

Positive geometry exercises: Matroids and amplitudes

1. Recall that the bilinear form $\langle \cdot, \cdot \rangle^{\text{dR}}$ on $A^r(M)$ is defined as a sum over flags in the lattice of flats of M . Show, or check in some examples, that when $a_E = \sum_{e \in E} a_e = 0$, the bilinear form $\langle \cdot, \cdot \rangle^{\text{dR}}$ descends to the quotient $A^r(M)/\omega A^{r-1}(M)$, where $\omega = \sum a_e e \in A^1(M)$.

2. (a) Compute $\langle \cdot, \cdot \rangle^{\text{dR}}$ on $A(M)$, $\overline{\langle \cdot, \cdot \rangle}^{\text{dR}}$ on $\bar{A}(M)$, and their inverses in the case $d = 1$ consisting of points on a line. Find determinants of these bilinear forms with respect to the different bases of bounded chambers.

(b) Repeat this for the $d = 2$ hyperplane arrangement consisting of four lines a, b, c, d , where a, b are parallel and a, c, d pass through a single point. Verify the deletion-contraction formula for the residue of an amplitude.

3. Prove that in the case of the hyperplane arrangement $M_{0,n}$, the matroid amplitude (of the complete graphic matroid K_{n-1}) indeed recovers $A_n^{\phi^3}$.

4. This exercise considers partial amplitudes for the complete graphic matroid.

- Show that the connected components of $M_{0,n}(\mathbb{R})$ are in bijection with permutations modulo dihedral symmetry, and thus $|\pi_0(M_{0,n}(\mathbb{R}))| = (n-1)!/2$.
- Let P_w denote the chamber indexed by the permutation w (up to dihedral symmetry). Show that the partial amplitude $A(P_w, P_v)$ is given by a sum of terms over cubic trees that are planar with respect to both w -order and v -order. How can you generate all such cubic trees? What is the number $a(w, v)$ of such trees?

5. The *Parke-Taylor form* on $M_{0,n}$ in physics literature is written

$$\text{PT} = \frac{dz_1 dz_2 \cdots dz_n}{\text{PGL}_2} \frac{1}{(z_1 - z_2)(z_2 - z_3) \cdots (z_n - z_1)},$$

where z_1, z_2, \dots, z_n denotes n distinct points on \mathbb{P}^1 . Make sense of this formula by using the action of PGL_2 to gauge-fix three of the points to $0, 1, \infty$ to obtain a $(n-3)$ -form on $M_{0,n}$. Show that the Parke-Taylor form is the canonical form of the positive component of $M_{0,n}(\mathbb{R})$.

6. The formula for a matroid amplitude has factors a_F in the denominator for various flats F . Prove that the rational function $A(P)$ has no pole at $a_F = 0$ when F is not connected. Start by checking this for $M_{0,5}$, that is $A_5^{\phi^3}$.

7. Open: find an explicit combinatorial formula for the inverse of $\overline{\langle \cdot, \cdot \rangle}^{\text{dR}}$ in the basis of canonical forms of bounded chambers. A special case follows from the formula for the inverse of $\langle \cdot, \cdot \rangle^{\text{dR}}$ given in the lectures. See Section 7.8 of [arXiv:2412.06705](https://arxiv.org/abs/2412.06705).

8. Open: let G be connected graph and $M = M(G)$ be the graphic matroid of G . The acyclic orientations of G are the topes of an oriented matroid $\mathcal{M}(G)$. Give explicit formulae for full and partial amplitudes for $\mathcal{M}(G)$. Explore these *graph ϕ^3 -amplitudes*.