Fall 2025 Math 565 Problem Set 6

1. (a) Let L be a lattice. Show that for any $x, y, z \in L$, we have
$$x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z).$$
   (b) Let L be a finite lattice. We say that L is modular if for any $x, y, z \in L$ we have
$$(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee (x \wedge z)).$$
   Show that if L is modular if and only if whenever $x \leq z$ we have $x \vee (y \wedge z) = (x \vee y) \wedge z$.

2. Let $v_1, v_2, \ldots, v_n \in \mathbb{R}^k$, and assume that these vectors span $\mathbb{R}^k$. Show that the set
$$\mathcal{M} := \{\{i_1, \ldots, i_k\} \mid v_{i_1}, \ldots, v_{i_k} \text{is a basis for } \mathbb{R}^k\}$$
is a matroid of rank k on $[n]$.

3. Let $\mathcal{M}$ be a matroid of rank k on the set X. Recall that a set $A \subset X$ is *independent* if it is a subset of some basis of $\mathcal{M}$.
   (a) Let $S \subset X$ be a nonempty subset. Show that the maximal (under inclusion) independent subsets of S all have the same cardinality.
   (b) Show that these maximal independent subsets form a matroid $\mathcal{M}|_S$ on S, that is, they satisfy the exchange axiom.

4. A poset P is *graded* (in the sense of the previous pset) if and only if we can assign an integer $\rho(x)$, called the rank, to each $x \in P$ so that if $x \lessdot y$ then $\rho(y) = \rho(x) + 1$. (You may assume this.)

   Let P be a finite graded poset with a $\hat{0}$ and $\hat{1}$. We say that P is *Eulerian* if each interval $[s, t]$ where $s < t$ has the same number of elements with odd rank as elements with even rank.
   (a) What do intervals of length 2 (that is, $[s, t]$ where $\rho(t) = \rho(s) + 2$) in Eulerian posets look like?
   (b) Verify that the Boolean algebra B_n is Eulerian.
   (c) Prove that a poset is Eulerian if and only if the Mobius function is given by $\mu(s, t) = (-1)^{\rho(t)-\rho(s)}$.

5. Let $\mathcal{A}$ be the hyperplane arrangement consisting of the n hyperplanes $x_i = 0$ in $\mathbb{R}^n$, for $i = 1, 2, \ldots, n$.
   (a) Show that the intersection poset $L(\mathcal{A})$ is isomorphic to the Boolean algebra.
   (b) Compute the Mobius function of $L(\mathcal{A})$.

(c) Compute the characteristic polynomial of $\mathcal{A}$.

6. Let G be a simple graph on $[n]$ and let $\mathcal{A}_G$ denote the corresponding graphical arrangement in $\mathbb{R}^n$. Prove that when G has no cycles, the poset $L(\mathcal{A}_G)$ is isomorphic to a Boolean algebra. Deduce a formula for the number of regions and bounded regions in $\mathcal{A}_G$ in this case.

7. Let $\mathcal{A}$ be the hyperplane arrangement in $\mathbb{R}^n$ consisting of all hyperplanes $x_i = x_j$ for $i \neq j$ and the hyperplanes $x_i = 0$ for $i = 1, 2, \ldots, n$. Prove that
$$\chi_{\mathcal{A}}(t) = (t-1)(t-2)(t-3)\cdots(t-n).$$