# Mirror Symmetry and Invertible Polynomials

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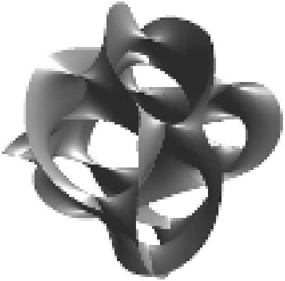

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- This week: Berglund-Hübsch-Krawitz mirror symmetry
- Research mathematics is not linear!
- This is a story, or a conversation.
- Algebra, geometry, physics, coding, experiments, explanations …

# What is Mirror Symmetry?

*Mirror Symmetry is a phenomenon first observed to occur in High-energy Theoretical Physics. Since the early 90s, mirror symmetry has been used to solve difficult mathematical problems known as curve-counting or enumerative questions in Algebraic Geometry. . . [Homological Mirror Symmetry] views two very different areas of mathematics, Symplectic Geometry and Algebraic Geometry, as opposite sides of the same categorical coin. Today, HMS is the cornerstone of an extremely active research field, reaching in influence far beyond its original formulation as a duality between Calabi-Yau manifolds.*

# A Quick Tour of Twentieth-Century Physics

- General relativity

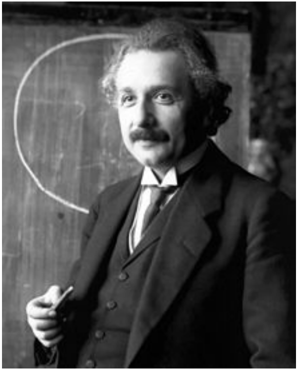


Figure: Albert Einstein

- Quantum mechanics

Figure: Fermilab

# General Relativity

## Features

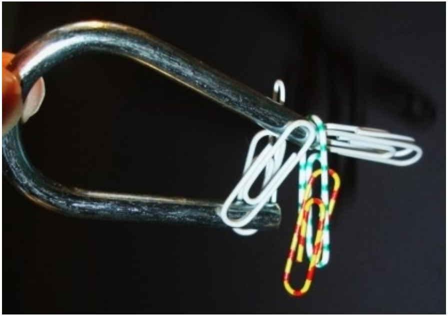
Figure: S. Bush et al.

- Measurements of time and distance depend on your relative speed.
- We specify events using coordinates in space-time.
- Space-time is curved.
- The curvature of space-time produces the effects of gravity.
- Useful for understanding large, massive objects such as stars and galaxies.

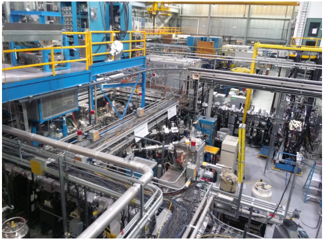
# General Relativity

## Question

- Why is the force of gravity so weak compared to other forces?



# Quantum Physics



- The smallest components of the universe behave randomly.
- Sometimes they act like waves and sometimes they act like particles.
- There are 61 elementary particles: electrons, neutrinos, quarks, photons, gluons, etc.
- Useful for understanding small objects at high energies.

# Quantum Physics

## Questions

- Why are there so many elementary particles?
- Why does the Standard Model depend on so many parameters?

# Where's the Theory of Everything?

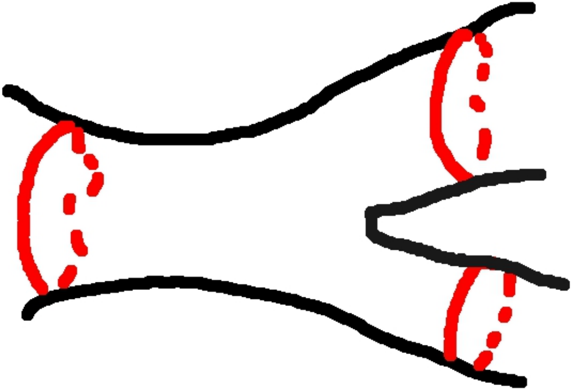
Can we build a theory of quantum gravity?

## Challenge

Quantum fluctuations in "empty" space create infinite energy!

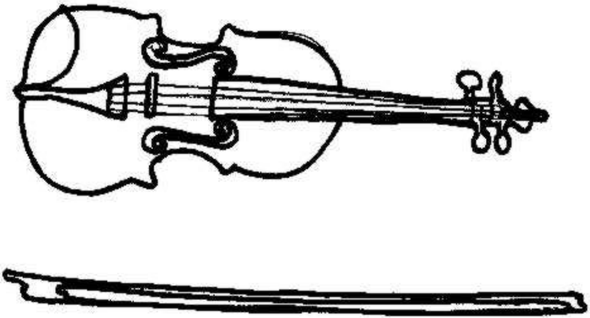
# Are Strings the Answer?

String Theory proposes that “fundamental” particles are strings.



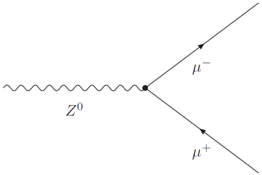
# Vibration Distinguishes Particles

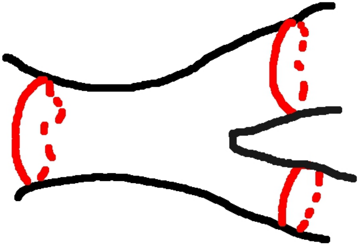
- Particles such as electrons and photons are strings vibrating at different frequencies.



# Finite Energy

String theory "smears" the energy created by creation and destruction of particles, producing finite space-time energy.



# Extra Dimensions

For string theory to work as a consistent theory of quantum mechanics, it must allow the strings to vibrate in extra, compact dimensions.

# Is Gravity Leaking?

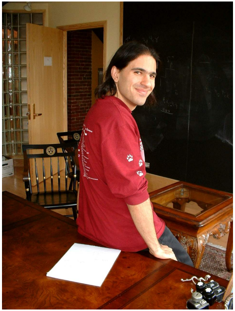


Figure: Nima Arkani-Hamed

If the electromagnetic force is confined to 4 dimensions but gravity can probe the extra dimensions, would this describe the apparent weakness of gravity?

# $T$-Duality

## Pairs of Universes

An extra dimension shaped like a circle of radius $R$ and an extra dimension shaped like a circle of radius $\alpha'/R$ yield indistinguishable physics! (The slope parameter $\alpha'$ has units of length squared.)

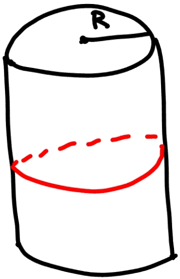



Figure: Large radius, few windings


Figure: Small radius, many windings

# Building a Model

At every point in 4-dimensional space-time, we should have 6 extra dimensions in the shape of a Calabi-Yau manifold.

# A-Model or B-Model?

## Choosing Complex Variables

A pair of fields describe how a string fits into the extra dimensions.

B-model $z = a + ib$, $w = c + id$

A-model $z = a + ib$, $\overline{w} = c - id$

# Mirror Symmetry

Physicists say . . .

- Calabi-Yau manifolds appear in pairs $(V, V^\circ)$.
- The universes described by $V$ and $V^\circ$ have the same observable physics.

# Mirror Symmetry for Mathematicians

The physicists' prediction led to mathematical discoveries!

Mathematicians say . . .

- Calabi-Yau manifolds appear in paired families $(V_\alpha, V_\alpha^\circ)$.
- The families $V_\alpha$ and $V_\alpha^\circ$ have dual geometric properties.

# Various Generalizations

- Categorical
- Enumerative
- Arithmetic

# BHK Duality

Figure: Per Berglund

Figure: Tristan Hübsch

Figure: Marc Krawitz

# A matrix polynomial

Consider a polynomial $F_A$ that is the sum of $n+1$ monomials in $n+1$ variables

$$F_A := \sum_{i=0}^{n} \prod_{j=0}^{n} x_j^{a_{ij}}.$$

We view $F_A$ as determined by an integer matrix $A = (a_{ij})$ (with rows corresponding to monomials).

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## Exercise

What are the matrices corresponding to the following polynomials?

1. $x^3 + xy^7$
2. $x^3 + y^3 + z^3$
3. $x^3 + y^2z + yz^2$
4. $x^2 + xy^3 + z^3$

# Invertible polynomials

We say $F_A$ is invertible if:

- The matrix $A$ is invertible
- There exist positive integers called weights $q_j$ so that $d := \sum_{j=0}^{n} q_j a_{ij}$ is the same constant for all $i$
- The polynomial $F_A$ has exactly one critical point, namely at the origin.

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## Exercise

Show these polynomials are invertible, and find the weights.

1. $x^3 + xy^7$
2. $x^3 + y^3 + z^3$
3. $x^3 + y^2 z + yz^2$
4. $x^2 + xy^3 + z^3$

# Keeping Ourselves Honest

Find (at least) three polynomials that are not invertible!

# The Calabi-Yau Condition

We say an invertible polynomial $F_A$ satisfies the Calabi-Yau condition if $d = \sum_{j=0}^{n} q_j$.

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### Exercise

Check whether these polynomials satisfy the Calabi-Yau condition.

1. $x^3 + xy^7$
2. $x^3 + y^3 + z^3$
3. $x^3 + y^2z + yz^2$
4. $x^2 + xy^3 + z^3$

# A Classification Project

The physicists Maximilian Kreuzer and Harald Skarke classified invertible polynomials.

Figure: The Vienna string theory group

(Kreuzer: center front; Skarke: sarcastic in the back)

# The Invertible Polynomial Classification

Kreuzer and Skarke proved that any invertible polynomial $F_A$ can be written as a sum of invertible potentials, each of which must be of one of the three *atomic types*:

$$\begin{aligned} W_{\text{Fermat}} &:= x^a, \\ W_{\text{loop}} &:= x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{m-1}^{a_{m-1}}x_m + x_m^{a_m}x_1, \text{ and} \\ W_{\text{chain}} &:= x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots x_{m-1}^{a_{m-1}}x_m + x_m^{a_m}. \end{aligned}$$

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## Fun Fact

In some of the early physics papers, chains are called tadpoles.

# Classification Practice

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## Exercise

Classify the invertible polynomials in 3 variables with weights $(1, 1, 1)$.