# Math 669 Winter 2026 Problem Set

This file will be updated intermittently throughout the semester. Hand in five problems by each of the following Tuesdays: February 17, March 17, April 14.

Some problems have starred (*) parts. To receive full credit for the problem (for the purposes of obtaining a grade in this course), you do not need to solve the starred parts.

Note that the problems are of **very different** difficulty and length.

**Problem 1.** Suppose $X$ is a $n \times n$ matrix. Fix $I, J \subset [n]$, $|I| = |J| = r$ and for $i \in [n]/I$, $j \in [n]/J$ let

$$y_{i,j} = |X_{I \cup i, J \cup j}|.$$

Recall Sylvester's identity:

$$\det(Y) = |X_{I,J}|^{n-r-1}|X|.$$

(1) Prove Sylvester's identity for $n - k = 2$.
(2) Prove Sylvester's identity by induction on $n - k$, with $n - k = 1, 2$ as base cases.

**Problem 2.** It may help to know Descartes' "rule of signs" for this problem.

(1) Suppose $0 < x_1 < x_2 < \cdots < x_n$. Prove that the $n \times n$ matrix $A = (a_{ij})$ with entries
$$a_{ij} = x_i^j$$
is TP.
(2) Suppose $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$. Prove that the $n \times n$ matrix $A = (a_{ij})$ with entries
$$a_{ij} = \exp(x_i y_j)$$
is TP. (Hint: argue that $\det(A)$ varies continuously with $x$ and $y$, and is never 0. Then reduce the problem to (a).)

**Problem 3.** Let

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

be the $n$-th Catalan number. So $C_1 = 1, C_2 = 2, C_3 = 5, \ldots$. Prove that the infinite matrix

$$\begin{pmatrix} C_1 & C_2 & C_3 & \cdots \\ C_2 & C_3 & C_4 & \cdots \\ C_3 & C_4 & C_5 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

is TP. What is the determinant of the top left $n \times n$ submatrix? (Hint: "Dyck path".)

**Problem 4.** Prove that the following four sets are identical:

(1) $GL_n(\mathbb{R})_{>0}$
(2) $GL_n(\mathbb{R})_{\geq 0} \cap Bw_0B \cap B_-w_0B_-$ where $w_0$ denotes the longest element $n(n-1)\cdots 21$ of $S_n$, and $B$ denotes the upper triangular matrices in $GL_n$.
(3) $U^-_{>0} \cdot T_{>0} \cdot U_{>0}$ where $U^-$ are the lower triangular matrices with 1-s on the diagonal, and $T_{>0}$ is the set of diagonal matrices with positive diagonal entries.
(4) $U_{>0} \cdot T_{>0} \cdot U^-_{>0}$ where $U^-$ are the lower triangular matrices with 1-s on the diagonal, and $T_{>0}$ is the set of diagonal matrices with positive diagonal entries.

**Problem 5.** Prove that the collection of row-initial column-solid minors $|X_{[1,j-i+1],[i,j]}|$ with $1 < i \leq j \leq n$ parametrizes the TP part $U_{>0}$ of the upper unipotent subgroup $U$. In other words, the map $X \mapsto \{|X_{[1,j-i+1],[i,j]}| \mid |1 < i \leq j \leq n\}$ is a homeomorphism $U_{>0} \cong \mathbb{R}_{>0}^{\binom{n}{2}}$.

**Problem 6.** Let $X$ be a $n \times n$ matrix of indeterminates $x_{ij}$. Let $V \subset \mathbb{C}[x_{ij}]$ be the subspace spanned by products of minors of the form

$$\det(X_{I,J}) \det(X_{\bar{I},\bar{J}})$$

where $\bar{I}$ denotes the complement of $I$ in $[n]$, and we have $|I| = |J|$. What is the dimension of $V$? Can you extend the answer to products of 3 of more minors?

**Problem 7.** Let $S_n$ denote the symmetric group on $n$ letters.

1. Prove that $S_n$ can be presented as the free group on the simple generators $s_1, s_2, \ldots, s_{n-1}$ modulo the relations
$$s_i^2 = 1, \qquad s_{i+1}s_is_{i+1} = s_is_{i+1}s_i, \qquad s_is_j = s_js_i \text{ if } |i-j| > 1,$$
where the last two identities are called the *braid move* and *commutation move* respectively.
2. List all reduced words of the longest permutation $w_0 \in S_4$.
3. Show that any two reduced words for $w \in S_n$ are related by a sequence of braid moves and/or commutation moves.

**Problem 8.** Let $r_{a,b}(w) := |\{i \leq a \mid w(i) \geq b\}|$. Prove the equivalence of the following descriptions of the Bruhat order on $S_n$.

1. the transitive closure of the relations $u \leq ut_{i,j}$ for $t_{i,j}$ a transposition and $\ell(u) < \ell(ut_{i,j})$.
2. $u \leq v$ if and only if $r_{a,b}(u) \leq r_{a,b}(v)$ for all $a, b$.
3. $u \leq v$ if and only if some reduced word for $v$ contains as a subword some reduced word for $u$.

**Problem 9.** In class, we proved that $\mathrm{GL}_n = \bigcup_{w \in S_n} B_-wB_-$. For a matrix $X \in \mathrm{GL}_n$ let $r_{a,b}(X)$ denote the rank of the submatrix $X_{[a],[b,n]}$. Prove that

$$B_-wB_- = \{X \in \mathrm{GL}_n \mid r_{a,b}(X) = r_{a,b}(w) \text{ for all } a, b\}.$$

Conclude that the union $\bigcup_{w \in S_n} B_-wB_-$ is disjoint.

**Problem 10.** This problem fills in one of the proofs from the lectures. Let $w \in S_n$, and recall that $S(w) := \{(i, w(i)) \mid i = 1, 2, \ldots, n\}$. Let $C$ be a $w$-NE-ideal and $C' \subset S(w)$ satisfy $|C| = |C'|$ and

$$I(C') \leq I(C), \qquad J(C) \leq J(C').$$

Show that this implies that $C' = C$.

**Problem 11.** Let $f(t)$ be a real analytic function with Taylor series expansion $f(t) = 1 + a_1t + a_2t^2 + \cdots$. We say that $f(t)$ is a totally positive function if the matrix

$$\begin{bmatrix} 1 & a_1 & a_2 & a_3 & \cdots \\ 0 & 1 & a_1 & a_2 & \ddots \\ 0 & 0 & 1 & a_1 & \ddots \\ 0 & 0 & 0 & \ddots & \ddots \end{bmatrix}$$

is totally nonnegative. Show that if $f(t)$ is a totally positive function, then so is $(f(-t))^{-1}$. Show that if $f(t)$ and $g(t)$ are totally positive functions, then so is $f(t)g(t)$.

**Problem 12.**

1. Suppose that $f(t)$ is a totally positive function. Show that $f(t)$ is meromorphic on the complex plane.
2. Show that the zeroes of $f(t)$ are real and negative while the poles of $f(t)$ are real and positive. Further show that $f(t)$ is holomorphic in the neighborhood of 0 in the complex plane.
3. (*) Give an example of a totally positive function that is entire (defined on the whole complex plane) and has no zeroes.
4. (*) Completely classify the totally positive functions in (b).

**Problem 13.** Let $A = (a_{ij})$ be a $k \times n$ matrix, where $k \leq n$. Denote the columns of $A$ by $c_1, c_2, \ldots, c_n \in \mathbb{C}^k$. For an ordered $k$-tuple $(j_1, j_2, \ldots, j_k) \subset \{1, 2, \ldots, n\}$ let $\Delta_{j_1,j_2,\ldots,j_k}$ be the determinant of the $k \times k$ matrix with columns given by $c_{j_1}, c_{j_2}, \ldots, c_{j_k}$, in that order. Fix an integer $1 \leq r \leq k$. Prove the Plücker relation:

$$\Delta_{i_1,i_2,\ldots,i_k}\Delta_{j_1,j_2,\ldots,j_k} = \sum \Delta_{i'_1,i'_2,\ldots,i'_k}\Delta_{j'_1,j'_2,\ldots,j'_k}$$

where the summation is over all exchanges of $j_1, j_2, \ldots, j_r$ with $r$ of the indices amongst $i_1, i_2, \ldots, i_k$, maintaining the order in each. (Hint: prove that the difference of the two sides is a multilinear expression that is alternating in the $k+1$ vectors $c_{i_1}, c_{i_2}, \ldots, c_{i_k}, c_{j_r}$.) For example, if $I = \{1, 2\}$ and $J = \{3, 4\}$ and $r = 1$, we get

$$\Delta_{12}\Delta_{34} = \Delta_{32}\Delta_{14} + \Delta_{13}\Delta_{24} = -\Delta_{23}\Delta_{14} + \Delta_{13}\Delta_{24}.$$

**Problem 14.** A polynomial function $p(x_{ij})$ in variables $x_{ij}$ is called *totally nonnegative* if $p(X) \geq 0$ for any TNN matrix $X$.

1. Let $w, v \in S_n$. Prove that
$$x_{1,w(1)} \cdots x_{n,w(n)} - x_{1,u(1)} \cdots x_{n,u(n)}$$
is TNN if $w \leq u$ in Bruhat order on $S_n$. (Hint: if $w < u$ in Bruhat order, then there is a chain $w = w_0 < w_1 < w_2 < \cdots < w_r = u$ where $w_i = w_{i+1}t_{i,j}$; that is, successive permutations in the chain differ by a transposition.)
2. Prove the converse of the previous statement.

**Problem 15.** A *complete matching* (just "matching" in this problem) on $[2n]$ is a set of edges in the complete graph $K_{2n}$ with vertex set $[2n]$ which uses each vertex exactly once.

1. Prove that the number of matchings on $[2n]$ is $(2n-1) \cdot (2n-3) \cdots 3 \cdot 1$.
2. Let $\pi$ be a matching. The *crossing number* $c(\pi)$ of $\pi$ is the number of (pairwise) intersections of edges when $\pi$ is drawn in a disk, with the vertices arranged in circular order on the boundary of the disk. For a skew-symmetric matrix $A$, define the pfaffian
$$\mathrm{pf}(A) = \sum_{\pi} (-1)^{c(\pi)} \prod_{(i,j)\in\pi} a_{ij}$$
where the sum is over all matchings on $[2n]$, and in the product we always take $i < j$. For example for $n = 2$, we have $\mathrm{pf}(A) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$.

   A proof of the following classical identity
$$\mathrm{pf}(A)^2 = \det(A)$$

can be found easily online.

Let $N$ be a planar acyclic directed network as usual, with $2n$ sources and an arbitrary number of sinks. Define a skew-symmetric $2n \times 2n$ matrix $A(N)$ by setting

$$a_{ij} = \sum_{p,q} \mathrm{wt}(p)\mathrm{wt}(q)$$

for $i < j$, where the summation is over all pairs of noncrossing paths from sources $i$ and $j$ to any pair of sinks. Prove Stembridge's Pfaffian-analogue of the Lindström Lemma:

$$\mathrm{pf}(A(N)) = \sum_{P} \mathrm{wt}(P)$$

where the summation is over all noncrossing families of paths $P$ using all the sources and any subset of sinks.

(3) Suppose $n = 2$ and $N$ is a planar acyclic directed network with nonnegative edge weights and 4 sources. Let $A = A(N)$. Show that $a_{13}a_{24} - a_{14}a_{23} \geq 0$. Conclude that subpfaffian positivity is not enough to guarantee that a skew-symmetric matrix $A$ is realizable by a network.

(4) **(*)** (This problem generalizes the previous one significantly.) Let $A = A(N)$ be a $2n \times 2n$ skew-symmetric matrix arising from a network $N$ with nonnegative edge weights.
   (a) Suppose $|I| = |J|$ is even. Prove that $|A_{I,J}| \geq 0$.
   (b) Suppose $|I| = |J|$ is odd. Prove that $|A_{I,J}| \geq 0$ for all networks if and only if $i_1 \leq j_1, i_2, \leq j_2, \ldots$.

(5) **(*)** (Open?) Find semialgebraic conditions on a $2n \times 2n$ skew-symmetric matrix that guarantee realizability by a network. For example, are the conditions of the previous problem, together with nonnegativity of subpfaffians enough to guarantee realizability?

(6) **(*)** (Open?) Find "generators" for the set of $2n \times 2n$ skew-symmetric matrices that are realizable as $A(N)$ by a planar network.

**Problem 16.** Prove that a collection of $\binom{n}{k}$ numbers $\Delta_I$, not all zero, are the Plücker coordinates of a point in the Grassmannian if and only if they satisfy the Plücker relation with one index swappers: for $i_1 < \cdots < i_{k-1}$ and $j_1 < \cdots < j_{k+1}$, we have

$$\sum_{s=1}^{k+1} (-1)^s \Delta_{i_1,\ldots,i_{k-1},j_s} \Delta_{j_1,\ldots,\hat{j}_s,\ldots,j_{k+1}} = 0.$$

**Problem 17.** Let $n = 2k$. Consider the quadratic monomials $\Delta_S = \Delta_I \Delta_J$ as $(I, J)$ varies over the columns of the standard Young tableaux $S$ of shape a $k \times 2$ rectangle. Thus each number $1, 2, \ldots, 2k$ is used once.

(1) For $k = 3, 4$, write down the transition matrix that expresses each $\Delta_S$ in terms of Temperley-Lieb immanants $F_{\tau,T}$, where $T = \emptyset$
(2) For $k = 3$, compute the inverse matrix that expresses $F_\tau$ in terms of $\Delta_S$.
(3) Prove that these matrices are upper-triangular, with one-s on the diagonal, under an appropriate ordering on $\{S\}$ and $\{\tau\}$.
(4) **(*)** Find a combinatorial formula for the entries of the inverse matrix.

**Problem 18.** For $I = \{i_1 < \cdots < i_k\}, J = \{j_1 < \cdots < j_k\} \in \binom{[n]}{k}$, define $\min(I,J) = (\min(i_1,j_1),\ldots,\min(i_k,j_k))$ and similarly $\max(I,J)$. Use Temperley-Lieb immanants to show that the difference $\Delta_{\min(I,J)}\Delta_{\max(I,J)} - \Delta_I\Delta_J$ is nonnegative on $\mathrm{Gr}(k,n)_{\geq 0}$.

**Problem 19.** (This problem is difficult and you will get credit if you make reasonable progress.)

(1) Let $\chi : S_n \to \mathbb{C}$ be an irreducible character of $S_n$. Prove that the polynomial

$$\sum_{w \in S_n} \chi(w) x_{1w(1)} x_{2w(2)} \cdots x_{nw(n)}$$

is a totally nonnegative polynomial; that is, it takes nonnegative values on TNN matrices. For example, if $\chi$ is the sign character, we are considering the determinant. If $\chi$ is the trivial character, we are considering the permanent.

(2) **(*)** (Open?) For a partition $\lambda$ of $n$, define $\eta_\lambda : S_n \to \mathbb{C}$ by $\eta_\lambda = \sum_\lambda a_{\lambda\mu}\chi^\mu$, where $\chi^\mu$ are the irreducible characters, and $a_{\lambda\mu}$ is the coefficient of the Schur function $s_\mu$ in the monomial symmetric function $m_\lambda$. Prove, or disprove that the function

$$\sum_{w \in S_n} \eta_\lambda(w) x_{1w(1)} x_{2w(2)} \cdots x_{nw(n)}$$

is totally nonnegative.

**Problem 20.** Let $X \in \mathrm{Gr}(k,n)$ and define $f_X$ by

$$f_X(i) := \min\{j \geq i \mid v_i \in \mathrm{span}(v_{i+1},\ldots,v_j)\}$$

where $v_1,\ldots,v_n$ are the columns of a matrix representative of $X$, and we define $v_i$ for $i \in \mathbb{Z}$ by $v_{i+n} = (-1)^{k-1}v_i$ for $i \in \mathbb{Z}$. Prove that $f_X$ is a $(k,n)$-bounded affine permutation.

**Problem 21.** Let $V \in \mathrm{Gr}(k,n)$ and define $V^\perp \in \mathrm{Gr}(n-k,n)$ to be the orthogonal complement. Find and prove a formula for the Plücker coordinates of $V^\perp$ in terms of those for $V$.

**Problem 22.** Express $f_{V^\perp}$ in terms of $f_V$. How are the Grassmann necklaces $\mathcal{I}_V$ and $\mathcal{I}_{V^\perp}$ related?

**Problem 23.** Let $n \geq 2$, and let $W$ denote the affine Coxeter group of type $\tilde{A}_{n-1}$. Thus $W$ has simple generators $s_0, s_1, \ldots, s_{n-1}$ and relations

$$s_i^2 = 1, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad s_i s_j = s_j s_i \text{ for } |i-j| > 1,$$

where indices are taken modulo $n$. Prove that $W$ is isomorphic to the group of affine permutations, that is, the set of bijections $f : \mathbb{Z} \to \mathbb{Z}$ satisfying $f(i+n) = f(i)$ and $\sum_{i=1}^n f(i) = \sum_{i=1}^n i$.

**Problem 24.** Recall from class that the face weights $y_F$ of a planar bipartite network are related to the edge weights by the formula:

$$y_F = \prod_{e \text{ on perimeter of } F} \mathrm{wt}(e)^{\pm 1}$$

where the exponent is $+1$ (resp. $-1$) if the edge is black to white (resp. white to black) while traversing the perimeter of $F$ clockwise. Show that the formula above gives a bijection

$$\{\text{edge weights modulo gauge equivalence}\} \cong \{\text{face weights satisfying } \prod_F y_F = 1\}.$$

In particular, the space of wedge weights modulo gauge equivalence is isomorphic to $\mathbb{R}_{>0}^{\#F-1}$.

**Problem 25.** Let $k = 3$ and $n = 7$, and consider the bounded affine permutation in window notation

$$f = [5, 3, 6, 9, 8, 11, 7].$$

Compute the corresponding Grassmann necklace and cyclic rank matrix. Find a reduced planar bipartite graph with $f$ as its affine trip permutation.

**Problem 26.** Let $\mathcal{M}$ be a matroid of rank $k$ on $[n]$, and suppose that $I \in \mathcal{M}$ is the minimal base of $\mathcal{M}$ in lexicographic order. Show that $I = \{i_1 < \cdots < i_k\}$ is also minimal in dominance order, that is, for $J = \{j_1 < \cdots < j_k\} \in \mathcal{M}$, we have $i_a \leq j_a$ for all $a$.

**Problem 27.** Let $\mathcal{M}$ be a matroid of rank $k$ on $[n]$, and let $\leq_a$ denote the total order $a <_a a+1 <_a \cdots <_a a+n-1$. For each $a \in [n]$, let $I_a$ be the lexicographically minimal element of $\mathcal{M}$ with respect to $\leq_a$. Show that $\mathcal{I}(\mathcal{M}) = (I_1, I_2, \ldots, I_n)$ is a Grassmann necklace.

**Problem 28.** Let $\mathcal{M}, \mathcal{M}'$ be matroids of rank $k$ on $[n]$. We say that $\mathcal{M}$ contains $\mathcal{M}'$ if $I \in \mathcal{M}'$ implies $I \in \mathcal{M}$.

1. Give an example of two distinct matroids with the same Grassmann necklace.
2. Show that among all matroids with a fixed Grassmann necklace $\mathcal{I}$, there is a maximal one under containment.

**Problem 29.** Let $\mathcal{M}$ be a matroid. The matroid polytope $P_{\mathcal{M}}$ of $\mathcal{M}$ is the convex hull of the vectors $e_I = e_{i_1} + \cdots + e_{i_k}$ as $I = \{i_1, \ldots, i_k\}$ varies over bases of $\mathcal{M}$.

1. Show that for any matroid $\mathcal{M}$, the edges of $P_{\mathcal{M}}$ are in the directions $e_i - e_j$ for some $i \neq j$.
2. Show that for a positroid $\mathcal{M}$, the facet hyperplanes of $P_{\mathcal{M}}$ are of the form $x_i + x_{i+1} + \cdots + x_j = a$ for some $i \leq j$ and $0 \leq a \leq k$.

**Problem 30.** If $I, J \in \binom{[n]}{k}$, define $\mathrm{sort}_1(I, J)$ and $\mathrm{sort}_2(I, J)$ as follows. Let the non-decreasing rearrangement of the multiset $I \cup J$ be $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_{2k}$. Then $\mathrm{sort}_1(I, J) = \{a_1, a_3, \ldots, a_{2k-1}\}$ and $\mathrm{sort}_2(I, J) = \{a_2, a_4, \ldots, a_{2k}\}$.

Prove that a matroid is a positroid if and only if it is sort-closed. That is, if $I, J \in \mathcal{M}$, then $\mathrm{sort}_1(I, J), \mathrm{sort}_2(I, J) \in \mathcal{M}$.

**Problem 31.** For a vector $v \in \mathbb{R}^n$ define $\mathrm{var}(v)$ to be the number of sign changes in the vector, ignoring zeroes. We declare that $\mathrm{var}(0) = -1$. Prove that $V \in \mathrm{Gr}(k, n)$ is TNN if and only if $\mathrm{var}(v) \leq k - 1$ for every $v \in V$.

**Problem 32.** For a vector $v \in \mathbb{R}^n$ define $\overline{\mathrm{var}}(v)$ to be the maximum of $\mathrm{var}(w)$ as $w$ varies over vectors obtained from $v$ by changing the zero components in $v$. Prove that $V \in \mathrm{Gr}(k, n)$ is TP if and only if $\overline{\mathrm{var}}(v) \leq k - 1$ for all nonzero $v \in V$.

**Problem 33.** Let $X \in \mathrm{GL}(n)_{>0}$ be totally positive. Prove that the eigenvalues of $X$ are real, positive, and distinct.

**Problem 34.** Recall that the space of edge weights on a planar bipartite graph, modulo gauge equivalence is denoted $(\mathcal{L}_G)_{>0}$. Find a necessary and sufficient condition for a subset $A \subset E(G)$ of edges to parametrize $(\mathcal{L}_G)_{>0}$. That is, we would like the restriction of the quotient map $\mathbb{R}_{>0}^{E(G)} \to (\mathcal{L}_G)_{>0}$ to $\mathbb{R}_{>0}^{A}$ to be an isomorphism.

**Problem 35.** Let $G$ be a reduced plabic (or planar bipartite) graph with trip bounded affine permutation $f$, and let $\mathcal{I} = \mathcal{I}(f)$ be the corresponding Grassmann necklace. For each trip strand $T_i$ with destination $i$, write $i$ inside all the faces of $G$ to the left of $T_i$. (Since $T_i$ has no self-intersections there is no ambiguity.)

1. Show that in this way every face of $G$ is labeled with a $k$-element subset.
2. Show that the labels of the boundary faces give the Grassmann necklace $\mathcal{I}$.

**Problem 36.** (Requires knowledge of cluster algebras). Let $G$ be a planar bipartite graph. Define a directed graph, called the *quiver* $Q(G)$ of $G$ as follows. The vertices of $Q(G)$ are the faces of $G$ that do not touch the boundary of the disk. There is a directed edge from $F \to F'$ if $F$ and $F'$ share a perimeter edges such that when going from $F$ to $F'$ the white vertex of the edge is on the left. If there are multiple edges between two faces $F$ and $F'$ we may cancel edges with opposite directions in pairs.

1. Show that the square (or spider) move on $G$ induces quiver mutation in the sense of cluster algebras.
2. Find a reduced planar bipartite graph $G$ such that $Q(G)$ is mutation equivalent to the quiver $E_6$.

**Problem 37.** Recall that we may think of a bounded affine permutation $f : \mathbb{Z} \to \mathbb{Z}$ as a juggling pattern. A ball thrown at time $t = i$ lands at time $t = f(i)$. The subset $I_a$ of the Grassmann necklace of $f$ is obtained by looking at the balls in the air at $t = a - \varepsilon$ and recording the times those balls will land.

Define the *dual Grassmann necklace* by recording instead the times those balls were thrown.

1. Write down an axiomatic characterization of $(k, n)$-dual Grassmann necklaces.
2. Explicitly describe the bijection between Grassmann necklaces and dual Grassmann necklaces.
3. How do we read the dual Grassmann necklace off the matroid?

**Problem 38.** The *top open positroid variety* $\mathring{\Pi}(k, n) \subset \mathrm{Gr}(k, n)$ is the open subset of the Grassmannian where the cyclic Plücker coordinates $\Delta_{[1,k]}, \Delta_{[2,k+1]}, \ldots, \Delta_{n,1,\ldots,k-1}$ are non-vanishing. Suppose that $\gcd(k, n) = 1$.

1. Show that
$$\#\mathring{\Pi}(k, n)(\mathbb{F}_q) = \frac{(q-1)^n}{q^n - 1} \binom{n}{k}_q .$$
2. **(*)** (1) can be interpreted as saying that a (uniformly random) point in $\mathrm{Gr}(k, n)(\mathbb{F}_q)$ lies in the top open positroid variety with probability $\frac{(q-1)^n}{q^n-1}$. What is the correct formulation of the statement: a point in the Grassmannian $\mathrm{Gr}(k, n)(\mathbb{R})$ lies in $\mathrm{Gr}(k, n)_{\geq 0}$ with probability ????

**Problem 39.** Let $a_n$ denote the number of bounded affine permutations of period $n$ (and any value of $k$ from 0 to $n$). Show that $a_n$ satisfies the recurrence $a_n = na_{n-1} + 1$. Find a closed formula for $a_n$.

**Problem 40.** Let $M$ be a square matrix, and suppose that the Schur complement $M/D$ is well-defined. Prove that

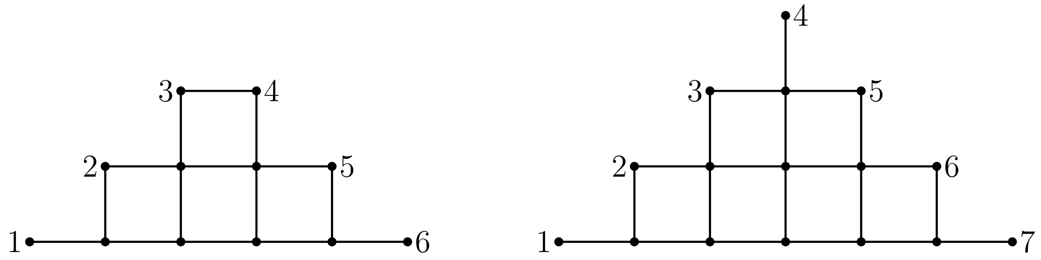
$$\mathrm{rk}(M) = \mathrm{rk}(D) + \mathrm{rk}(M/D).$$

**Problem 41.** Prove the quotient formula $M/D = (M/E)/(D/E)$ whenever all the Schur complements are well-defined.

**Problem 42.** Prove that $M$ is positive definitive if and only if $D$ and $M/D$ are both positive definite.

**Problem 43.** Let $\Gamma$ and $\Gamma'$ be related by a $Y$-$\Delta$ move. Prove directly that a circular pair $(P, Q)$ is connected through $\Gamma$ if and only if it is connected through $\Gamma'$.

**Problem 44.** Prove that the *standard graphs* $\Gamma_n$ with $n$ boundary nodes are well-connected and critical (or reduced). We illustrate $\Gamma_6$ and $\Gamma_7$.



**Problem 45.** Let $G$ be a medial graph. Suppose that two strands $S, S'$ intersect twice. Prove that after some Yang-Baxter moves, the graph will have an empty lens formed by $S$ and $S'$.

**Problem 46.** Let $G$ and $G'$ be two lensless medial graph in the disk, each with $n$ strands. Suppose the starting and ending points of the strands of $G$ and of $G'$ are identical. Prove that $G$ and $G'$ are related by a sequence of Yang-Baxter move.

**Problem 47.** Let $\Gamma$ be a connected electrical network with boundary $B \subset V(\Gamma)$, and apply a chosen voltage vector to the boundary nodes. Show that the voltages among all vertices of $\Gamma$ is maximized at one of the boundary vertices.

**Problem 48.** Let $\Gamma$ be an electrical network with boundary $B \subset V(\Gamma)$. Show that the current flowing through any edge $e$ of $\Gamma$ is less than or equal to the sum of all the positive currents flowing into the boundary vertices.

**Problem 49.** Prove the following result stated in class. Let $\Gamma$ be a circular planar electrical network and $(P, Q)$ be a circular pair with $|P| = k = |Q|$. Then $(-1)^k \det \Lambda(P; Q) \geq 0$ with equality if and only if $(P, Q)$ are not connected through $\Gamma$.

**Problem 50.** Prove the identity

$$-\Lambda_{i,j}(\Gamma) = \frac{L_{(i,j|\text{ rest singleton })}(\Gamma)}{L_{\text{ all singleton }}(\Gamma)}.$$

**Problem 51.** Let $\sigma, \tilde{\sigma}$ be dual noncrossing set partitions on $[n]$. Prove that $|\sigma| + |\tilde{\sigma}| = n + 1$, where $|\sigma|$ denotes the number of blocks in $\sigma$.

**Problem 52.** Let $\Gamma$ and $\Gamma'$ be cactus networks related by one of the five moves: $Y$-$\Delta$, series, parallel, pendant removal, and loop removal. Check that $\mathcal{L}(\Gamma) = \mathcal{L}(\Gamma')$.

**Problem 53.** Let $\Gamma$ be an electrical network. Let $u_{2k-1}(t)(\Gamma)$ be the electrical network obtained from $\Gamma$ by adding a a boundary spike at $k$ with weight $1/t$. Let $u_{2k}(t)(\Gamma)$ be the electrical network obtained from $\Gamma$ by adding a boundary bridge from $k$ to $k+1$ with weight $t$. Prove that the operations $u_i$ satisfy the following relations:

$$u_i(a)u_i(b) = u_i(a+b), \qquad u_i(a)u_j(b) = u_j(b)u_i(a) \qquad \text{for } |i-j| > 1,$$
$$u_i(a)u_{i+1}(b)u_i(c) = u_{i+1}(bc/(a+c+abc))u_i(a+c+abc)u_{i+1}(ab/(a+c+abc)).$$

**Problem 54.** Let $P_n$ denote the set of matchings on $2n$ points on the boundary of the disk. Recall that we make $P_n$ into a poset by declaring that $\tau' < \tau$ if $\tau'$ can be obtained from $\tau$ by uncrossing one of the crossings of $\tau$.

1. What are the covering relations in $P_n$?
2. Prove that $P_n$ is a graded poset, with grading given by the number of crossings.

**Problem 55.** Recall the stratification $E_n = \bigsqcup_\tau E_\tau$. Prove that $\overline{E_\tau} = \bigsqcup_{\tau' \leq \tau} E_{\tau'}$.

**Problem 56.** Show that the circular planar networks are the subset of cactus networks corresponding to medial pairings that form a particular upper order ideal in $P_n$.

**Problem 57.** Recall that to each $\tau \in P_n$ we have defined a bounded affine permutation $f_\tau \in \mathcal{B}(n+1, 2n)$. Show that $\tau \leq \tau'$ if and only if $f_\tau \geq f_{\tau'}$ in affine Bruhat order.

**Problem 58.** In class we stated two ways to embed $E_n$ into $\mathrm{Gr}(n+1, 2n)$. The first one is via the map $\Gamma \mapsto N(\Gamma)$ on cactus networks, and the second is to describe embedding the response matrix of an electrical network directly $\Lambda \mapsto X(\Lambda)$ (and then extend to cactus networks by continuity). Show that these maps agree.

**Problem 59.** Let $\mathrm{OG}(n, 2n) \subset \mathrm{Gr}(n, 2n)$ be the subvariety cut out by the equations $\Delta_I = \Delta_{[2n]\setminus I}$ for all $I$. Consider the bilinear form $\eta$ on $\mathbb{R}^{2n}$ given by the diagonal matrix $(1, -1, 1, -1, \ldots, -1)$. Show that the maximal isotropic Grassmannian with respect to $\eta$ has two components, one of which is $\mathrm{OG}(n, 2n)$.

**Problem 60.** Prove that for a planar Ising network we have $\langle \sigma_A \rangle \geq 0$ for any subset $A \subseteq [n]$.

**Problem 61.** Figure out the transformation of edge weights for the $Y$-$\Delta$ move on planar Ising networks.

**Problem 62.** Let $N$ denote the $Y$-network where $n = 3$ with three unknown edge weights. Verify the claim in class that if $N^*$ is the planar dual Ising network to $N$ then $\phi(M^*), \phi(M) \in \mathrm{Gr}(3, 6)$ are related by cyclic shift.