Fall 2025 Math 565 Problem Set 1

You may use theorems in the textbook from sections that we have already covered. If you use the result of a problem from the textbook that was not previously assigned, you have to solve that problem too.

Note that there are hints at the back of the textbook for many of the problems.

(1) (Problem 1C) Show that a connected graph on n vertices is a tree if and only if it has $n-1$ edges.

(2) (Problem 3J) The edges of K_n are colored red and blue in such a way that a red edge is in at most one red triangle. Show that there is a subgraph K_k with $k \geq \lfloor \sqrt{2n} \rfloor$ that contains *no* red triangle.

(3) (Problem 3E(a)) A *tournament* on n vertices is an orientation of the edges of K_n. A *transitive tournament* is a tournament such that the vertices can be numbered in such a way that (i, j) is an edge if and only if $i < j$. Show that if $k \leq \log_2 n$, every tournament on n vertices has a transitive subtournament on k vertices.

(4) The *degree* $\deg(v)$ of a vertex v is the number of edges incident to v, with the convention that a loop at v contributes two to the degree of v.

A graph G is called k-regular if all its vertices have degree exactly equal to k. Determine all pairs (k, n) such that there exists a k-regular simple graph on n vertices.

(5) Let $G = (V, E)$ be a simple graph with no triangles. Prove that there is a partition $V = X \sqcup Y$ of the vertices into two disjoint parts, so that for any $x \in X$, we have $d(x) \leq |Y|$ and for any $y \in Y$, we have $d(y) \leq |X|$. Here $d(x)$ denotes the degree of a vertex x. Hint: Make a careful choice of a vertex x, and let Y be its set of neighbors.