Anosov representations of cubulated hyperbolic groups

Teddy Weisman
University of Michigan

STDC
March 6, 2025

joint with Sami Douba, Balthazar Flechélles, Feng Zhu

Convex cocompactness in $\mathrm{SL}(2, \mathbb{R})$

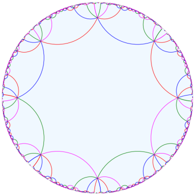
Definition

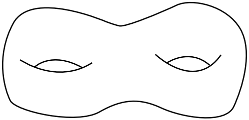
A discrete faithful representation $\rho : \Gamma \to \mathrm{SL}(2, \mathbb{R})$ is *convex cocompact* if $\rho(\Gamma)$ acts with compact quotient on a nonempty convex subset of $\mathbb{H}^2$.

Convex cocompactness in $\mathrm{SL}(2,\mathbb{R})$

Definition

A discrete faithful representation $\rho : \Gamma \to \mathrm{SL}(2,\mathbb{R})$ is *convex cocompact* if $\rho(\Gamma)$ acts with compact quotient on a nonempty convex subset of $\mathbb{H}^2$.

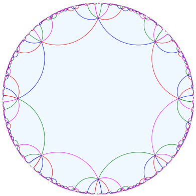


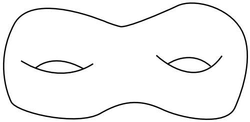


Convex cocompactness in $\mathrm{SL}(2,\mathbb{R})$

Definition

A discrete faithful representation $\rho : \Gamma \to \mathrm{SL}(2,\mathbb{R})$ is *convex cocompact* if $\rho(\Gamma)$ acts with compact quotient on a nonempty convex subset of $\mathbb{H}^2$.





Theorem

Let Γ be a finitely generated group. Then $\rho : \Gamma \to \mathrm{SL}(2,\mathbb{R})$ is convex cocompact if and only if an orbit map $x \mapsto \rho(\gamma)x$, $x \in \mathbb{H}^2$, is a quasi-isometric embedding.

In particular, Γ is always Gromov-hyperbolic.

Convex cocompactness in $\mathrm{SL}(n, \mathbb{R})$

Let $\mathbb{X} = \mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n)$.

Convex cocompactness in $\mathrm{SL}(n, \mathbb{R})$

Let $\mathbb{X} = \mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n)$.

Claim

There is a unique $\mathrm{SL}(n, \mathbb{R})$-invariant Riemannian metric on $\mathbb{X}$, making it a Hadamard manifold (in particular, a CAT(0) space).

Any discrete $\Gamma < \mathrm{SL}(n, \mathbb{R})$ acts properly discontinuously on $\mathbb{X}$.

Convex cocompactness in $\mathrm{SL}(n, \mathbb{R})$

Let $\mathbb{X} = \mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n)$.

Claim

There is a unique $\mathrm{SL}(n, \mathbb{R})$-invariant Riemannian metric on $\mathbb{X}$, making it a Hadamard manifold (in particular, a CAT(0) space).

Any discrete $\Gamma < \mathrm{SL}(n, \mathbb{R})$ acts properly discontinuously on $\mathbb{X}$.

Possible notions of "convex cocompactness" when $n > 2$:

Γ is QI embedded in $\mathbb{X}$

Γ acts cocompactly on convex $C \subseteq \mathbb{X}$.

Convex cocompactness in $\mathrm{SL}(n, \mathbb{R})$

Let $\mathbb{X} = \mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n)$.

Claim

There is a unique $\mathrm{SL}(n, \mathbb{R})$-invariant Riemannian metric on $\mathbb{X}$, making it a Hadamard manifold (in particular, a CAT(0) space).

Any discrete $\Gamma < \mathrm{SL}(n, \mathbb{R})$ acts properly discontinuously on $\mathbb{X}$.

Possible notions of "convex cocompactness" when $n > 2$:

Γ is QI embedded in $\mathbb{X}$

Γ acts cocompactly on convex $C \subseteq \mathbb{X}$.

These do *not* agree!

Convex cocompactness in $\mathrm{SL}(n, \mathbb{R})$

Let $\mathbb{X} = \mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n)$.

Claim

There is a unique $\mathrm{SL}(n, \mathbb{R})$-invariant Riemannian metric on $\mathbb{X}$, making it a Hadamard manifold (in particular, a CAT(0) space).

Any discrete $\Gamma < \mathrm{SL}(n, \mathbb{R})$ acts properly discontinuously on $\mathbb{X}$.

Possible notions of "convex cocompactness" when $n > 2$:

Γ is QI embedded in $\mathbb{X}$
Badly behaved ($\mathbb{X}$ contains flats)

Γ acts cocompactly on convex $C \subseteq \mathbb{X}$.

These do *not* agree!

Convex cocompactness in $\mathrm{SL}(n, \mathbb{R})$

Let $\mathbb{X} = \mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n)$.

Claim

There is a unique $\mathrm{SL}(n, \mathbb{R})$-invariant Riemannian metric on $\mathbb{X}$, making it a Hadamard manifold (in particular, a CAT(0) space).

Any discrete $\Gamma < \mathrm{SL}(n, \mathbb{R})$ acts properly discontinuously on $\mathbb{X}$.

Possible notions of "convex cocompactness" when $n > 2$:

Γ is QI embedded in $\mathbb{X}$
Badly behaved ($\mathbb{X}$ contains flats)

Γ acts cocompactly on convex $C \subseteq \mathbb{X}$.
Very few examples with Γ hyperbolic

These do *not* agree!

Anosov representations

Let $M \in \mathrm{SL}(n, \mathbb{R})$. Then $\sigma_k(M)$ is the kth largest *singular value* of M.

Anosov representations

Let $M \in \mathrm{SL}(n, \mathbb{R})$. Then $\sigma_k(M)$ is the kth largest *singular value* of M.

Definition

Let Γ be a finitely generated group. A representation $\rho : \Gamma \to \mathrm{SL}(n, \mathbb{R})$ is *k-Anosov* if there are $A, B > 0$ such that

$$\alpha_k(\rho(\gamma)) := \log\left(\frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))}\right) \geq A|\gamma| - B$$

for all $\gamma \in \Gamma$.

Anosov representations

Let $M \in \mathrm{SL}(n, \mathbb{R})$. Then $\sigma_k(M)$ is the kth largest *singular value* of M.

Definition

Let Γ be a finitely generated group. A representation $\rho : \Gamma \to \mathrm{SL}(n, \mathbb{R})$ is *k-Anosov* if there are $A, B > 0$ such that

$$\alpha_k(\rho(\gamma)) := \log\left(\frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))}\right) \geq A|\gamma| - B$$

for all $\gamma \in \Gamma$.

Fix a basepoint $x_0 \in \mathbb{X}$. For any $g \in \mathrm{SL}(d, \mathbb{R})$, we have

$$\alpha_k(g) \leq d_{\mathbb{X}}(x_0, gx_0).$$

So, a k-Anosov representation induces a quasi-isometric embedding $\Gamma \to \mathbb{X}$.

Theorem (Kapovich-Leeb-Porti, Bochi-Potrie-Sambarino)

Suppose that Γ is a finitely generated group and $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ is k-Anosov. Then Γ is Gromov-hyperbolic.

Theorem (Kapovich-Leeb-Porti, Bochi-Potrie-Sambarino)

Suppose that Γ is a finitely generated group and $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ is k-Anosov. Then Γ is Gromov-hyperbolic.

Question

Which hyperbolic groups admit Anosov representations?

In particular, is it all of them?

Theorem (Kapovich-Leeb-Porti, Bochi-Potrie-Sambarino)

Suppose that Γ *is a finitely generated group and* $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ *is* k*-Anosov. Then* Γ *is Gromov-hyperbolic.*

Question

Which hyperbolic groups admit Anosov representations?

In particular, is it all of them? (No.)

Theorem (Kapovich-Leeb-Porti, Bochi-Potrie-Sambarino)

Suppose that Γ is a finitely generated group and $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ is k-Anosov. Then Γ is Gromov-hyperbolic.

Question

Which hyperbolic groups admit Anosov representations?

In particular, is it all of them? (No.)

Question

Which linear hyperbolic groups admit Anosov representations?

Is it all of them?

Theorem (Kapovich-Leeb-Porti, Bochi-Potrie-Sambarino)

Suppose that Γ is a finitely generated group and $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ is k-Anosov. Then Γ is Gromov-hyperbolic.

Question

Which hyperbolic groups admit Anosov representations?

In particular, is it all of them? (No.)

Question

Which linear hyperbolic groups admit Anosov representations?

Is it all of them? (Nobody knows.)

Question

Which linear hyperbolic groups admit Anosov representations?

Some answers:

Question

Which linear hyperbolic groups admit Anosov representations?

Some answers:

- Groups acting convex cocompactly on negatively curved symmetric spaces

Question

Which linear hyperbolic groups admit Anosov representations?

Some answers:

- Groups acting convex cocompactly on negatively curved symmetric spaces
- Hyperbolic Coxeter groups (Danciger-Guéritaud-Kassel, Lee-Marquis)

Question

Which linear hyperbolic groups admit Anosov representations?

Some answers:

- Groups acting convex cocompactly on negatively curved symmetric spaces
- Hyperbolic Coxeter groups (Danciger-Guéritaud-Kassel, Lee-Marquis)
- Amalgams (Dey-Kapovich-Leeb, Danciger-Guéritaud-Kassel)

Question

Which linear hyperbolic groups admit Anosov representations?

Some answers:

- Groups acting convex cocompactly on negatively curved symmetric spaces
- Hyperbolic Coxeter groups (Danciger-Guéritaud-Kassel, Lee-Marquis)
- Amalgams (Dey-Kapovich-Leeb, Danciger-Guéritaud-Kassel)
- Other sporadic constructions (e.g. Kapovich, Douba-Tsouvalas)

Theorem (Douba-Flechélles-W.-Zhu)

Let Γ be a hyperbolic group. If Γ acts properly discontinuously and cocompactly on a CAT(0) cube complex, then Γ admits an Anosov representation.

Theorem (Douba-Flechélles-W.-Zhu)

Let Γ be a hyperbolic group. If Γ acts properly discontinuously and cocompactly on a CAT(0) cube complex, then Γ admits an Anosov representation.

A few examples of cubulated hyperbolic groups:

- Random groups at density $< 1/6$ (Ollivier-Wise)

Theorem (Douba-Flechélles-W.-Zhu)

Let Γ be a hyperbolic group. If Γ acts properly discontinuously and cocompactly on a CAT(0) cube complex, then Γ admits an Anosov representation.

A few examples of cubulated hyperbolic groups:

- Random groups at density $< 1/6$ (Ollivier-Wise)
- Output of "strict hyperbolization" process (Lafont-Ruffoni), allows for many examples with "exotic" Gromov boundary.

Where do the representations come from?

Where do the representations come from?

Theorem (Agol, Haglund-Wise)

Let Γ be a hyperbolic group. Then Γ is cubulated if and only if it is virtually a quasiconvex subgroup of a right-angled Coxeter group.

Where do the representations come from?

Theorem (Agol, Haglund-Wise)

Let Γ be a hyperbolic group. Then Γ is cubulated if and only if it is virtually a quasiconvex subgroup of a right-angled Coxeter group.

Right-angled Coxeter groups admit (many) discrete faithful representations generated by *linear reflections*.

Definition

An involution $g \in \mathrm{GL}(d, \mathbb{R})$ is a *reflection* if its 1-eigenspace is a hyperplane.

Where do the representations come from?

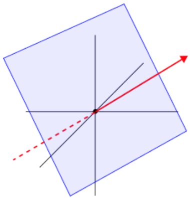
Theorem (Agol, Haglund-Wise)

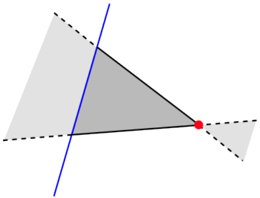
Let Γ be a hyperbolic group. Then Γ is cubulated if and only if it is virtually a quasiconvex subgroup of a right-angled Coxeter group.

Right-angled Coxeter groups admit (many) discrete faithful representations generated by *linear reflections*.

Definition

An involution $g \in \mathrm{GL}(d, \mathbb{R})$ is a *reflection* if its 1-eigenspace is a hyperplane.



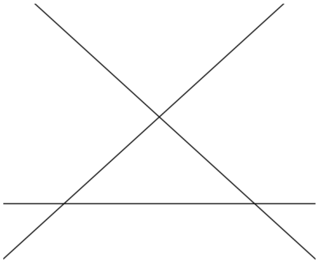


Theorem (Tits, Vinberg)

Any Coxeter group C has a faithful representation into $\mathrm{PGL}(d, \mathbb{R})$ *generated by reflections, which preserves a convex domain Ω in $\mathbb{P}(\mathbb{R}^d)$, with fundamental domain a simplex.*

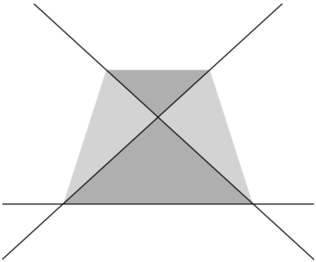
Theorem (Tits, Vinberg)

Any Coxeter group C has a faithful representation into $\mathrm{PGL}(d, \mathbb{R})$ generated by reflections, which preserves a convex domain Ω in $\mathbb{P}(\mathbb{R}^d)$, with fundamental domain a simplex.



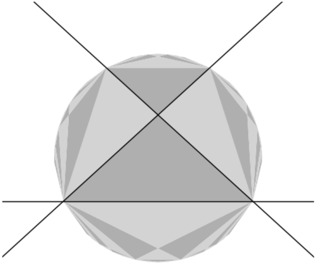
Theorem (Tits, Vinberg)

Any Coxeter group C has a faithful representation into $\mathrm{PGL}(d, \mathbb{R})$ *generated by reflections, which preserves a convex domain Ω in $\mathbb{P}(\mathbb{R}^d)$, with fundamental domain a simplex.*



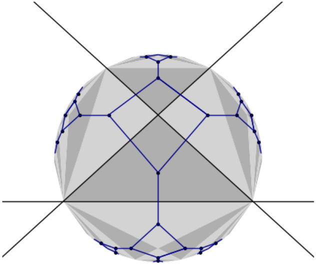
Theorem (Tits, Vinberg)

Any Coxeter group C has a faithful representation into $\mathrm{PGL}(d, \mathbb{R})$ generated by reflections, which preserves a convex domain Ω in $\mathbb{P}(\mathbb{R}^d)$, with fundamental domain a simplex.



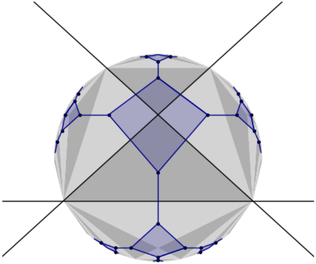
Theorem (Tits, Vinberg)

Any Coxeter group C has a faithful representation into $\mathrm{PGL}(d, \mathbb{R})$ generated by reflections, which preserves a convex domain Ω in $\mathbb{P}(\mathbb{R}^d)$, with fundamental domain a simplex.



Theorem (Tits, Vinberg)

Any Coxeter group C has a faithful representation into $\mathrm{PGL}(d, \mathbb{R})$ generated by reflections, which preserves a convex domain Ω in $\mathbb{P}(\mathbb{R}^d)$, with fundamental domain a simplex.



The dual graph is $\mathrm{Cay}(C)$, the 1-skeleton of a CAT(0) cube complex $D(C)$.

Overall goal: use combinatorics of the Davis complex $D(C)$ to estimate $\log\left(\frac{\sigma_1}{\sigma_2}\right)$ when $\Gamma < C$ is a hyperbolic quasiconvex subgroup.

Overall goal: use combinatorics of the Davis complex $D(C)$ to estimate $\log\left(\frac{\sigma_1}{\sigma_2}\right)$ when $\Gamma < C$ is a hyperbolic quasiconvex subgroup.

Some important ideas:

- Generalized ping-pong in real projective space

Overall goal: use combinatorics of the Davis complex $D(C)$ to estimate $\log\left(\frac{\sigma_1}{\sigma_2}\right)$ when $\Gamma < C$ is a hyperbolic quasiconvex subgroup.

Some important ideas:

- Generalized ping-pong in real projective space
- "Bottlenecking" for geodesics in quasiconvex hyperbolic subgroups

Overall goal: use combinatorics of the Davis complex $D(C)$ to estimate $\log\left(\frac{\sigma_1}{\sigma_2}\right)$ when $\Gamma < C$ is a hyperbolic quasiconvex subgroup.

Some important ideas:

- Generalized ping-pong in real projective space
- "Bottlenecking" for geodesics in quasiconvex hyperbolic subgroups
- Hierarchical nature of C

Overall goal: use combinatorics of the Davis complex $D(C)$ to estimate $\log\left(\frac{\sigma_1}{\sigma_2}\right)$ when $\Gamma < C$ is a hyperbolic quasiconvex subgroup.

Some important ideas:

- Generalized ping-pong in real projective space
- "Bottlenecking" for geodesics in quasiconvex hyperbolic subgroups
- Hierarchical nature of C
- Higher-rank Morse lemma, local-to-global principle for quasigeodesics in symmetric space $\mathbb{X}$ (Kapovich-Leeb-Porti)