Positive geometry exercises: Polytopes

1. Pick a lattice quadrilateral P, for example, $\mathrm{conv}((0,0),(2,0),(2,1),(1,2))$. Verify some basic results about $\Omega(P)$:

   (1) Compute the normal fan of P and write down the dual volume function.
   (2) Compute the residues of $\Omega(P)$ along each of the facet lines.
   (3) Check that $\Omega(P) = \Omega(T_1) + \Omega(T_2)$, where T_1, T_2 is obtained by triangulating P via a diagonal.
   (4) Prove the integral formula for dual volume function:

$$\mathrm{Vol}_{\mathbf{z}}^{\vee}(P) = \int_{\mathbb{R}^d} \exp(-h_P(\mathbf{v}) - \langle \mathbf{v}, \mathbf{z} \rangle) d\mathbf{v}.$$

   (5) Check that the numerator of $\Omega(P)$ vanishes on the residual arrangement.
   (6) Verify that $\Omega(P)$ takes constant sign in the interior of P.

2. The *dual mixed volume* of polytopes $\mathbf{P} = (P_1, P_2, \dots, P_r)$ is the rational function

$$m_{\mathbf{P}}(\mathbf{x}) := \mathrm{Vol}^{\vee}(x_1 P_1 + x_2 P_2 + \cdots + x_r P_r).$$

   (1) Show that the dual volume function $\mathrm{Vol}_{\mathbf{z}}^{\vee}$ is a special case (of a specialization) of the dual mixed volume.
   (2) Show that the dual mixed volume, when it is defined, is a rational function. What is its degree?
   (3) Open: What can you say about dual mixed volumes for convex bodies that are not polytopes?

3. Consider the vector space K_n spanned by symbols s_{ij}, $1 \le i, j \le n$, modulo the relations

$$s_{ij} = s_{ji}, \qquad s_{ii} = 0, \qquad \sum_{j=1}^{n} s_{ij} = 0, \text{ for } i = 1, 2, \dots, n.$$

Let

$$X_{i,j} := \sum_{i \le a < b \le j-1} s_{a,b}$$

as (i,j) vary over the $\binom{n}{2} - n$ diagonals of the n-gon. Show that $X_{i,j}$ form a basis of K_n and that $\dim(K_n) = \binom{n}{2} - n$. Show that $\sum_{1 \le a \le b \le n-2} s_{a,b+1} = 0$.

4. Verify the relation between the dual mixed volume of the associahedron

$$P(s) = \sum_{1 \le a \le b \le n-2} s_{a,b+1} \Delta_{[a,b]}, \qquad \Delta_{[a,b]} := \mathrm{conv}(e_a, e_{a+1}, \dots, e_b)$$

and ϕ^3-amplitudes:

$$\mathrm{Vol}(P(s)^{\vee}) = A_n^{\phi^3}(s),$$

in the $n = 5$ case, where the $s_{i,j}$ satisfy the relations from Problem 3; that is,

$$s_{12} + s_{13} + s_{14} + s_{23} + s_{24} + s_{34} = 0.$$

The definition of the associahedron is a codimension one polytope lying in the hyperplane $\sum_i x_i = \sum_{1 \le a \le b \le n-2} s_{a,b+1}$.

5. Open: What can we say about the support of the adjoint polynomial, or more generally, the numerator of the dual mixed volume rational function? What is the Newton polytope of this numerator? When are the coefficients positive?

6. (a) Recall that the Brunn-Minkowski inequality states that

$$\mathrm{Vol}((A+B)/2)^2 \geq \mathrm{Vol}(A)\mathrm{Vol}(B), \text{ for non-empty compact sets } A, B.$$

Prove the "dual" inequality

$$\mathrm{Vol}^\vee((P+Q)/2)^2 \leq \mathrm{Vol}^\vee(P)\mathrm{Vol}^\vee(Q)$$

for polytopes P, Q containing the origin in the interior. Here, $\mathrm{Vol}^\vee := \mathrm{Vol}^\vee_{\mathbf{z}=0}$.
(b) Open: What is the correct *dual* analogue of the Alexandrov-Finchel inequalities?

7. Let C be a polyhedral cone in $\mathbb{R}^{d+1}$ with extremal rays $V(C)$ and let $\mathcal{T}$ denote a triangulation of C using the same set of extremal rays. The adjoint polynomial of C is given by

$$\mathrm{adj}_C(\mathbf{z}) = \sum_{F\in\mathcal{T}} |\det(F)| \prod_{\mathbf{v}\in V(C)\setminus V(F)} \langle \mathbf{v}, \mathbf{z}\rangle,$$

where if F is generated by $\mathbf{v}_1, \ldots, \mathbf{v}_{d+1}$ then $\det(F) = \det(\mathbf{v}_1, \ldots, \mathbf{v}_{d+1})$. The adjoint polynomial of the cone $\mathrm{cone}(P)^*$ is the homogenization of the numerator of $\mathrm{Vol}_{\mathbf{z}}(P)$.

Explore the following theorem of Aluffi and Li: "the adjoint polynomial of a cone in the positive orthant is a covolume polynomial." In particular, such adjoint polynomials $A(z_0, z_1, z_2, \ldots, z_n)$ are *sectional log-concave*: after the substitution $z_i = a_i x + b_i y$ we obtain a two-variable polynomial with log-concave coefficients. How is this related to the positive geometry of polytopes?