MU*MU AND BP*BP DO NOT ADMIT COEFFICIENT PRESERVING ANTIAUTOMORPHISMS OF RINGS

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This note establishes a conjecture posed by Boardman in [Boa82]: The rings of stable operations of **MU** and **BP** do not admit coefficient preserving antiautomorphisms.

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1. The conjecture

Definition 1.1. Let R be an associative ring. An antiautomorphism of rings of R is a bijective map $\sigma : R \to R$ which satisfies the following properties:

(1) It preserves the unit element: $\sigma(1_R) = 1_R$.
(2) It preserves sums: $\sigma(a + b) = \sigma(a) + \sigma(b)$, for all $a, b \in R$.
(3) It swaps the order of products: $\sigma(ab) = \sigma(b)\sigma(a)$, for all $a, b \in R$.

In [Boa82] Boardman formulates the following conjecture about the complex cobordism spectrum **MU** and Brown-Peterson spectra **BP**.

Conjecture 1.2. For $E = \mathbf{MU}$ or $\mathbf{BP}$, we conjecture that there exists no antiautomorphism of the ring E^*E that preserves E^*. We have no proof at this time, mainly for lack of interest. Note that these groups E^*E are large and the statement is algebraic; we are not restricting to continuous antiautomorphisms, whatever they might be.

Our goal in this note is to show that Boardman's intuition was correct.

Theorem 1.3 (3.1)**.** *The Conjecture 1.2 is true.*

2. The supplies

We will need the following notion of finiteness.

Notation 2.1. Going forward, A will denote a commutative ring and M, N will denote A-modules. We denote by $\mathrm{D}_A(M)$ the dual A-module of M

$$\mathrm{D}_A(M) = \underline{\mathrm{Hom}}_A(M, A).$$

For an A-algebra Γ, the dual $\mathrm{D}_A(\Gamma)$ will be equipped with the natural A-algebra structure. When the ring and modules above are graded, we will use the notation $\mathrm{D}_A^{\mathrm{gr}}$ and $\underline{\mathrm{Hom}}_A^{\mathrm{gr}}$ to mean the respective graded modules of graded morphisms.

Definition 2.2. We call a sequence of morphisms $(x_n)_{n\in\mathbf{N}} \in \underline{\mathrm{Hom}}_A(M, N)^{\mathbf{N}}$ *pointwise finite*, if it satisfies the following property:

- for each $m \in M$, there is only a finite number of $n \in \mathbf{N}$ such that $x_n(m) \neq 0$.

The set of pointwise finite sequences forms a left A-submodule of $\underline{\mathrm{Hom}}_A(M, N)^{\mathbf{N}}$. We will denote this *submodule of finite sequences* by $\underline{\mathrm{Hom}}_A(M, N)^{\mathbf{N},\mathrm{pf}}$.

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Remark 2.3. A sequence of morphisms $(x_n)_{n\in\mathbf{N}} \in \underline{\mathrm{Hom}}_A(M,N)^{\mathbf{N}}$ is pointwise finite if and only if evaluation induces a morphism of left A-modules $M \to \oplus_{\mathbf{N}} N$. It follows that we have a canonical isomorphism of left A-modules

$$\underline{\mathrm{Hom}}_A(M,N)^{\mathbf{N},\mathrm{pf}} \xrightarrow{\simeq} \underline{\mathrm{Hom}}_A(M, \oplus_{\mathbf{N}} N).$$

Remark 2.4. A sequence of morphisms $(x_n)_{n\in\mathbf{N}} \in \underline{\mathrm{Hom}}_A(M,N)^{\mathbf{N}}$ is pointwise finite if and only if its image under the forgetful map $\underline{\mathrm{Hom}}_A(M,N)^{\mathbf{N}} \to \underline{\mathrm{Hom}}_{\mathbf{Z}}(M,N)^{\mathbf{N}}$ is.

Example 2.5. Let $\mathbf{MU}$ be the complex cobordism spectrum. Recall [Boa82, §8]

$$\begin{aligned} \mathbf{MU}_* &\simeq \mathbf{Z}[x_1, x_2, \cdots] \qquad |x_i| = 2i, \\ \mathbf{MU}_*\mathbf{MU} &\simeq \mathbf{MU}_*[b_1, b_2, \cdots] \qquad |b_i| = 2i, \\ \mathbf{MU}^*\mathbf{MU} &\simeq \mathrm{D}^{\mathrm{gr}}_{\mathbf{MU}_*}(\mathbf{MU}_*\mathbf{MU}). \end{aligned}$$

Let $B_n \in \mathbf{MU}^*\mathbf{MU}$ be the dual of b_1^n, for $n \in \mathbf{N}$. Then the sequence $(x_1^n B_n)_{n\in\mathbf{N}}$ is clearly pointwise finite. On the other hand, the sequence $(B_n x_1^n)_{n\in\mathbf{N}}$ is not pointwise finite. Indeed, by [Boa82, 8.6], we may choose generators x_i of $\mathbf{MU}_*$ such that $\eta_R(x_1) = x_1 + 2b_1$, and thus

$$(B_n x_1^n)(1) = \langle B_n x_1^n, 1\rangle = \langle B_n, \eta_R(x_1)^n\rangle = \langle B_n, (x_1+2b_1)^n\rangle = \langle B_n, 2^n b_1^n\rangle = 2^n \neq 0,$$

for all $n \in \mathbf{N}$.

Example 2.6. Fix a prime p and let $\mathbf{BP}$ be the p-local Brown-Peterson spectrum. Recall [Boa82, §9]

$$\begin{aligned} \mathbf{BP}_* &\simeq \mathbf{Z}_{(p)}[v_1, v_2, \cdots] \qquad |v_i| = 2(p^i-1), \\ \mathbf{BP}_*\mathbf{BP} &\simeq \mathbf{BP}_*[t_1, t_2, \cdots] \qquad |t_i| = 2(p^i-1), \\ \mathbf{BP}^*\mathbf{BP} &\simeq \mathrm{D}^{\mathrm{gr}}_{\mathbf{BP}_*}(\mathbf{BP}_*\mathbf{BP}). \end{aligned}$$

Let $T_n \in \mathbf{BP}^*\mathbf{BP}$ be the dual of t_1^n, for $n \in \mathbf{N}$. Then the sequence $(v_1^n T_n)_{n\in\mathbf{N}}$ is readily seen to be pointwise finite. On the other hand, the sequence $(T_n v_1^n)_{n\in\mathbf{N}}$ is not pointwise finite. Indeed, with Hazewinkel generators we have $\eta_R(v_1) = v_1 + pt_1$. By the same argument as in Example 2.5, it follows

$$(T_n v_1^n)(1) = p^n \neq 0,$$

for all $n \in \mathbf{N}$.

To keep track of pointwise finiteness, we will work with the notion of *slender* groups.

Definition 2.7. Let S be an abelian group. We say S is *slender* if the natural map

$$\mathrm{Hom}_{\mathbf{Z}}(M^{\mathbf{N}}, S) \to \oplus_{\mathbf{N}} \mathrm{Hom}_{\mathbf{Z}}(M, S)$$

is a bijection, for every abelian group M.

We have the following examples; see [Fuc15, 13.2.5].

Example 2.8. The abelian group of integers $\mathbf{Z}$ is slender.

Example 2.9. Let p be a prime number. The abelian group of (p)-local integers $\mathbf{Z}_{(p)}$ is slender.

Example 2.10. Direct sums of slender groups are slender.

Let us denote by S a slender group and by $s : N \to S$ a homomorphism of groups. Pointwise finite sequences have the following finiteness preservation property.

Proposition 2.11. *Let $(x_n)_{n\in\mathbf{N}} \in \underline{\mathrm{Hom}}_A(M,N)^{\mathbf{N}}$ be pointwise finite, and let $\sigma : \underline{\mathrm{Hom}}_A(M,N) \to \underline{\mathrm{Hom}}_A(M,N)$ be an endomorphism of groups. Then $((s_* \circ \sigma)(x_n))_{n\in\mathbf{N}} \in \underline{\mathrm{Hom}}_{\mathbf{Z}}(M,S)^{\mathbf{N}}$ is pointwise finite.*

Proof. Let $(x_n)_{n\in\mathbf{N}} \in \underline{\mathrm{Hom}}_A(M,N)$ be pointwise finite and let $m \in M$ be fixed. Then the composition

$$\mathbf{Z}^{\mathbf{N}} \xrightarrow{\sum_n x_n(-)_n} \underline{\mathrm{Hom}}_A(M,N) \xrightarrow{\sigma} \underline{\mathrm{Hom}}_A(M,N) \xrightarrow{\mathrm{ev}_m} N \xrightarrow{s} S$$

is an element of

$$\underline{\mathrm{Hom}}_{\mathbf{Z}}(\mathbf{Z}^{\mathbf{N}}, S) \simeq \oplus_{\mathbf{N}} \underline{\mathrm{Hom}}_{\mathbf{Z}}(\mathbf{Z}, S) \simeq \oplus_{\mathbf{N}} S.$$

Therefore, $((s_* \circ \sigma)(x_n a_n))_{n\in\mathbf{N}}$ is pointwise finite for every sequence $(a_n)_{n\in\mathbf{N}} \in \mathbf{Z}^{\mathbf{N}}$. The claim follows. $\square$

Remark 2.12. An equivalent formulation of Proposition 2.11 is: The map of abelian groups $(s_* \circ \sigma)^{\mathbf{N}} : \underline{\mathrm{Hom}}_A(M,N)^{\mathbf{N}} \to \underline{\mathrm{Hom}}_{\mathbf{Z}}(M,S)^{\mathbf{N}}$ restricts to a map on the corresponding submodules of pointwise finite sequences. If the underlying group of N is slender, then Proposition 2.11 together with Remark 2.4 imply that any group endomorphism of $\underline{\mathrm{Hom}}_A(M,N)^{\mathbf{N}}$ preserves pointwise finite sequences.

Remark 2.13. The proof of Proposition 2.11 also applies in the graded setting under the additional hypothesis that pointwise finite sequences only have components in finitely many degrees.

3. The truth

We now come to the main theorem. We will reduce the failure of existence of antiautomorphisms of rings to the failure of preservation of pointwise finite sequences. This failure is essentially the phenomenon observed in Example 2.5 and Example 2.6:

Reversing the order of products does generally not preserve pointwise finite sequences.

Theorem 3.1. *Let E be the complex cobordism spectrum* **MU** *or the Brown-Peterson spectrum* **BP***. Then the ring E^*E does not admit an antiautomorphism of rings $\sigma : E^*E \to E^*E$ satisfying $\sigma(E^*) = E^*$.*

Proof. Let $B_n \in \mathbf{MU}^*\mathbf{MU}$ be the dual of b_1^n, for $n \in \mathbf{N}$. Then the sequence $(x_1^n B_n)_{n\in\mathbf{N}}$ is pointwise finite by Example 2.5. Since $\eta_R(x_1) = x_1 + 2b_1$, we have

$$B_n x_1 = x_1 B_n + 2B_{n-1}. \tag{3.2}$$

Assume by contradiction that σ does exist. An application of σ to (3.2) implies the equation

$$\sigma(x_1)\sigma(B_n) = \sigma(B_n)\sigma(x_1) + 2\sigma(B_{n-1}).$$

In particular, induction shows

$$\sigma(B_n)\sigma(x_1^n) \equiv (-2)^n \mod \sigma(x_1)\mathbf{MU}^*\mathbf{MU}.$$

Let $a : \mathbf{MU}_* \to \mathbf{Z}$ be the canonical augmentation, and define $s = a \circ (\sigma|_{MU_*})^{-1} : \mathbf{MU}_* \to \mathbf{Z}$. Then the above shows $(s_* \circ \sigma)(x_1^n B_n)(1) = (-2)^n \neq 0$, which contradicts Remark 2.13. An analogous argument applies to **BP** with the sequence $(v_1^n T_n)_{n\in\mathbf{N}}$ from Example 2.6. ☐

Remark 3.3. An analogous argument applies to the motivic spectra **MGL** and **BP** over any base field.

References

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