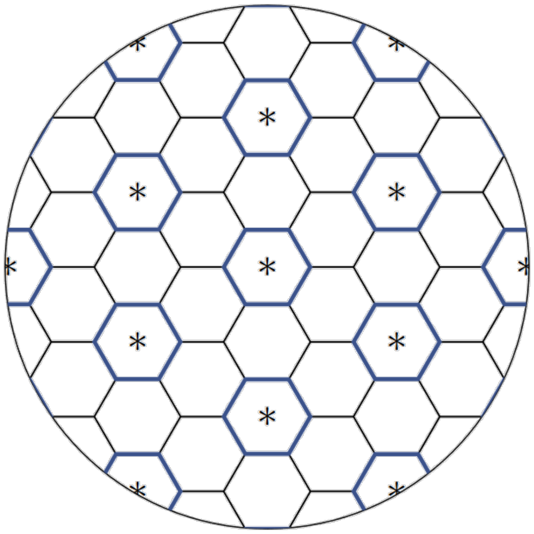
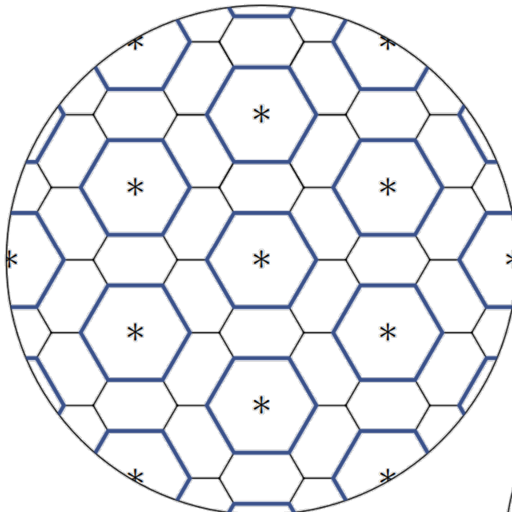
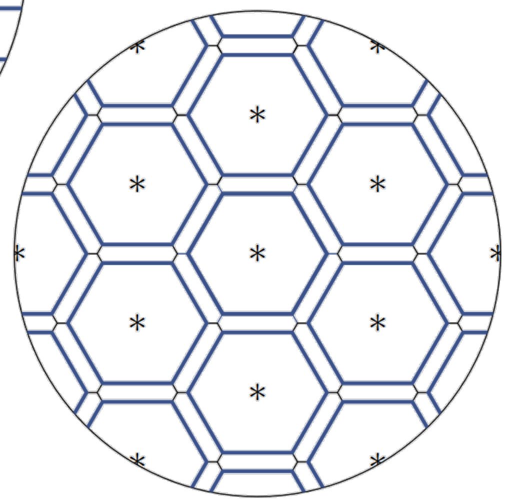
The squish map, the $SL_2(\mathbb{C})$ double dimer model, and stone-bone-snake tilings.





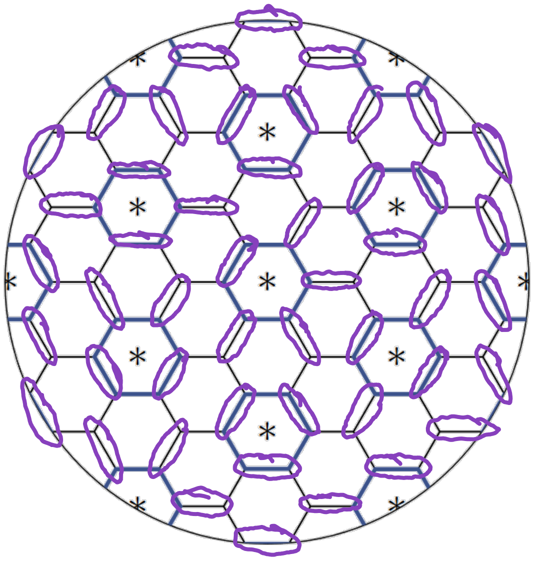


Leigh Foster (UO, soon Waterloo)

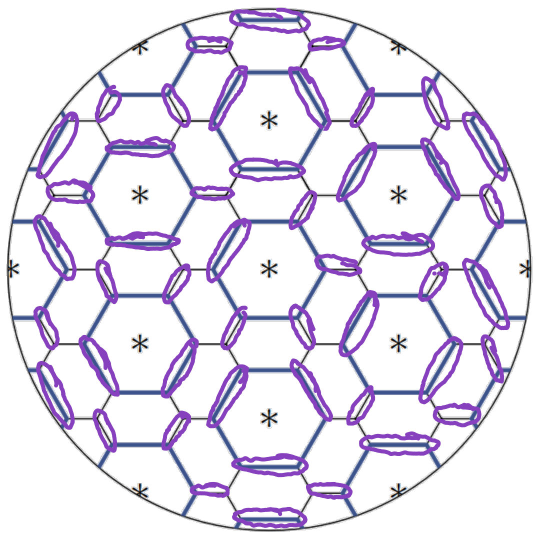
Ben Young (UO)

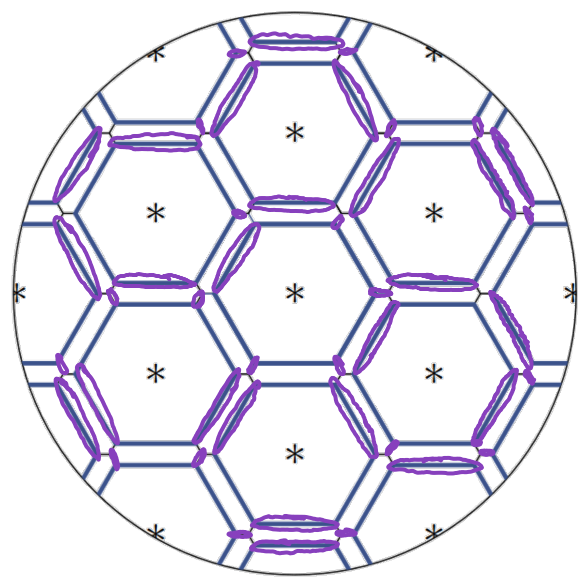
JJM - Statistical & Dynamical Combinatorics - MIT, 2024

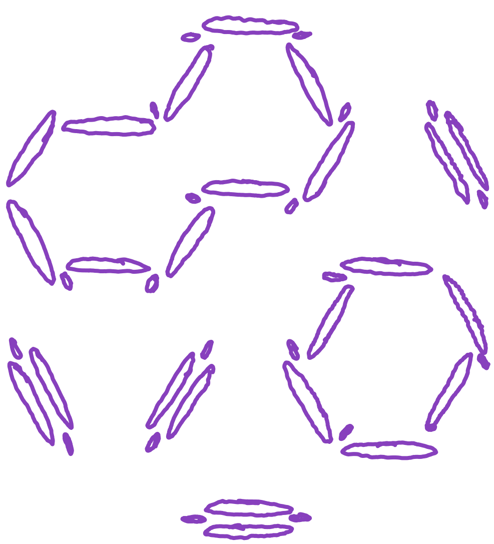
Squish map sends:



dimer configs / perfect matchings on the hex graph . . .

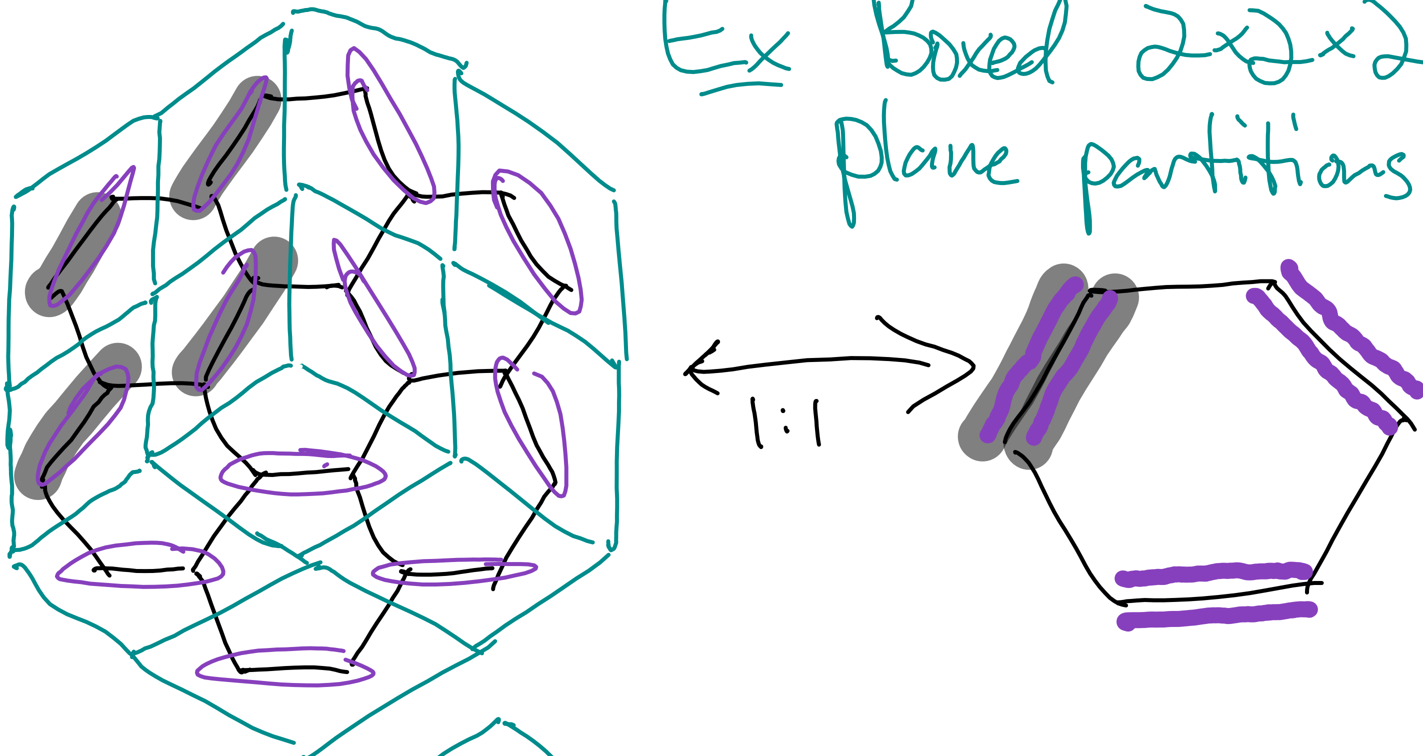


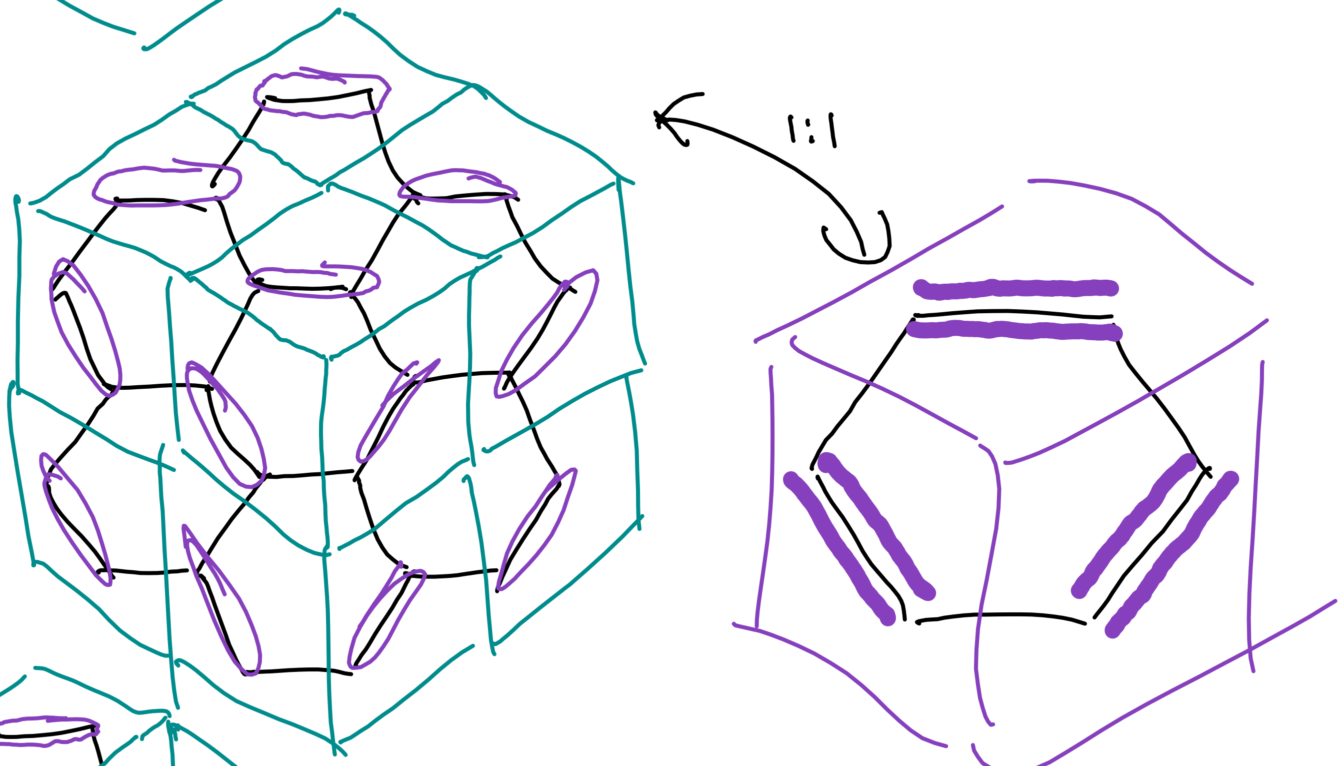




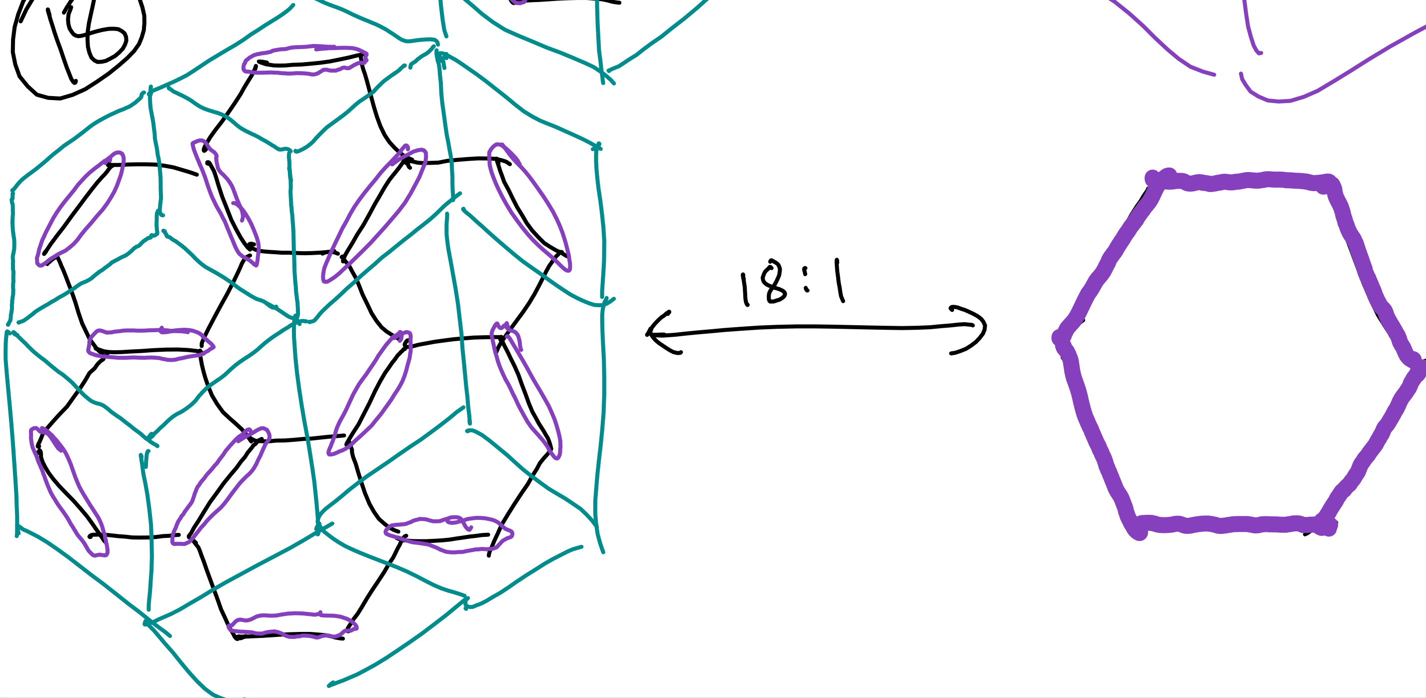
. . . to double dimer configs on a coarser hex graph.

Ex Boxed $2\times2\times2$ plane partitions



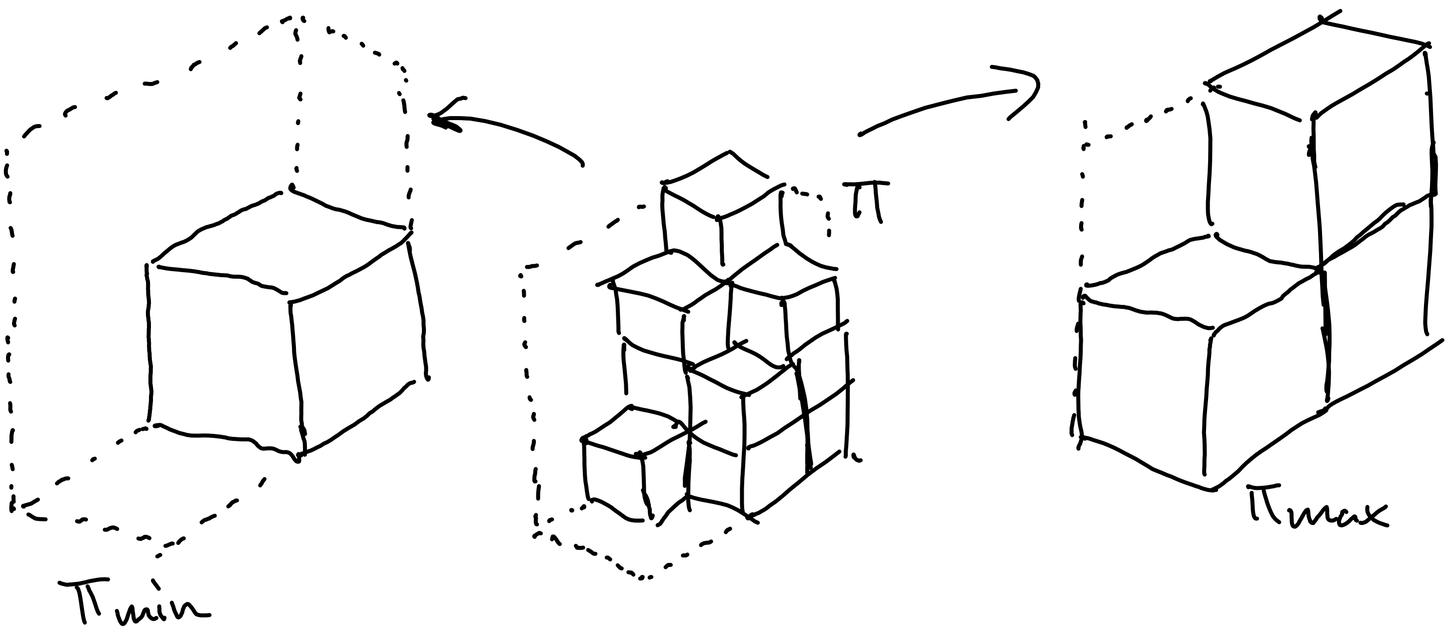


18

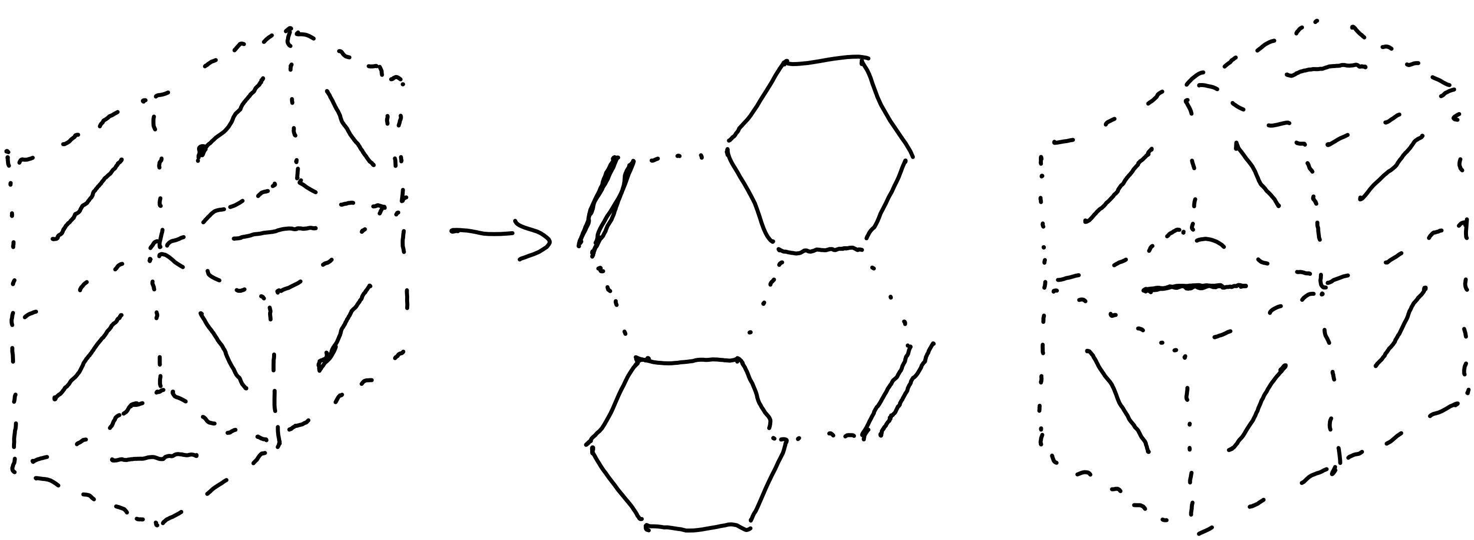


In general: Plane partition π is made with unit cubes.

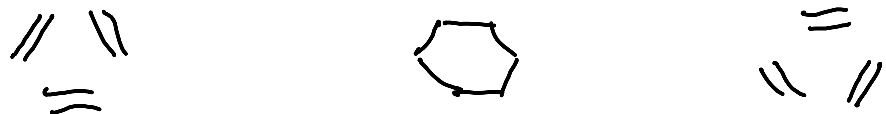
Approximate it with π_{min}, π_{max} — two plane partitions made with $2\times2\times2$ cubes and $\pi_{min} \subseteq \pi \subseteq \pi_{max}$ ...



... then superimpose the dimer configs.



— Consider first uniform measure on boxed $2\times2\times2$ plane partitions. Under squish map:



$1/20$ $\qquad$ $18/20$ $\qquad$ $1/20$

Technically this could be realized as a trivial double dimer model, with weird weights.

Any larger example: get some nontrivial double dimer model.

Thm (Foster–Y. 2024) it is an instance of the $SL_2(\mathbb{C})$ double dimer model. (Kenyon 2014)

(Kasteleyn 1961)
Kasteleyn theory (single dimer model)

$G = (V,E)$ planar, bipartite

weight: $w: E \to \mathbb{C}$

There's a matrix K (variant of adjacency matrix) such that

$$\det K = \pm \sum_{\pi} w(\pi)$$

Moreover: • K^{-1} encodes correlation functions of the model (Kenyon, Fisher)

• If G, w are particularly nice: one can extract limit shapes and universal asymptotics

$SL_2(\mathbb{C})$ double dimer model (Kenyon 2014).

On each edge of the graph $G=(V,E)$ (planar, bipartite):

- a weight, $w: E \to \mathbb{C}$
- a connection $\Gamma: E \to SL_2(\mathbb{C})$

Given a double dimer config π its measure is

$$\frac{1}{Z} \prod_{e \in \pi} w(e) \prod_{C} \operatorname{tr}\Big(\overrightarrow{\prod_{e \in C}} \Gamma(e) \Big)$$

taken over loops in π

Thm (Kenyon) $Z = \text{det } K$

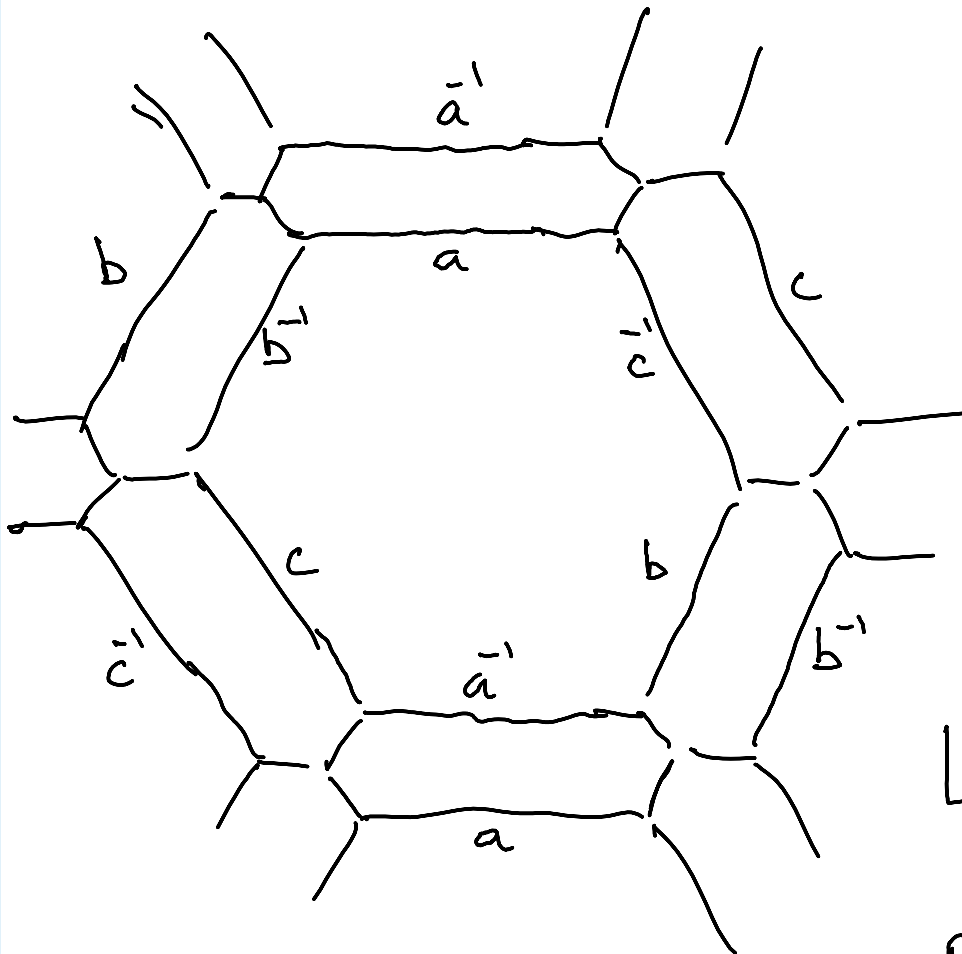
- K is a matrix with entries in $SL_2(\mathbb{C})$
- "det" is "Quaternion Determinant"

  - less is true about it.
    e.g. the thing with K^{-1} does not work, as far as I know?
    I am currently confused about this point.

- use Squish map to learn about correlations in (some of) $SL_2(\mathbb{C})$ double dimer model.

- Analogy: "Free Fermion line" in parameter space of 6 vertex model.

Idea of proof — 2-periodic weights



$$A = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$$

sim. B, C

$$L = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

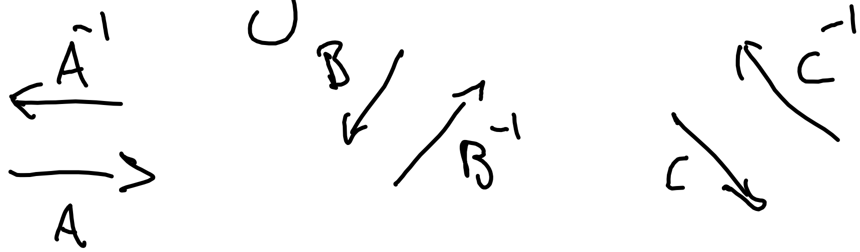
$$R = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Given a loop in the double-dimer model (ex. the boundary of a single hexagon) compute the weight of all single dimer configs that squish to it by

$$\operatorname{tr}\left(L C^{-1} L B L A^{-1} L C L B^{-1} L A\right)$$

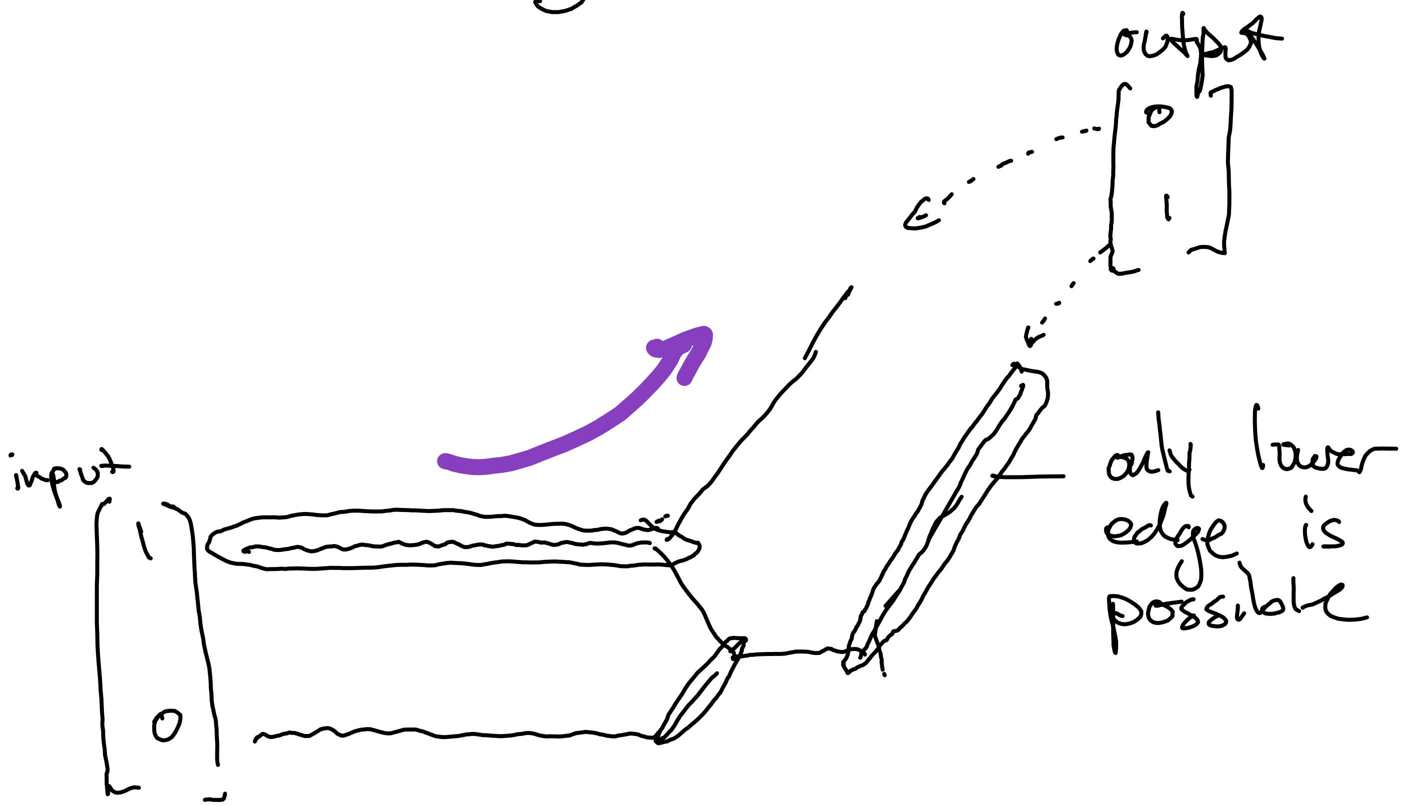
That is: traverse the loop

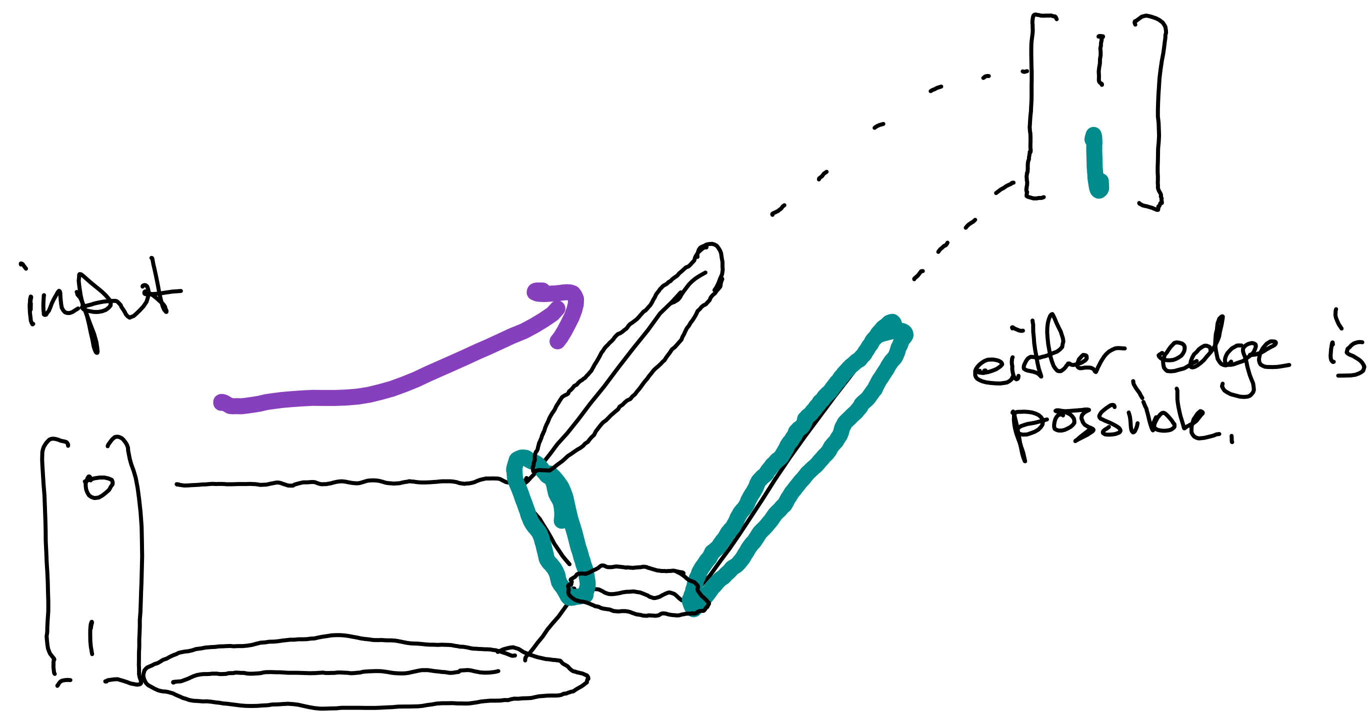
write down edge matrices:



and $\{L$ for left turns$\}$
$\{R$ for right turns$\}$

Note: L is the transfer matrix for turning left



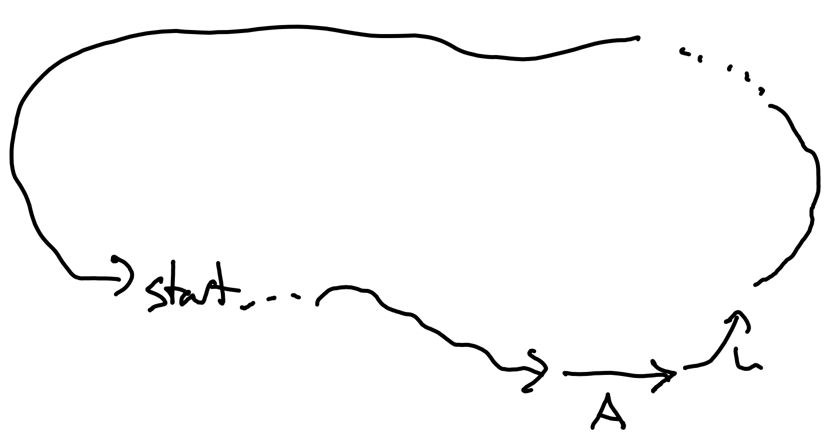


i.e. if $\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$ is total weight of all configs so far along the loop,

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}_{L}\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_0 + x_1 \end{bmatrix}$$

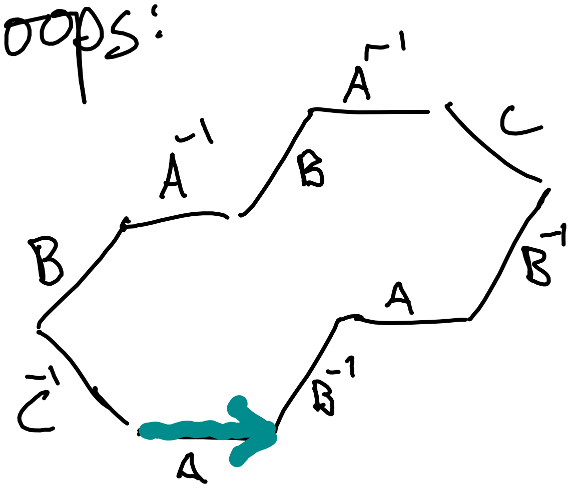
is the weight after 1 more step.

taking the trace is asking to start and end on the same edge.



Check: $\mathrm{tr}(L^6) = 18.$ $(a=b=c=1)$

Other loops:



$$\mathrm{tr}\left(L C^{-1} L B L A^{-1} R B L A^{-1} L C L B^{-1} L A R B^{-1} L A\right)$$

etc.

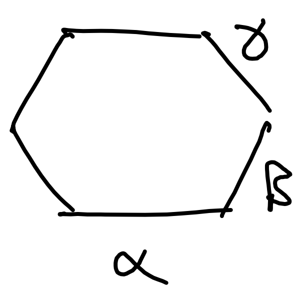
Aim: "Move" the L, R matrices on to the edges; ensure all edge matrices have det. $\underline{1}$

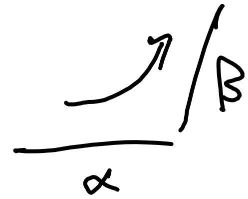
Do this by setting: $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\alpha = i A J,$$

$$\beta = i R B^{-1} L J$$

$$\gamma = -i L C R J$$

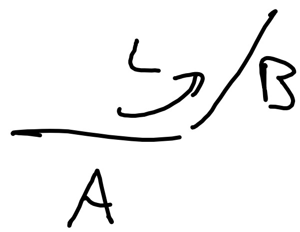


So for instance: 

$$\cdots\ \beta\,\alpha\ \cdots = \cdots\ (iRBLJ)(iAJ)\ \cdots$$

$$= -\ \cdots\ BLJ^2A\ \cdots$$

$$= -\ \cdots\ BLA\ \cdots$$



and the overall sign dies, as the path is even-length.

"Why" it works, loosely:

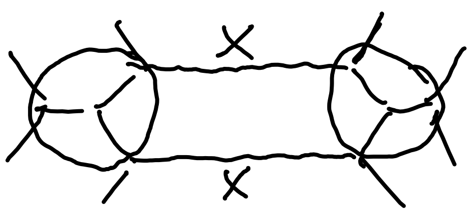
$$(LJ)^3 = \begin{bmatrix} & -1 \\ 1 & -1 \end{bmatrix}^3 = -I = (RJ)^3$$

so, with J in there, 2 lefts $\approx$ 1 right, just as on the hex graph.

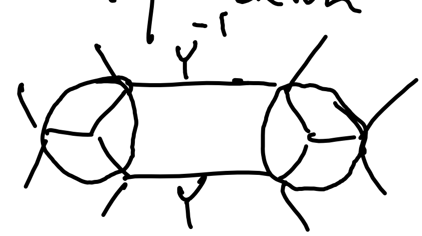
Result: a periodic $SL_2(\mathbb{C})$ double dimer model with:

- interesting connection
- trivial weight function.

If your single-dimer weight function w was something else, but still 2-periodic: Let w_{sq} be the weight above and write $w = w_{sq}\, w_{extra}$



w_{extra}

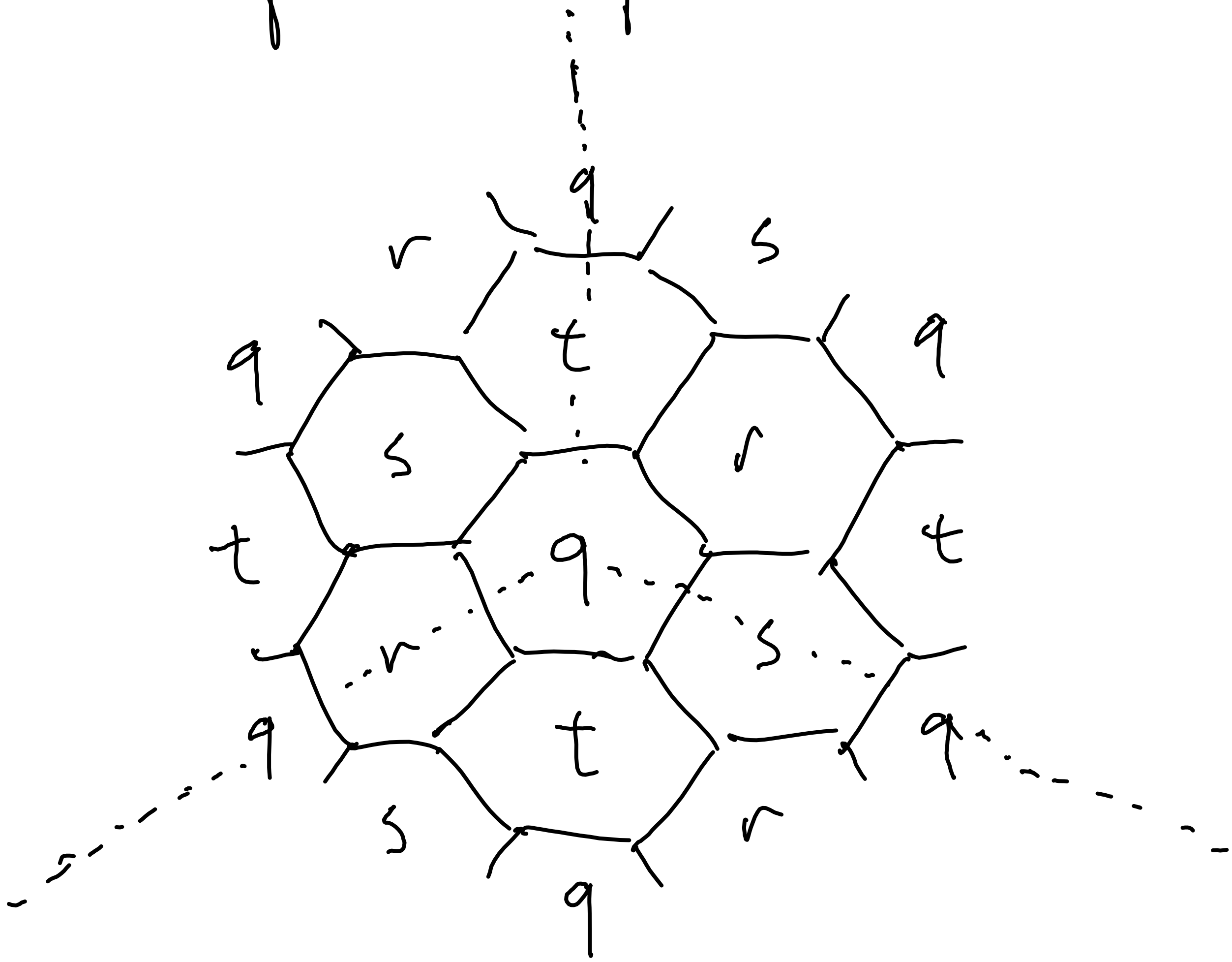


w_{sq}

Then: $w_{sq} \rightsquigarrow \Gamma$ and you get an $SL_2(\mathbb{C})$ double dimer config with weight function w_{extra}.

Why / How:

Original motivation was study of 2-periodic weight on plane partitions.



$$Z = 1 + q + (qr + qs + qt) + (qrs + qst + qrt + q^2 r + q^2 s + q^2 t + \cdots) + \cdots$$

My thesis: define $Q = qrst$

$$M(x,y) = \prod_{i \geq 1} \left(\frac{1}{1-xy^i} \right)^i$$

$$\tilde{M}(x,y) = M(x,y)\, M\left(\tfrac{1}{x}, y\right)$$

then

$$Z = \frac{M(1,Q)^4 \cdot \tilde{M}(rs,Q)\,\tilde{M}(st,Q)\,\tilde{M}(rt,Q)}{\tilde{M}(-r,Q)\,\tilde{M}(-s,Q)\,\tilde{M}(-t,Q)\,\tilde{M}(-rst,Q)}$$

So, this is now also a partition function of the double dimer model.

Note: If $q = r = s = t$, $Z = M(1,q)$

Side note: compare

$$\frac{\tilde{M}(1,Q)^4 \cdot \tilde{M}(rs,Q)\tilde{M}(st,Q)\tilde{M}(rt,Q)}{\tilde{M}(-r,Q)\tilde{M}(-s,Q)\tilde{M}(-t,Q)\tilde{M}(-rst,Q)}$$

to MacMahon's enumeration of boxed $a \times b \times c$ plane partitions written in the form which Propp and Kuperberg attribute to each other:

$$\frac{H(a)H(b)H(c)H(a+b+c)}{H(a+b)H(b+c)H(c+a)}$$

Why so similar??

(MacMahon's GF for plane partitions)

If $q=t$, $r=s$ $\quad Z = M(1,qr)^2\,\tilde{M}(r,qr)$

(Checkerboard coloring of plane partitions).

q	r	q	
r	q	r	
q	r	q	$\ddots$

Can prove this via Schur process / vertex operators.

Whole computation of Z: breaks all the tricks one would use to compute correlations.

Also note: in

$$\frac{M(1,Q)^4 \cdot \tilde{M}(rs,Q)\tilde{M}(st,Q)\tilde{M}(rt,Q)}{\tilde{M}(-r,Q)\tilde{M}(-s,Q)\tilde{M}(-t,Q)\tilde{M}(-rst,Q)}$$

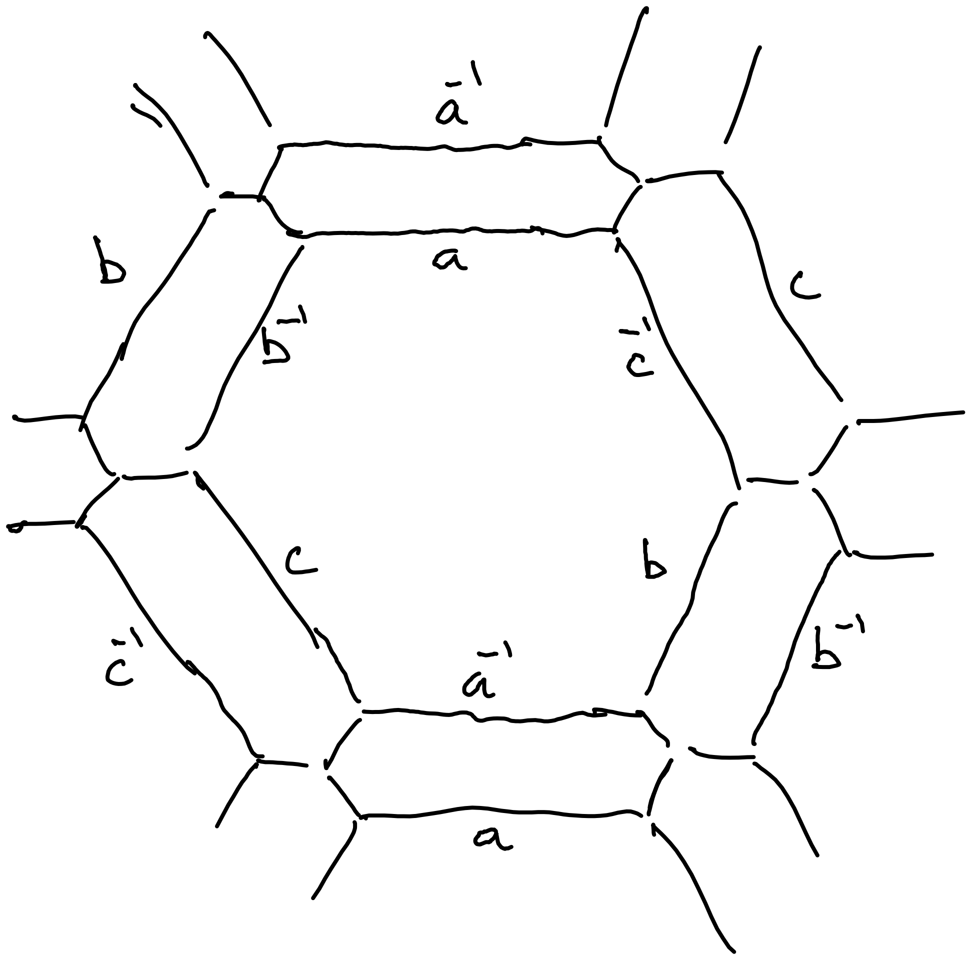
put $r=s=t=-1$, you get

$$M(1,-q)^2.$$

- original motivation of squish map: prove this.

- This specialization in hindsight suggests the matrix J

- in our notation, $a = b = c = i$
(where $i^2 = -1$)

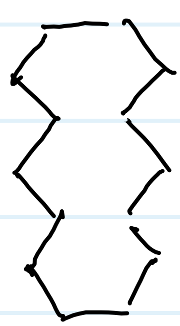


What about other roots of unity?

If $a = b = c = \xi$ with $\xi^{12} = 1$, we noticed that all loops appear to contribute 2, 0, or -2.

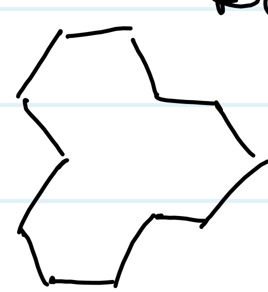
meaning they don't occur.

stone



2

bone



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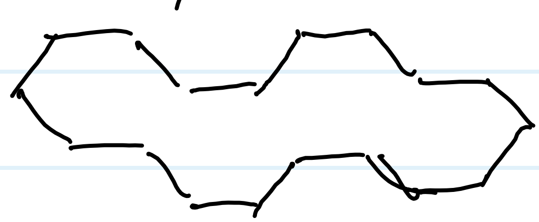
Conj:

A Loop L appears in some nonzero-weight double dimer config

$$\Updownarrow$$

L is signed-tileable by stones, bones, and snakes



Such L contributes

$$2 \cdot (-1)^{\#\{\text{stones needed}\}}$$

To prove $\Searrow$ we would need some result about one of the Conway-Lagarias tiling groups; Foster gives extensive empirical results, and a conjectural presentation, in her thesis.

Thank you!

Happy birthday Jim!