Fall 2025 Math 565 Problem Set 3

1. Let $(A_1, A_2, \ldots, A_n)$ be a family of sets with a system of distinct representatives. Let $a \in A_1$. Prove that there is an SDR containing a, but show by example that it may not be possible to find an SDR $(a_1, \ldots, a_n)$ in which $a_1 = a$.

2. (Problem 5A) A *perfect* matching in a (possibly not bipartite) graph is a collection of edges so that every vertex is incident to one edge of the matching.

   (a) Show that a finite regular bipartite graph of degree $d > 0$ has a perfect matching.

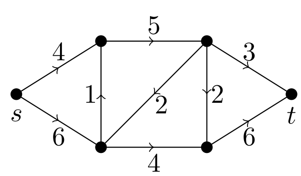
   (b) Find a simple graph, regular of degree 3, that does not have a perfect matching.

   (c) Suppose G is bipartite with parts X and Y. Further assume that every vertex in X has the same degree $s > 0$ and every vertex in Y has the same degree t. Prove that if $s \geq t$, then there is a complete matching M of X into Y.

3. (Problem 5D) Let S be the set $\{1, 2, \ldots, mn\}$. We partition S into m sets $A_1, \ldots, A_m$ of size n. We also partition S into m sets of $B_1, \ldots, B_m$ of size n. Show that the sets A_i can be renumbered so that $A_i \cap B_i \neq \emptyset$.

4. (Problem 6B) Let A_i, $1 \leq i \leq k$ be distinct subsets of $\{1, 2, \ldots, n\}$. Suppose that $A_i \cap A_j \neq \emptyset$ for all i and j. Show that $k \leq 2^{n-1}$ and give an example where equality occurs.

5. Find the max-flow and min-cut in the following network.



6. (Problem 7C) Let (X_1, Y_1) and (X_2, Y_2) be two mincuts. Show that $(X_1 \cup X_2, Y_1 \cap Y_2)$ is also a mincut.

7. (Problem 7D) Prove Hall's theorem from the maxflow-mincut theorem.