Some Facts Concerning Algebraic Relative Interior and Algebraic Closure

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1. Definition

Consider any topological vector space. Interior and closure of a set are typically defined using the topological structure, but they can also be defined using the algebraic structures instead. In the context of finite-dimensional Euclidean spaces, if attention is restricted to convex sets, the two approaches coincide.[1] But in infinite dimensional topological vector spaces they don't. Therefore, the following results are mostly of interest if the underlying vector space is infinite dimensional.

In the context of the interior, the *relative* interior is often of most interest. Therefore, we focus on the relative interior. The following definitions are taken from (Ok, 2007, p. 438 and p. 448).

Definition 1. *The "algebraic relative interior" of a subset S of a vector space X is defined as:*

$$ri(S) = \{x \in S | \forall y \in \text{aff}(S)\ \exists \bar{\alpha} \in (0,1)\ \forall \alpha \in [0, \bar{\alpha}] :\ (1-\alpha)x + \alpha y \in S\}.$$

Definition 2. *The "algebraic closure" of a subset S of a vector space X is defined as:*

$$cl(S) = \{x \in X | \exists y \in S\ \forall \alpha \in (0,1] :\ (1-\alpha)x + \alpha y \in S\}$$

Some of our results below extend results in (Rockafellar, 1970) where they are proved for finite-dimensional Euclidean spaces, using the topological notions of interior and closure. We prove versions of these results for potentially infinite dimensional vector spaces, using the algebraic notions of interior and closure. Specifically, Lemma

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[1]See (Ok, 2007, p. 442 and p.450).

2 extends Theorem 6.1, Proposition 4 extends Theorem 6.3, Corollary 1 extends Corollary 6.3.1, and Propositions 5 and 6 extend Theorem 6.6 in (Rockafellar, 1970). Our proofs sometimes incorporate ideas from Rockafellar's proofs.

2. Basic Properties

Lemma 1. *For all subsets S, S_1, S_2 of a vector space X:*

(a) $ri(S) \subseteq S \subseteq cl(S) \subseteq \text{aff}(S)$.
(b) If $S_1 \subseteq S_2$ *and* $\text{aff}(S_1) = \text{aff}(S_2)$ *then* $ri(S_1) \subseteq ri(S_2)$.
(c) If $S_1 \subseteq S_2$ *then* $cl(S_1) \subseteq cl(S_2)$.
(d) $\text{aff}(ri(S)) \subseteq \text{aff}(S) \subseteq \text{aff}(cl(S))$.
(e) $\text{aff}(cl(S)) \subseteq \text{aff}(S))$.
(f) If S is convex and $ri(S) \neq \emptyset$ *then* $\text{aff}(S) \subseteq \text{aff}(ri(S))$.

Remark: The condition $\text{aff}(S_1) = \text{aff}(S_2)$ in (b) cannot be dropped. For example, in $\mathbb{R}^2$, $ri([0,1] \times \{0\}) = (0,1) \times \{0\}$ but $ri([0,1]^2) = (0,1)^2$. In (f) we do not know whether the condition that S is convex can be dropped.

We offer proofs only for those parts of Lemma 1 for which we suspect that the proofs might not be obvious.

Proof. (a) Proof of $cl(S) \subseteq \text{aff}(S)$: If $x \in cl(S)$ then either $x \in S$, in which case $x \in \text{aff}(S)$ is obvious, or $x \notin S$ and there is some $y \in S$ such that: $z \equiv 0.5x + 0.5y \in S$. By the definition of affine hull: $-y + 2z \in \text{aff}(S)$, and notice that $-y + 2z = x$.

(f) Consider any $x \in S$. We are going to argue that x can be written as an affine combination of elements of $ri(S)$. This obviously implies the claim. We are going to use Lemma 2 below. By Lemma 2, $(1-\alpha)x + \alpha y \in ri(S)$ for all $y \in ri(S)$ and all $0 < \alpha \leq 1$. Because we have assumed $ri(S)$ is non-empty, at least one $y \in ri(S)$ exists. Then, $z \equiv 0.5x + 0.5y \in ri(S)$. But note that $x = -y + 2z$, and thus x is an affine combination of y and z.

□

3. Algebraic Relative Interior and Algebraic Closure Of Convex Sets

Proposition 1. *For all subsets S of a vector space X:*

(a) If S is convex, then $ri(S)$ is convex.
(b) If S is convex, then $cl(S)$ is convex.

Proof. (a) Consider $x, y \in ri\,(S)$ and $z \in \text{aff}\,(S)$. Then there exist $\alpha_x \in (0,1)$ such that for all $\alpha \in [0, \alpha_x]$, $(1-\alpha)\,x + \alpha z \in S$ and $\alpha_y \in (0,1)$ such that for all $\alpha \in [0, \alpha_y]$, $(1-\alpha)\,y + \alpha z \in S$. Now consider $\lambda x + (1-\lambda)\,y$ for any $\lambda \in [0,1]$. Note that:

$$\begin{aligned} & (1-\alpha)\,[\lambda x + (1-\lambda)\,y] + \alpha z \\ = \; & \lambda\,[(1-\alpha)\,x + \alpha z] + (1-\lambda)\,[(1-\alpha)\,y + \alpha z]\,. \end{aligned}$$

For all $\alpha \in [0, \min\{\alpha_x, \alpha_y\}]$, $(1-\alpha)\,x + \alpha z \in S$ and $(1-\alpha)\,y + \alpha z \in S$. The convexity of S implies $(1-\alpha)\,[\lambda x + (1-\lambda)\,y] + \alpha z \in S$. So $\lambda x + (1-\lambda)\,y \in ri\,(S)$.

(b) Let $x, y \in cl\,(S)$ and $\lambda \in [0,1]$. We want to prove that $\lambda x + (1-\lambda) y \in cl(S)$, that is, that there there exists $z \in S$ such that $(1-\alpha)[\lambda x + (1-\lambda)y] + \alpha z \in S$ for all $\alpha \in (0,1]$. Because $x, y \in cl\,(S)$ there exist $z_x, z_y \in S$ such that $(1-\alpha)x + \alpha z_x \in S$ and $(1-\alpha)y + \alpha z_y \in S$ for all $\alpha \in (0,1]$. We set $z = \lambda z_x + (1-\lambda) z_y$. Note that:

$$\begin{aligned} & (1-\alpha)[\lambda x + (1-\lambda)y] + \alpha z \\ = \; & (1-\alpha)[\lambda x + (1-\lambda)y] + \alpha[\lambda z_x + (1-\lambda)z_y] \\ = \; & \lambda[(1-\alpha)x + \alpha z_x] + (1-\lambda)[(1-\alpha)y + \alpha z_y] \end{aligned}$$

and this is contained in S for all $\alpha \in (0,1]$ because S is convex.

□

Lemma 2. *If S is a convex subset of a vector space X, and if $y \in cl\,(S)$, then $(1-\lambda)\,x + \lambda y \in ri\,(S)$ for all $x \in ri(S)$ and all $0 \le \lambda < 1$.*

Proof. Step 1: We first show $(1-\lambda)\,x + \lambda y \in S$. $y \in cl\,(S)$ means that there exists $x_1 \in S$ such that $(1-\alpha)y + \alpha x_1 \in S$ for all $\alpha \in (0,1]$. Also, $x \in ri\,(S)$ implies that $(1-\alpha)\,x + \alpha\,(2x - x_1) \in S$ for all $\alpha \in [0, \bar{\alpha}]$ for some $\bar{\alpha} \in (0,1)$. This holds because

$2x - x_1 \in \text{aff}(S)$. Now pick any $\alpha \in (0, \bar{\alpha})$ and note that $(1-\lambda)x + \lambda y$ can be written as the convex combination:

$$\frac{\lambda+\alpha}{1+\alpha}\left\{\left[1-\frac{(1-\lambda)\,\alpha}{\lambda+\alpha}\right]y+\frac{(1-\lambda)\,\alpha}{\lambda+\alpha}x_1\right\}$$
$$+\left(1-\frac{\lambda+\alpha}{1+\alpha}\right)\left\{(1-\alpha)\,x+\alpha\,(2x-x_1)\right\}$$

The two vectors in curled brackets are contained in S. By the convexity of S, therefore also their convex combination is contained in S. Hence, $(1-\lambda)\,x + \lambda y \in S$.

Step 2: We next show $(1-\lambda)\,x + \lambda y \in ri(S)$. Let $z \in \text{aff}(S)$. We have to show that there exists $\bar{\alpha} \in (0,1)$ such that for all $\alpha \in [0, \bar{\alpha}]$:

$$(1-\alpha)\,[(1-\lambda)\,x+\lambda y]+\alpha z \in S.$$

$y \in cl\,(S)$ means that there exists $x_1 \in S$ such that $(1-\alpha)y + \alpha x_1 \in S$ for all $\alpha \in (0,1]$. $x \in ri\,(S)$ implies that there exists $\bar{\alpha} \in (0,1)$ such that for all $\alpha \in (0, \bar{\alpha}]$,

$$(1-\alpha)\,x+\alpha\left[-\frac{\lambda}{1-\lambda}x_1+\frac{1}{1-\lambda}z\right]\in S.$$

This is because $-\frac{\lambda}{1-\lambda}x_1 + \frac{1}{1-\lambda}z \in \text{aff}(S)$. Now pick any $\alpha \in (0, \bar{\alpha})$ and note that $(1-\alpha)\,[(1-\lambda)\,x + \lambda y] + \alpha z$ can be written as the convex combination:

$$\lambda\left\{(1-\alpha)y+\alpha x_1\right\}+(1-\lambda)\left\{(1-\alpha)\,x+\alpha\left[-\frac{\lambda}{1-\lambda}x_1+\frac{1}{1-\lambda}z\right]\right\}.$$

The two vectors in curled brackets are contained in S. By the convexity of S, therefore also their convex combination is contained in S. Hence, $(1-\alpha)\,[(1-\lambda)\,x + \lambda y] + \alpha z \in S$. □

4. Iterating Operators

Proposition 2. *If S is a convex subset of a vector space X then $ri(ri(S)) = ri(S)$.*

Remark: The condition that S be convex can not be dropped from this Proposition. The set S in Figure 6 on page 441 in Ok (2007) is a counterexample. The origin is in $ri(S)$ but not in $ri(ri(S))$, because the affine hull of $ri(S)$ is in this example $\mathbb{R}^2$, but for $x = (0,0)$ and $y = (1,0)$ there is no $\bar{\alpha} > 0$ such that $(1-\alpha)x + \alpha(1,0) \in ri(S)$. This point is also made in Exercise 22 in Ok (2007).

Proof. Step 1: First we prove $ri\,(riS) \subseteq ri(S)$. The claim is obviously true if $ri(S)$ is empty. Therefore, we now assume that $ri(S)$ is not empty. $x \in ri\,(ri(S))$ implies that for every $y \in \text{aff}\,(ri(S))$, there exists $\bar{\alpha} \in (0,1)$ such that for all $\alpha \in [0, \bar{\alpha}]$:

$$(1-\alpha)\,x + \alpha y \in ri(S).$$

Recall from part (f) in Lemma 1 that convexity of S and non-emptiness of $ri\,(S)$ imply: $\text{aff}\,(S) \subseteq \text{aff}\,(ri(S))$ and that according to part (a) in Lemma 1: $ri(S) \subseteq S$. Therefore we can conclude that for every $y \in \text{aff}\,(S)$ there exists $\bar{\alpha} \in (0,1)$ such that for all $\alpha \in [0, \bar{\alpha}]$:

$$(1-\alpha)\,x + \alpha y \in S,$$

and hence $x \in ri(S)$.

Step 2: Next we prove $ri(S) \subseteq ri\,(ri(S))$.The claim is obviously true if $ri(S)$ is empty. Therefore, we now assume that $ri(S)$ is not empty. $x \in ri\,(S)$ implies that for every $y \in \text{aff}\,(S)$ there exists $\bar{\alpha} \in (0,1)$ such that for all $\alpha \in [0, \bar{\alpha}]$:

$$(1-\alpha)\,x + \alpha y \in S.$$

Because S is convex, we can appeal to Lemma 2 and conclude that for all λ with $0 < \lambda \leq 1$ and for all $\alpha \in [0, \bar{\alpha}]$:

$$(1-\lambda)[(1-\alpha)\,x + \alpha y] + \lambda x \in ri(S).$$

But this means that for all $\alpha \in (0, \bar{\alpha})$:

$$(1-\alpha)\,x + \alpha y \in ri(S).$$

This argument applies to all $y \in \text{aff}\,(S)$. But now recall part (d) in Lemma 1 implies: $\text{aff}\,(ri(S)) \subseteq \text{aff}\,(S))$. So $x \in ri\,(ri(S))$. ☐

Proposition 3. *If S is a convex subset of a vector space X and $ri(S) \neq \emptyset$ then $cl(cl(S)) = cl(S)$.*

Remark: The condition that S be convex can not be dropped from this Proposition. Here is a counterexample.

$$S = (\mathbb{Q} \setminus \{0\} \times (0, +\infty)) \cup (\mathbb{R} \setminus \mathbb{Q} \times (-\infty, 0))\,.$$

On the one hand, $(0,0) \notin cl\,(S)$. On the other hand, for all $t \neq 0$, $(t,0) \in cl\,(S)$ and thus $(0,0) \in cl\,(cl\,(S))$.

Proof. Parts (a) and (c) of Lemma 1 imply that for any set S we have: $cl(S) \subseteq cl(cl(S))$. It remains to show that $x \in cl(cl(S))$ implies $x \in cl(S)$. Since $ri\,(S) \neq \emptyset$, take any $y \in ri\,(S) \subseteq S$. By Proposition 4 below we have: $ri(S) = ri(cl(S))$. So $y \in ri(cl(S))$. Because S is convex, we know from part (b) in Proposition 1 that $cl(S)$ is convex. Applying Lemma 2 to $cl\,(S)$, we get $(1-\lambda)y + \lambda x \in ri(cl(S)) = ri\,(S) \subseteq S$ for all $0 \leq \lambda < 1$. So $x \in cl\,(S)$. □

5. Swapping the Order of Operators

Proposition 4. *If S is a convex subset of a vector space X and $ri(S) \neq \emptyset$ then $cl(ri(S)) = cl(S)$ and $ri(cl(S)) = ri(S)$.*

Proof. Step 1: $cl\,(ri\,(S)) \subseteq cl\,(S)$ follows directly from (a) and (c) in Lemma 1. To prove $cl\,(S) \subseteq cl\,(ri\,(S))$ we consider any $x \in cl\,(S)$. Since $ri\,(S) \neq \emptyset$, take any $y \in ri\,(S)$. Then Lemma 2 implies$(1-\lambda)y + \lambda x \in ri\,(S)$ for all $0 \leq \lambda < 1$ and thus $x \in cl\,(ri\,(S))$.

Step 2: $ri\,(S) \subseteq ri\,(cl\,(S))$ follows directly from (b), (d) and (e) in Lemma 1. It remains to show: $ri\,(cl\,(S)) \subseteq ri\,(S)$. Suppose $x \in ri\,(cl\,(S))$. Since $ri\,(S) \neq \emptyset$, take $y \in ri\,(S)$. If $x = y$, then the claim holds trivially. So we focus on the case where $x \neq y$. Note that $x \in cl\,(S)$ and $y \in cl\,(S)$. So $2x - y \in \text{aff}\,(cl\,(S))$. $x \in ri\,(cl\,(S))$ implies that there exists $\bar{\alpha} \in (0,1)$ such that for all $\alpha \in [0, \bar{\alpha}]$,

$$x + \alpha\,(x-y) = (1-\alpha)\,x + \alpha\,(2x-y) \in cl\,(S)\,.$$

Pick any $\alpha \in (0, \bar{\alpha})$. Note that

$$x = \frac{1}{1+\alpha}\left[x + \alpha\,(x-y)\right] + \left(1 - \frac{1}{1+\alpha}\right) y.$$

Lemma 2 then implies $x \in ri\,(S)$. □

Corollary 1. *Let S_1 and S_2 be two subsets of a vector space X and suppose $ri(S_1) \neq \emptyset$ and $ri(S_2) \neq \emptyset$. Then $ri(S_1) = ri(S_2)$ if and only if $cl(S_1) = cl(S_2)$.*

Proof. If $ri(S_1) = ri(S_2)$, we have: $cl(ri(S_1)) = cl(ri(S_2))$, and therefore, by Lemma 4, $cl(S_1) = cl(S_2)$. If $cl(S_1) = cl(S_2)$, we have: $ri(cl(S_1)) = ri(cl(S_2))$, and therefore, by Lemma 4, $ri(S_1) = ri(S_2)$. □

6. 6. Linear Mappings

In this section we fix two vector spaces X and Y and a linear mapping $f : X \to Y$.

Proposition 5. *For all sets $S \subseteq X$: $f(cl(S)) \subseteq cl(f(S))$.*

Proof. Suppose $y \in f(cl(S))$. We need to show $y \in cl(f(S))$. Note that $y \in f(cl(S))$ means that there is some $x \in cl(S)$ for which $y = f(x)$. Moreover, $x \in cl(S))$ means that there is some $z \in S$ such that $\lambda z + (1-\lambda)x \in S$ for all $\lambda \in (0, 1]$. Because f is linear, we obtain: $\lambda f(z) + (1-\lambda)f(x) \in f(S)$ for all $\lambda \in (0, 1]$. Because $f(z) \in f(S)$ this means that $f(x) \in cl(f(S))$, which is what we had to prove. □

Proposition 6. *For all convex sets $S \subseteq X$ with $ri(S) \neq \emptyset$: $f(ri(S)) = ri(f(S))$.*

Proof. We first prove that $y \in f(ri(S))$ implies $y \in ri(f(S))$. Note that $y \in f(ri(S))$ means that there is some $x \in ri(S)$ for which $y = f(x)$. Moreover, $x \in ri(S)$ means that for every $z \in \text{aff}(S)$ there is some $\bar{\alpha} \in (0, 1)$ such that for all $\alpha \in [0, \bar{\alpha}]$: $(1-\alpha)x + \alpha z \in S$. Because f is linear, we obtain: $(1-\alpha)f(x) + \alpha f(z) \in f(S)$. Now $f(z) \in f(\text{aff}(S))$ (by definition) and $f(\text{aff}(S)) = \text{aff}(f(S))$ (which one can easily verify) imply $f(z) \in \text{aff}(f(S))$. Thus, we can conclude $f(x) \in ri(f(S))$.

Next we prove: $ri(f(S)) \subseteq f(ri(S))$. A slightly stronger claim is: $ri(f(S)) = ri\left(f(ri(S))\right)$. This is what we shall show. In the argument that follows, we employ Corollary 1 as well as Propositions 3, 4, and 5. We postpone checking the assumptions of these results until the end of the proof.

By Corollary 1 what we have to show is equivalent to: $cl(f(S)) = cl\left(f(ri(S))\right)$. Now $cl\left(f(ri(S))\right) \subseteq cl(f(S))$ is obvious because $f(ri(S)) \subseteq f(S)$. It thus remains to show that $cl(f(S)) \subseteq cl\left(f(ri(S))\right)$. By Proposition 3 this is equivalent to: $cl(f(S)) \subseteq cl(cl\left(f(ri(S))\right))$. This follows from: $f(S) \subseteq cl\left(f(ri(S))\right)$. By Proposition 5 this is implied by: $f(S) \subseteq f(cl(ri(S)))$. By Proposition 4 this is the same as: $f(S) \subseteq f(cl(S))$, which is obviously true.

We now verify that the assumptions of Corollary 1 and Propositions 3, 4 and 5 hold. This amounts to verifying that the sets $f(S)$ and $f(ri(S))$ are both convex and have a non-empty relative interior. Convexity of the two sets follows from the convexity of S and $ri(S)$ (by part (a) in Proposition 1) and the linearity of f. To prove $ri(f(S)) \neq \emptyset$ we note that we showed in the first step of this proof that $f(ri(S)) \subseteq ri(f(S))$. By assumption $ri(S) \neq 0$. Therefore, $f(ri(S)) \neq \emptyset$, and the claim follows. To show that $ri(f(ri(S))) \neq \emptyset$ we note again that we showed in the first step of this proof that $f(ri(ri(S))) \subseteq ri(f(ri(S)))$. By Lemma 2: $f(ri(ri(S))) = f(ri(S))$, and this is non-empty because $ri(S) \neq \emptyset$, by assumption. □

References

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