Fall 2025 Math 565 Problem Set 4

1. (Problem 17H) Describe a bijective correspondence between symmetric Latin squares of order n in which all symbols appear on the main diagonal and symmetric Latin squares of order $n+1$ with all $(n+1)$'s on the diagonal.

2. Suppose that A is a symmetric Latin square of order n such that every symbol appears on the main diagonal. Prove that n is odd.

3. Let $(X = \{x_1, x_2, \ldots, x_{n^2+n+1}\}, \mathcal{L} = \{L_1, L_2, \ldots, L_{n^2+n+1}\})$ be a projective plane of order n. Define a $(n^2+n+1) \times (n^2+n+1)$ *incidence matrix* M by
$$m_{ij} = \begin{cases} 1 & \text{if } x_i \text{ lies on } L_j \\ 0 & \text{otherwise.} \end{cases}$$
Show that $\det(M) = \pm(n+1)n^{(n^2+n)/2}$. (Hint: what is MM^T?)

4. Let $(X, \mathcal{L})$ be a projective plane of order n. Suppose we are given a set of points $A \subseteq X$ such that no 3 points in A lie on a common line. Prove that $|A| \leq n+2$.

5. An affine plane is a pair $(X, \mathcal{L})$ of points and lines satisfying:

   (a) there exist 3 points not all on one line,

   (b) any two points belong to a unique line,

   (c) given a point p not on a line L, there exists a unique L' passing through p such that $L \cap L' = \emptyset$.

   Convince yourself that $\mathbb{R}^2$ is an affine plane.

   (i) For any projective plane, show that one can construct an affine plane by removing one of the lines and all the points on it.

   (ii) Prove that a finite affine plane has n^2+n lines and n^2 points, for some integer n.

6. Prove that the Fano plane cannot be embedded in the Euclidean plane, that is, one cannot find 7 points and 7 lines in $\mathbb{R}^2$ with three points in each line, three lines through each point, and such that each pair of lines intersect at one of the points, and each pair of points lies on one of the lines.

7. Let X be a set with n^2+n+1 elements, where $n \geq 2$. Let $\mathcal{L}$ a collection of n^2+n+1 subsets of X, such that each $L \in \mathcal{L}$ has size $n+1$. Suppose that for any two distinct $L, L' \in \mathcal{L}$, we have $|L \cap L'| \leq 1$. Prove that $(X, \mathcal{L})$ is a projective plane of order n. (Hint: start by showing, via double counting, that any pair of points belong to exactly one of the $L \in \mathcal{L}$.)