Fall 2025 Math 565 Problem Set 5

1. (Problem 23B) Let G be a simple connected graph G on n vertices. Define a geometric lattice $L(G)$ whose points correspond to the edge set $E(G)$. The elements of L are all partitions Π of $V(G)$ such that the subgraph of G induced by each block of Π is connected.

   Prove that the bases of $L(G)$ are exactly the edge sets of spanning trees in G.

2. Let L be a geometric lattice, and $x \leq y$ be two elements. Show that the interval

   $$[x, y] := \{z \in L \mid x \leq z \leq y\}$$

   is again a geometric lattice.

3. Let $(X, \mathcal{F})$ be a combinatorial geometry. A *basis* B of a flat F is an independent set $B \subseteq F$ such that $\overline{B} = F$.

   Now let F be a flat of a combinatorial geometry and suppose $B \subseteq F$ satisfies $\overline{B} = F$ and B is minimal satisfying his property. Show that B is a basis of F.

4. Let L be a finite geometric lattice. Prove that L is *graded*: for any $x \leq y$ in L, any two maximal chains from x to y have the same length. You must prove this directly from the axioms of a geometric lattice; do not use the correspondence with combinatorial geometries or matroids proved in class.

5. A collection $\mathcal{M}$ of k-element subsets of $[n]$, satisfies the *exchange axiom* if: given $I, J \in \mathcal{M}$ and $i \in I$, there exists $j \in J$ such that $(I - \{i\} \cup \{j\}) \in \mathcal{M}$. (Thus $\mathcal{M}$ is the bases of a matroid.)

   Suppose $\mathcal{M}$ satisfies the exchange axiom. Show that $\mathcal{M}$ satisfies the *dual exchange axiom*: if $I, J \in \mathcal{M}$ and $j \in J$ there exists $i \in I$ such that $(I - \{i\} \cup \{j\}) \in \mathcal{M}$.

6. Let $M = U_{k,n}$ be the uniform matroid of rank k on $[n] = \{1, 2, \ldots, n\}$. Thus the bases of M are all k-element subsets of $[n]$. Find the independent sets, circuits, rank function, closure operator, and flats of $U_{k,n}$.

   A *circuit* is a subset $S \subset [n]$ that is dependent (= not independent), and minimal under inclusion.