Fall 2025 Math 565 Problem Set 2

(1) (Part of Problem 3D) Prove that $N(4,4;2) = 18$.
(2) (Problem 3F) Prove that for all integers $r \geq 1$, there is a minimal number $N(r)$ with the following property. If $n \geq N(r)$ and the integers in $\{1, 2, \ldots, n\}$ are colored with r colors, then there are three elements x, y, z (not necessarily distinct) with the same color and $x + y = z$. Determine $N(2)$ and show that $N(3) > 13$.
(3) (Problem 3G) Let m be given. Show that if n is large enough, every $n \times n$ $(0,1)$-matrix has a principal submatrix of size m (i.e., a submatrix obtained by removing $n - m$ rows and the same $n - m$ columns), in which all the elements below the diagonal are the same, and all the elements above the diagonal are the same.
(4) Let XYZ be an equilateral triangle, and let S be the set of all points on the three segments XY, YZ, XZ. Prove that in any two-coloring of S, we can find a right-angled triangle all of whose vertices have the same color.
(5) (Problem 33B) Find the chromatic polynomial of the n-cycle C_n for $n \geq 3$. Find the chromatic polynomial of the n-wheel W_n (this is the graph obtained from C_n by adding a new vertex and joining it to all vertices of C_n).
(6) Prove that any graph G has at least $\binom{\chi(G)}{2}$ edges.
(7) Let G be a n-vertex, triangle-free, simple planar graph such that $n \geq 3$. Show that $\#E \leq 2n - 4$.