Counting Odd Dashings of Cubes

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Preliminaries

An *odd dashing* on a graph is a labelling of the edges as “solid” and “dashed” such that every 4-cycle includes an odd number of dashed edges.

Given an odd dashing on a graph, a *node flip* changes the label of all edges connected to a particular node. Since a node touches an even number of edges on each face, performing a node flip on an odd dashing results in another odd dashing.

We consider two odd dashings to be equivalent if one can be obtained from the other by a sequence of node flips.

The question we address is finding the number of inequivalent odd dashings of cubes in n dimensions.

Dimensions 1 & 2

Small dimensions may be done by hand. There is a single edge in a 1-dimensional cube, and a node flip changes the label, so clearly there is one odd dashing of the 1-cube up to equivalence.

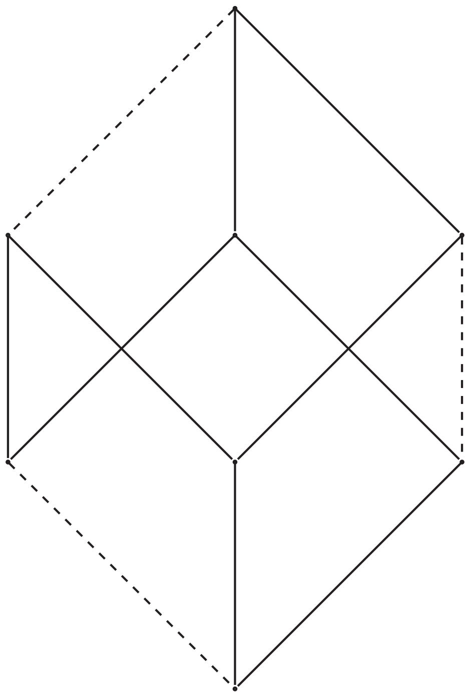
Similarly, the 2-cube is a single 4-cycle, and has 8 odd dashings (4 with three dashed edges and 4 with one), which can all be seen to be equivalent under node flips.

Dimension 3

Consider an odd dashing of the 3-cube. Since each of the six faces must have at least one dashed edge, and each edge touches two faces, there must be at least three dashed edges in an odd dashing.

If an odd dashing of the 3-cube has five (or more) dashed edges, then ten (or more) vertices (with multiplicity) touch a dashed edge. A 3-cube has eight vertices, so by the pigeonhole principle there must be a node touching more than one dashed edge. If an odd dashing has four dashed edges, then eight faces (with multiplicity) touch a dashed edge. A 3-cube has six faces, so by the pigeonhole principle there must be a face with three dashed edges, and again some node touches more than one dashed edge.

If two dashed edges meet at a node, then since each node touches three edges, we can reduce the number of dashed edges by flipping that node. Thus any odd dashing of the 3-cube may be reduced to an odd dashing with exactly three dashed edges.



Dimension 3

Every odd dashing with three dashed edges has one dashed edge belonging to each set of parallel edges. Each dashed edge can be moved to a neighboring parallel edge by performing two node flips, one on each side of the edge connecting the dashed edge to the parallel edge it will be moved to. Such a maneuver fixes one of the three dashed edges, and results in another three edge odd dashing.

By a sequence of such moves, any three edge odd dashing of the 3-cube can be transformed into any other. Thus there is exactly one odd dashing of the 3-cube up to node flips.