Finding Sub-Chromotopologies

Ethan Ross[1], Aaron Bagheri[2], Naba Mukhtar[3], Theo Van Den Hurk[4]

[1]University of Alberta
[2]Harvey Mudd College
[3]Bucknell University
[4]University of Toronto

Introduction

Motivated by the fruitful discussion on the category-theoretic properties of Adinkras, our group thought we would analyze the problem of enumerating sub-objects. We decided to start by enumerating sub-chromotopologies first, since that seemed to be a reasonable first step. Thus, our main objective is, given any (N, k)-chromotopology, to enumerate every sub-(N', k')-chromotopology, where $N' \leq N$ and $N' - k' \leq N - k$.

Definitions

Cayley Graph: Given a group H and a set of generators $S \subset H$, we define the (right)Cayley graph of H and S to be the graph $\mathfrak{G}(H, S)$ whose vertex set is H and vertices $v_1, v_2 \in H$ are connected if and only if there exists $s \in S$ such that $v_1 s = v_2$.

We know that all chromotopologies come from quotients of Hamming cubes by doubly even error correcting codes, and so we can state what chromotopologies are neatly as follows. Given $\mathbb{F}_2^N$ and a doubly even code $\mathcal{C} \subset \mathbb{F}_2^N$ with dim $\mathcal{C} = k$, let $\pi : \mathbb{F}_2^N \to \mathbb{F}_2^N/\mathcal{C}$ be the canonical quotient map and $S_N = \{e_1, \ldots, e_N\}$ be the set of standard basis vectors of $\mathbb{F}_2^N$.

(N, k)-Chromotopology: A (N, k)-chromotopology is the Cayley graph $\mathfrak{G}(\mathbb{F}_2^N/\mathcal{C}, \pi(S_N))$.

Sub (N, k)-Chromotopology: An sub-(N', k')-chromotopology of an (N, k)-chromotopology, G, is any subgraph of G that is also an (N', k')-chromotopology.

Example

Our method for counting sub-chromotopologies is as follows: Erase however many colors you would like, count the number of disconnected components, then do this for every possible distinct combination of colors. This works, since if we have any sub-chromotopology obtained from any subset of the colors, it will be one of the connected components described. We illustrate this method with the (3,0)-chromotopology.

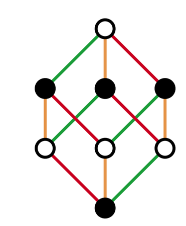

Figure 1: Full $(3, 0)$-Chromotopology

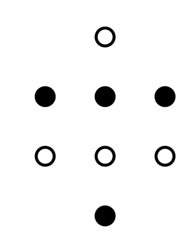

Figure 2: All sub-(0,0)-chromotopologies

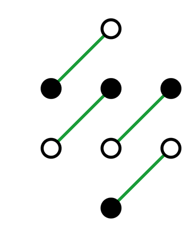

Figure 3: sub-(1,0)-chromotopologies obtained from erasing all colors except green.

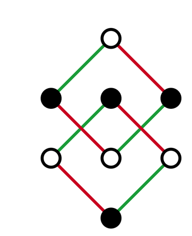

Figure 4: sub-(2,0)-chromotopologies colored in green and red.

Erasing Colors Returns Chromotopologies

An easy to check fact, is that, given any (N, k)-chromotopology, by erasing any set of colors, we are left with a disjoint union of chromotopologies. By counting the number of vertices in each connected component, we can determine which chromotopologies we are left with.

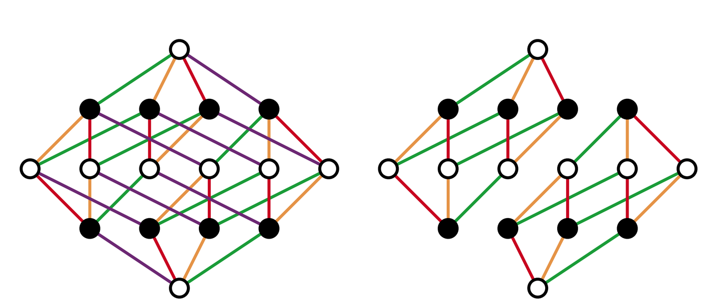

Figure 5: Erasing purple gives two disjoint cubes.

Counting Sub-Cubes

Let $\mathcal{A}_N$ be any $(N, 0)$-chromotopology, with $N \geq 1$. In order to show that all the sub-chromotopologies are of the form $\mathcal{A}_{N'}$, $N' \leq N$, it will suffice to only check that all sub-$(N-1, k)$-chromotopologies are of the form $k = 0$. Since this obviously holds for $\mathcal{A}_1$ this provides a proof by induction. Once we have this, we can begin to count how many distinct copies of $\mathcal{A}_{N'}$ are contained in $\mathcal{A}_N$.

In the proof stated above, it becomes apparent that for any $\mathcal{A}_N$, by erasing one color, we obtain two disjoint copies of $\mathcal{A}_{N-1}$. So by cycling through all N colors in this fashion, we obtain that there are $2N$ distinct copies of $\mathcal{A}_{N-1}$.

Counting Sub-Cubes Continued

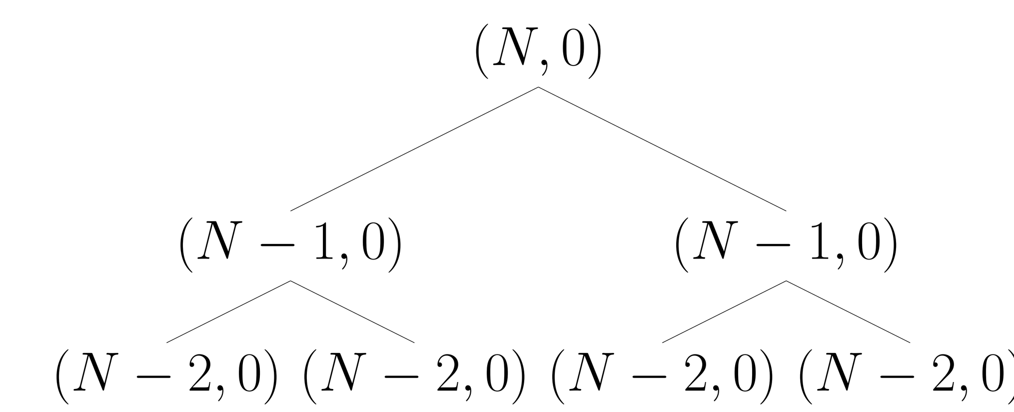

Figure 6: The branching behavior of sub-chromotopologies of cubes

We can thus see that every $\mathcal{A}_{N-1}$ obtained by erasing a single color will itself have $2(N-1)$ copies of $\mathcal{A}_{N-2}$. Thus, summing over every single possible one we get $4N(N-1)$ copies of $\mathcal{A}_{N-2}$ in $\mathcal{A}_N$. In general, we get for any $N' \leq N$, there are

$$2^{N-N'}\binom{N}{N'}$$

distinct copies of $\mathcal{A}_{N'}$

Conclusion

We completely solved the problem of finding sub-chromotopologies of the N-cubes, but they are the easiest case. In non-cubical examples, we saw that it really depends on which code you quotient by to determine what happens when you erase a color. However, just as in our case, knowing the number of connected components really helps with the calculation. And so, it would be worthwhile to explore the homological properties of subgraphs of Adinkras.

References

[1] Delfosse, N., et. al. "A Construction of Quantum LDPC Codes from Cayley Graphs" arXiv:1206.2656v3