \mathbb{Z}/p^j$, and in associated Witt Mackey functor, restrictions are projections of rings and corestrictions are multiplications by the appropriate power of p.) $\square$

Comment: Despite the fact that the homology computed in Lemma 2 consists of $\underline{\mathbb{Z}/p^{(r-j+1)}}$-modules, it turns out that $\mathcal{W}_{r,j}$ is not in general equivalent to a chain complex in the category of $\underline{\mathbb{Z}/p^{(r-j+1)}}$-modules.

For example for $r = j = 1$, we have a $\underline{\mathbb{Z}/p}$-module

$$\mathcal{W}_{1,1} \otimes_{\mathbb{Z}} \mathbb{Z}/p. \tag{10}$$

Attempting to construct $\mathcal{W}_{1,1}$ using Bockstein obstruction theory, we get a non-zero obstruction at the $(p-1)$'st step, indicating a non-zero higher Massey power of the generator of an odd-degree cohomology group (one can also see this effect on the Tate complex, where the computation is simpler).

Nevertheless, we have the following:

3. **Theorem.** *Let R be a perfect $\mathbb{F}_p$-algebra. Then in the category of Mackey modules $D(\underline{\mathbb{Z}}_p\text{-}Mod)$, one has*

$$THH_{\mathbb{Z}/p^r}(R) = \mathcal{W}_{r,0} \otimes_{\mathbb{Z}} W(R) \tag{11}$$

(where the tensor product is calculated in each isotropy separately).

Proof. We first prove that both sides have the same homology. The proof is a complete rehash of the case of $R = \mathbb{F}_p$ treated by Hesselholt and Madsen [15]. We use induction on r. The computation for $r = 1$ is obtained from the material of [15], 5.2 by replacing $H\mathbb{F}_p$ with HR, which corresponds to tensoring the answer with $W(R)$. (There is also a more modern treatment using one instead of two spectral sequences given in [17].)

The induction step is given in 5.3 and 5.4 of [15]. In Section 5.5. of [15], it is actually remarked that the proof works over any perfect field, but only perfectness is used (not the field structure). The main idea is using the cyclotomic structure, and determining the differential in the Borel homology spectral sequence, which follows from the Tate spectral sequence. This also gives the identification of the connecting map and, by induction, the $\underline{\mathbb{Z}}_p$-module model of $THH_{\mathbb{Z}/p^r}(R)$. A more modern treatment using only the "Tate square" also exists, see [22].

Now again from the calculation of coefficients, by taking homotopy limits, it follows that, as a $\mathbb{Z}/p^s$-equivariant $H\underline{\mathbb{Z}}_p$-module,

$$TR(R) = H\underline{W(R)} \tag{12}$$

(where by the right hand side, we mean homology with coefficients in a constant Mackey functor). Now again we can argue that there is a canonical morphism

$$TR(R) \to THH_{\mathbb{Z}/p^r}(R).$$

Now apply Φ^{s-r}. Again, on coefficients, we get an isomorphism on coefficients in degree 0. Further, both sides are isomorphic polynomial algebras on a generator in dimension 2, which is inverted by passing to Tate cohomology. The comparison map is multiplicative, and hence has to be an isomorphism. □

It is also useful to record the morphisms between the objects $\mathcal{W}_{r,j}$ of the (unbounded) derived category of $\mathcal{W}_{r,0}$-modules, which can be thought of as a partial "rigidity" statement:

4. **Lemma.** *The graded module of morphisms from $\mathcal{W}_{r,i}$ to $\mathcal{W}_{r,j}$ in the derived category $D\mathcal{W}_{r,0}$ of $\mathcal{W}_{r,0}$-E_∞-modules is isomorphic to the $\mathbb{Z}/p^r$ isotropy part of the graded Mackey functor $H_*\mathcal{W}_{r,j}$ when $i \leq j$ and is* 0 *else.*

Proof. Recall the $\mathbb{Z}/p^s$-equivariant based CW-complexes

$$\Phi_{s,i} = \widetilde{E\mathcal{F}[\mathbb{Z}/p^i]},$$

$0 < i \leq s$, which are equivalent to S^0 in isotropies $\mathbb{Z}/p^j$ for $j \geq i$ and contractible in smaller isotropies. Then the suspension spectra $\Sigma^\infty \Phi_{s,j}$ are E_∞-ring spectra, where

$$\Phi_{s,i} \wedge \Phi_{s,j} \sim \Phi_{s,\max(i,j)}. \tag{13}$$

(The E_∞-ring structure follows from an E_∞-structure on $\Phi_{s,i}$ with respect to the smash product, where the operad structure can be constructed from the infinite representation sphere model, or directly using obstruction theory.)

The groups in question can be expressed as derived $\mathbb{Z}/p^r$-equivariant maps between $\mathcal{W}_{r,0}$-modules of the form $\mathcal{W}_{r,0} \wedge \Phi_{s,i}$ for different i. (Again, we may pass fluidly between chain-level and spectral models using the results of Greenlees and Shipley [11].) Thus, the positive part of our statement (the case of $i \leq j$) follows from (13).

For the case $i > j$, consider a map of $\mathcal{W}_{r,0}$-modules

$$\Sigma^n \mathcal{W}_{r,i} \to \mathcal{W}_{r,j}. \tag{14}$$

Composing with the n-suspension of the projection

$$\mathcal{W}_{r,j} \to \mathcal{W}_{r,i}, \tag{15}$$

we obtain a map of $\mathcal{W}_{r,0}$-modules

$$\Sigma^n \mathcal{W}_{r,j} \to \mathcal{W}_{r,j}, \tag{16}$$

which we already classified. In particular, we claim that the composition (16) is 0. Otherwise, it would be injective on homotopy groups in all isotropies (by the part $i \leq j$ of our statement, which we already proved, applied to $i = j$). However, we also know that the projection (15) is 0 on homotopy groups in degrees $n \gg 0$.

Thus (16) is 0. Thus, the map (14) factors through the cofiber C of the projection (15). However, there are no non-zero morphisms of $H\underline{\mathbb{Z}}$-modules

$$\Sigma^n C \to \mathcal{W}_{r,j}, \tag{17}$$

since C has a $H\underline{\mathbb{Z}}$-module cellular structure where cells are principal projectives with generators in isotropies in which $\mathcal{W}_{r,j}$ has 0 homotopy (see (8)).

However, then there are also no non-zero morphisms (17) of $\mathcal{W}_{r,0}$-modules, since

$$C \wedge_{H\underline{\mathbb{Z}}} \mathcal{W}_{r,0} \sim C,$$

(which follows from

$$\mathcal{W}_{r,0} \wedge_{H\underline{\mathbb{Z}}} \mathcal{W}_{r,0} \sim \mathcal{W}_{r,0},$$

which, again, holds by (13).) $\square$

4. The case of a semiperfect $\mathbb{F}_p$-algebra

We now treat the case of a quasiregular semiperfect $\mathbb{F}_p$-algebra R. We will give the definition of this concept, which is somewhat technical, following [3], but for our purposes, we can focus on the case

$$R = C \otimes_A C \otimes_A \cdots \otimes_A C \tag{18}$$

where A is a smooth $\mathbb{F}_p$-algebra and C is its colimit perfection (i.e. colimit of a sequence of homomorphisms, each of which is the Frobenius).

The reason for this is that if A is a smooth $\mathbb{F}_p$-algebra, we can consider cosimplicial descent from A to (18). A crucial fact used in [3] is that this preserves (1). In other words, forming a cosimplicial object (the Godement "standard construction") from the levels (18) and then applying TR^r is equivalent to applying TR^r to (18) and then taking the realization of the Godement standard construction. This is the key ingredient of the crystalline descent method of [3], which, for the purpose of this paper, we can treat as a black box. Therefore, to give a $\underline{\mathbb{Z}}_p$-module chain model of $THH(A)$ for a smooth $\mathbb{F}_p$-algebra A, it suffices to give it, in a functorial way, for quasiregular semiperfect algebras, specifically those of the form (18).

Denote by

$$S_R = \lim(\dots \xrightarrow{\phi} R \xrightarrow{\phi} R)$$

the limit perfection of R, where $\phi : R \to R$ is the Frobenius.

Denote by J the kernel of the projection $S_R \to R$ and denote also

$$I = Ker(\phi : R \to R).$$

One has

$$J/J^2 \cong I/I^2,$$

since $\phi(I) \subseteq I^2$. On the element level, I consists of elements $u \in R$ such that $u^p = 0$. Since we can take p'th roots by semiperfectness, the projection $J \to I$ is surjective, and hence the induced projection $\pi : J/J^2 \to I/I^2$ is surjective. Now suppose an element is in the kernel. We can represent any element of J as a sequence

$$(\dots, v_2, v_1),\ v_i^p = v_{i-1} \tag{19}$$

where v_1 is its projection in I. In particular, $v_1^p = 0$. Since every element of I^2 lifts to an element of J^2, an element of $Ker(\pi)$ can be represented by an element of the form (19) with $v_1 = 0$. But such an element can be expressed as the product

$$(\dots, v_3^{p-1}, v_2^{p-1}) \cdot (\dots, v_3, v_2),$$

and thus is in J^2.

Using the notation $L_{B/A}$ of Quillen [24] for the derived cotangent complex for a (non-derived) commutative A-algebra B, we have

$$L_{S_R/\mathbb{F}_p} = 0,$$

which, by Theorem 5.1 of [24], gives

$$L_{R/\mathbb{F}_p} = L_{R/S_R}.$$

Then Theorem 6.3 of [24] gives

$$H_1 L_{R/S_R} = Tor_1^{S_R}(R, R) = J/J^2,$$

which gives

$$H_1 L_{R/\mathbb{F}_p} = I/I^2. \tag{20}$$

So far, we only used the fact that R is a semiperfect $\mathbb{F}_p$-agebra. Now by definition ([3], Definition 8.8), a semiperfect $\mathbb{F}_p$-algebra R is called *quasiregular* when I/I^2 is a flat R-module and

$$H_n L_{R/\mathbb{F}_p} = 0 \text{ for } n \neq 1.$$

In our situation of interest, we have a somewhat stronger property. Call a quasiregular semiperfect $\mathbb{F}_p$-algebra R *quasismooth* when J/J^2 is a free R-module of finite rank and

$$Sym_R^\ell(J/J^2) = J^\ell/J^{\ell+1}. \tag{21}$$

5. **Lemma.** *(see also* [3, 16]*) Let A be a smooth $\mathbb{F}_p$-algebra and let P be its colimit perfection. Then*

$$R = P \otimes_A \cdots \otimes_A P \tag{22}$$

is a quasismooth semiperfect $\mathbb{F}_p$-algebra.

Proof. First of all, it is well known (and easy to show) that P is perfect. A tensor product over $\mathbb{F}_p$ of two perfect $\mathbb{F}_p$-algebras is perfect (since $\phi\otimes\phi$ is a composition of the injective maps $\phi\otimes 1$ and $1\otimes\phi$; this proves injectivity of the Frobenius, while surjectivity is trivial). Thus,

$$P \otimes_{\mathbb{F}_p} \cdots \otimes_{\mathbb{F}_p} P \tag{23}$$

is perfect. Thus, (22) is semiperfect.

To prove that (22) is quasiregular, first note that (22) is a colimit of a sequence of locally complete intersections, which implies the vanishing of its Quillen homology in degrees > 1. On the other hand, 0'th Quillen homology always vanishes for semiperfect $\mathbb{F}_p$-algebra.

For the flatness, note that in the arguments at the beginning of this section, S_R could be replaced by any perfect $\mathbb{F}_p$-algebra which has R as a quotient. In particular, denoting by Q the kernel of the projection from (23) to (22), we have

$$I/I^2 = Q/Q^2.$$

Now once again, we are in the situation of a limit of a sequence of regular $\mathbb{F}_p$-algebras, for which the flatness holds. Thus, it holds in the colimit. $\square$

Now let R be a quasismooth semiperfect $\mathbb{F}_p$-algebra, and let J/J^2 be as above. Assume that it is a free R-module on a finite basis B

$$J/J^2 = \bigoplus_B R. \tag{24}$$

We shall write

$$W_r^{B_\ell}(R) = \bigoplus_{B_\ell} W_r(R) \tag{25}$$

where B_ℓ is the set of unordered ℓ-tuples (with possible repeats) of elements of B (i.e. the quotient of the Cartesian product by the symmetric group Σ_ℓ acting by permutation of coordinates).

We can write this construction more functorially and more generally as follows. Recalling the notion of Witt vectors with coefficients in a module [6, 7, 25], we have

$$W_r^{B_\ell}(R) \cong W_r(S_R, J^\ell)/W_r(S_R, J^{\ell+1}). \tag{26}$$

(Recall that the elements of $W_r(S_R/J^\ell)$ are tuples

$$(a_0, \dots, a_\ell)$$

where $a_i \in (J^\ell)^{p^i}$, with the usual Witt vector addition and - in this case scalar - multiplication.)

We see that the right-hand side does not depend on the choice of a basis, and, in fact, makes sense for every quasiregular semiperfect $\mathbb{F}_p$-algebra R. In that generality, this motivates in fact writing

$$\mathcal{W}_{r,j}(R,\ell) := W(S_R, J^\ell)/W(S_R, J^{\ell+1}) \otimes_{\mathbb{Z}} \mathcal{W}_{r,j}.$$

Thus, we may also write

$$\mathcal{W}_r(R,\ell) := \operatorname{holim} \mathcal{D}_r(R,\ell),$$

where $\mathcal{D}_r(R,\ell)$ is the diagram

$$\begin{tikzcd}
 & & & \mathcal{W}_{r,r}(R,\ell) \arrow[d, "\phi"] \\
 & & \mathcal{W}_{r,r-1}(R,\ell) \arrow[r, "\pi"] \arrow[d, "\phi"] & \mathcal{W}_{r,r}(R,\ell) \\
 & \dots \arrow[r, "\pi"] \arrow[d, "\phi"] & \mathcal{W}_{r,r-1}(R,\ell) & \\
\mathcal{W}_{r,0}(R,\ell) \arrow[r, "\pi"] & \dots & &
\end{tikzcd}$$

where ϕ is the map induced by the p-th power (i.e. Frobenius) on R. (Compare with analogous constructions in [22].)

6. **Theorem.** *For a quasiregular semiperfect $\mathbb{F}_p$-algera R, one has*

(27)

$$THH^{\mathbb{Z}/p^r}(R)_{2\ell} = \bigoplus_{i=0}^{\ell} W_{r+1}(S_R, J^i)/W_{r+1}(S_R, J^{i+1}) \otimes_{W_{r+1}(S_R)} W_{r+1}(R),$$

$$THH^{\mathbb{Z}/p^r}(R)_{2\ell+1} = 0.$$

Further, in the category $D\underline{\mathbb{Z}}_p$-Mod, one has

$$THH_{\mathbb{Z}/p^r}(R) = \bigoplus_{\ell \geq 0} \mathcal{W}_r(R,\ell)[2\ell]. \tag{28}$$

It is useful to point out in particular the following:

7. **Corollary.** *If R is a quasiregular semiperfect $\mathbb{F}_p$-algebra, then*

$$THH_{\mathbb{Z}/p^r}(R)$$

is equivalent to a direct sum of even suspensions of the $\underline{\mathbb{Z}}_p$-module complexes $\mathcal{W}_{r,i}$.

Proof of Theorem 6. We first consider the case $r = 0$. Apply [24], Theorem 5.1, to the sequence

$$\mathbb{F}_p \to R \otimes R \to R.$$

This gives a "Mayer-Vietoris" exact triangle

$$J/J^2 \oplus J/J^2 \to J/J^2 \to L_{R/R\otimes R} \to (J/J^2 \oplus J/J^2)[1]$$

where the first map is the codiagonal. Thus, we obtain

$$L_{R/R\otimes R} = J/J^2[2].$$

By [24], Theorem 6.3, we then have

$$Tor_{2\ell}^{R\otimes R}(R, R) = Sym_R^{\ell} J/J^2 = J^{\ell}/J^{\ell+1},$$
$$Tor_{2\ell+1}^{R\otimes R}(R, R) = 0.$$

Now following [15], we have a spectral sequence

$$Tor_*^{A_*\otimes R\otimes R}(R, R) \Rightarrow THH_*(R)$$

where A_* is the dual Steenrod algebra which collapses for $p = 2$ and, using the standard differentials of [15] for $p > 2$, implies (27) (and hence (28)) for $r = 0$.

For $r \geq 1$, one then also repeats the method of [15], considering the Borel homology spectral sequence, which has the differential induced from the case $R = \mathbb{F}_p$ and consequently collapses to even degrees. Extensions are given by multiplication by p composed with the map induced by the Frobenius on R.

One then again gets the total fixed points by using the fundamental cofibration

$$E\mathbb{Z}/p^r_+ \wedge E \to E \to \widetilde{E\mathcal{F}[\mathbb{Z}/p]} \wedge E$$

for a $\mathbb{Z}/p^r$-equivariant spectrum E, and identifying the last term by induction using the cyclotomic structure. The identification of the connecting map also mimics the perfect case, thus giving the result as well as the decomposition (28).

The induction described is sufficient to prove the statement of Corollary 7, and to count the number of copies of the complexes $\mathcal{W}_{r,i}$. However, we stated our answer functorially in R, and the functoriality must be right in a coherent sense to be used for descent. For this purpose, we invoke Lemma 4. Recall that, similarly as the $\mathbb{Z}/p^r$-equivariant spectrum $TR(R)$, we also have a similarly defined $\mathbb{Z}/p^r$-equivariant spectrum $TF(R)$ where the homotopy limit is taken with respect to the Frobenius. This, in our case, translates to maps induced by the algebraic Frobenius. Thus, we observe that in the limit, we have

$$TF_{\mathbb{Z}/p^r}(R) \sim \bigoplus_{\ell \geq 0} \underline{W(S_R, J^\ell/J^{\ell+1})}[2\ell]. \tag{29}$$

Since $W(S_R, J^\ell/J^{\ell+1})$ are free $\mathbb{Z}_p$-modules (since we are at the infinite level, the whole $W(S_R)$ is a free $\mathbb{Z}_p$-module, as is the case for any perfect $\mathbb{F}_p$-algebra), there are no higher derived maps between the $\underline{\mathbb{Z}}_p$-modules (29). Now let $R \to R'$ be a homomorphism of quasiregular semiperfect $\mathbb{F}_p$-algebras. Then the functoriality we describe fits into the diagram of $H\underline{\mathbb{Z}}_p$-modules

$$\begin{tikzcd} TF_{\mathbb{Z}/p^r}(R) \arrow[r] \arrow[d] & TF_{\mathbb{Z}/p^r}(R') \arrow[d] \\ THH_{\mathbb{Z}/p^r}(R) \arrow[r] & THH_{\mathbb{Z}/p^r}(R') \end{tikzcd} \tag{30}$$

where the horizontal arrows are given by our functoriality, while the vertical arrows are the natural projections. We may extend scalars over $H\underline{\mathbb{Z}}_p = TF_{\mathbb{Z}/p^r}(\mathbb{F}_p)$ to $THH_{\mathbb{Z}/p^r}(\mathbb{F}_p)$ to obtain a diagram of $\mathcal{W}_{r,0}$-modules

$$\begin{tikzcd} TF_{\mathbb{Z}/p^r}(R) \otimes_{\underline{\mathbb{Z}}_p} \mathcal{W}_{r,0} \arrow[r] \arrow[d] & TF_{\mathbb{Z}/p^r}(R') \otimes_{\underline{\mathbb{Z}}_p} \mathcal{W}_{r,0} \arrow[d] \\ THH_{\mathbb{Z}/p^r}(R) \arrow[r] & THH_{\mathbb{Z}/p^r}(R'). \end{tikzcd} \tag{31}$$

By Lemma 4, however, the bottom horizontal arrow (30) is uniquely determined (in the A_∞-sense) by the remaining arrows. This completes the proof of our functoriality statement. $\square$

Comment: Theorem 6 can be thought of as specifying an A_∞-sheaf on the crystalline site of $\mathbb{F}_p$-algebras considered in [3]. Therefore, we can use faithfully flat quasiregular semiperfect descent to obtain Mackey chain models for $THH_{\mathbb{Z}/p^r}(R)$ for a smooth $\mathbb{F}_p$-algebra R.

The "rigidity" of Lemma 4, however, does not imply that, in the case of smooth $\mathbb{F}_p$-algebras, we would get again sums of copies of the complexes $\mathcal{W}_{r,i}$. In fact, in the case of the polynomial algebra $\mathbb{F}_p[x]$, it is known ([10, 13, 14, 16]) that one obtains summands of finite $RO(\mathbb{Z}/p^r)$-graded suspensions of the complexes $\mathcal{W}_{r,i}$, which are non-trivial extensions of the complexes $\mathcal{W}_{r,i}$. In the case of quasiregular semiperfect algebras, this behavior is ruled out in part by the fact that the chain homology is contained in even degrees.

However, it is worth noting that by generalizing the Illusie complex to tensor algebras, Hesselholt [13] was able to describe TC of an $\mathbb{F}_p$-algebra as a derived functor. Hahn, Raksit, and Wilson [12] recently used a similar alternative to quasiregular semiperfect descent to define a version of the motivic filtration on commutative ring spectra.

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