Positive geometry exercises: Hyperplane arrangements

Given an affine arrangement $\bar{\mathcal{A}}$ and a hyperplane $H \in \bar{\mathcal{A}}$, we can *restrict* $\bar{\mathcal{A}}$ to H to obtain an arrangement $\bar{\mathcal{A}}|_H$ inside H. The hyperplanes of $\bar{\mathcal{A}}|_H$ are obtained by intersecting hyperplanes in $\bar{\mathcal{A}}$ with H.

1. Explicitly verify the results about the Orlik-Solomon algebra:

- compute the lattice of flats and the characteristic polynomial;
- compare with the Hilbert series of the Orlik-Solomon algebra and the topology of the hyperplane arrangement complement (if possible);
- compare the characteristic polynomial with the point count over a finite field;
- check that the stated collection of canonical forms of bounded chambers form a basis of the top component.

for
(a) the case $d = 1$ of an arrangement of points on a line.
(b) the $d = 2$ hyperplane arrangement consisting of four lines a, b, c, d, where a, b are parallel and a, c, d pass through a single point. (Realize this over the integers when doing the finite field computation.)
(c) the moduli space $M_{0,5}$ discussed in the lectures.

2. Let $\bar{\mathcal{A}}$ denote an affine hyperplane arrangement. In the lecture, we defined the Orlik-Solomon algebra $\bar{A}(\bar{\mathcal{A}})$ of $\bar{\mathcal{A}}$ as a subspace of $A(M)$ where M is the matroid of the corresponding projective arrangement. Show that $\bar{A}(\bar{\mathcal{A}})$ also has the explicit description

$$\bar{A}(\bar{\mathcal{A}}) := \bigwedge(\bar{e} \mid H_e \in \bar{\mathcal{A}})/J$$

where J is the ideal generated by the relations

$$\partial(\bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_k) = 0 \text{ if } H_{e_1} \cap \cdots \cap H_{e_k} \neq \emptyset \text{ has codimension} < k,$$
$$\bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_k = = 0 \text{ if } H_{e_1} \cap \cdots \cap H_{e_k} = \emptyset.$$

3. Pick a flag of (affine) subspaces $W_0 \subset W_1 \subset \cdots \subset W_{d-1} \subset W_d = \mathbb{C}^d$ in generic position. Thus W_{d-1} is a hyperplane. We say that a chamber C is *k-bounded* if it intersects W_k non-trivially, but does not intersect W_{k-1}. Verify in the examples above the following statements:

- the number of k-bounded chambers is equal to the absolute value of the coefficient of t^{d-k} in $\bar{\chi}_M(t)$;
- the canonical forms of the k-bounded chambers intersected with W_k (viewed as chambers of $\bar{\mathcal{A}}|_{W_k}$) form a basis of $\bar{A}^k(M)$. (Implicit in this statement is that the canonical forms of chambers of $\bar{\mathcal{A}}|_{W_k}$ can be viewed as elements of $\bar{A}(\bar{\mathcal{A}})$).

4. Find an explicit formula for $\bar{\Omega}(C)$ as an element of $\bar{A}^\bullet(M)$. Avoid triangulating C as this will introduce new hyperplanes that do not correspond to any element of $\bar{A}^\bullet(M)$. (Hint: if you know about triangulations of oriented matroids, that should help!)

5. Let $\mathcal{A}$ be a central hyperplane arrangement and let $\mathcal{A}|_H$ be the restriction of $\mathcal{A}$ to the hyperplane $H \in \mathcal{A}$. Show that there is a well-defined residue map

$$\text{Res}_H : A(\mathcal{A}) \to A(\mathcal{A}|_H)$$

given by

$$e_1 \wedge \cdots \wedge e_k \longmapsto \begin{cases} e_1 \wedge \cdots \wedge e_{k-1} & \text{if } H = H_{e_k}, \\ 0 & \text{if none of the } e_i \text{ correspond to } H. \end{cases}$$

6. (Assumes knowledge of *broken circuits*.) Give simple formulae for the transition matrices between **nbc**-bases and canonical form bases.

7. Let $\omega = \sum_{e \in E} a_e e \in \bar{A}^1(M)$ be a generic degree one element of the Orlik-Solomon algebra. The *Aomoto complex* is the complex $(\bar{A}^\bullet(M), \omega)$:

$$0 \to \bar{A}^0(M) \xrightarrow{\omega\wedge} \bar{A}^1(M) \xrightarrow{\omega\wedge} \bar{A}^2(M) \xrightarrow{\omega\wedge} \bar{A}^3(M) \cdots$$

Express the action of ω in the canonical form basis of $\bar{A}^\bullet(M)$; see problem 3. (cf. Yoshinaga `arXiv:math/0703733`.)

8. Open: What are the multiplicative structure constants of $A^\bullet(M)$ with respect to the canonical form basis?