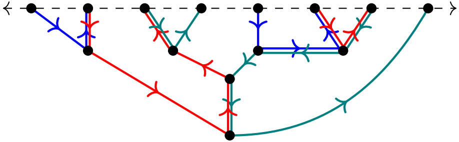
Webs and quantum representations

Heather M. Russell (University of Richmond)

Julianna Tymoczko (Smith College)



Propp $2^{\binom{4}{2}}$

June 29, 2024

Plan of talk

1. Setting the stage: some Schubert calculus and Springer fibers
2. Noncrossing matchings and Springer fibers: Talia Goldwasser, Meera Nadeem, and Garcia Sun
3. $\mathfrak{sl}_3$-webs and Springer fibers: Emily Hafken, Veronica Lang, Orit Tashman
4. $\mathfrak{sl}_n$-webs and strandings: Felicia Flores, Emily Hafken, Annabelle Hendrickson, Orit Tashman, and Heather Russell

Flags

The flag variety is G/B

If $G = GL_n(\mathbb{C})$ and B is upper-triangular matrices then each flag is

- ... a coset gB
- ... a nested subspace $V_1 \subseteq V_2 \subseteq \cdots \subseteq \mathbb{C}^n$

$$\left\langle \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\rangle \subseteq \left\langle \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle \subseteq \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

- ... a matrix with zeros to the right and below a permutation

$$\begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Schubert cells

Each gB has a representative in exactly one of the following:

$$\begin{pmatrix} * & * & 1 \\ * & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} * & 1 & 0 \\ * & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} * & * & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

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These are the **Schubert cells** BwB/B. They are parametrized by permutation matrices w.

Springer fibers

Fix a linear operator $X : \mathbb{C}^n \to \mathbb{C}^n$

Definition

The Springer fiber $\mathcal{S}_X$ of X consists of flags $gB = V_\bullet$ for which

- ... $g^{-1}Xg$ is upper-triangular
- ... the image XV_i of each subspace is contained in V_i

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For example: When X is a single Jordan block

$XV_1 \subseteq V_1$ means V_1 is spanned by an eigenvector $\vec{e}_1$

The Springer fiber is the flag IB corresponding to the identity I

Nilpotent Springer fibers and the partition $\lambda(X)$

We focus on the case when X is nilpotent

- ....$X^m = 0$ for some m
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- ....$X^m = 0$ for some m
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Fact: The Springer fiber of X is homeomorphic to the Springer fiber of each conjugate of X.

The conjugacy class of X has a unique representative in Jordan canonical form with blocks arranged in nondecreasing order.

Jordan blocks partition n into $\lambda(X)$.

If $\lambda(X)$ has 2 rows then we call $\mathcal{S}_X$ a 2-row Springer fiber.

Why Springer fibers? Representation theory

Theorem

Suppose X is nilpotent with Jordan type $\lambda(X)$

- *S_n acts naturally on the cohomology $H^*(\mathcal{S}_X)$*
- *The top-dimensional cohomology of $H^*(\mathcal{S}_X)$ is irreducible*
- *In fact $H^{top}(\mathcal{S}_X)$ is irreducible of type $\lambda(X)$*
- *The set $\{H^{top}(\mathcal{S}_\lambda)\}$ is precisely the collection of irreducible representations of S_n*

(λ ranges over nilpotent conjugacy classes, or partitions of n)

Springer '76, Kazhdan-Lusztig '80, Borho-MacPherson '83, Hotta '82, Lusztig '84, Garsia-Procesi '92, and others...

Example of Springer fiber for $\lambda(X) = (2, 1)$

Example: $X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

The Springer fiber $\mathcal{S}_X$ has cells

$$\begin{pmatrix} a & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

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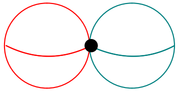

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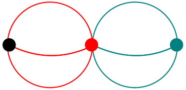


Springer Schubert cells

Suppose λ is a partition of n.

Definition

The Young diagram of shape λ is an arrangement of boxes (left- and top-aligned) with λ_i boxes in the i^{th} row. A Young tableau is a Young diagram that has been filled with numbers according to some rule:

- **row-strict** means the numbers increase in each row (L to R)
- **standard** means row-strict and that numbers increase in each column (top to bottom)

1	3
2	4
5	

2	4
1	3
5	

Springer Schubert cells

Theorem

Springer fibers are paved by affine cells $C_w \cap \mathcal{S}_X$. *The cells* $C_w \cap \mathcal{S}_X$

are enumerated by *tableaux of shape* $\lambda(X)$.

Springer Schubert cells

Theorem

Springer fibers are paved by affine cells $C_w \cap \mathcal{S}_X$. The cells $C_w \cap \mathcal{S}_X$ that are nonempty

are enumerated by row-strict tableaux of shape $\lambda(X)$.

- Fillings tell you when to add each basis vector to flag

Springer Schubert cells

Theorem

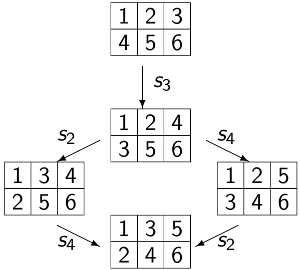
Springer fibers are paved by affine cells $C_w \cap \mathcal{S}_X$. The cells $C_w \cap \mathcal{S}_X$ that
- *are nonempty*
- *have maximal dimension*

are enumerated by row-strict / standard *tableaux of shape $\lambda(X)$.*

- Fillings tell you when to add each basis vector to flag
- Dimension comes from certain inversions—not all entries are free

What's known about the geometry of Springer fibers

- Spaltenstein described components for the Springer fiber using dense smooth subsets corresponding to standard tableaux.
- Shimomura partitioned the Springer fiber into affines with a recursive construction.
- Tymoczko, Fresse, Precup all pave Springer fibers
- Pagnon and Ressayre showed the closures are contained in the following partial order.



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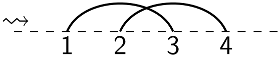
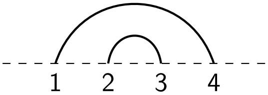
Definition

A matching M on $\{1, 2, \ldots, kn\}$ is a subset of disjoint pairs $(i < j)$ called arcs. A matching is noncrossing if it contains no pairs $(i_1, j_1), (i_2, j_2)$ with $i_1 < i_2 < j_1 < j_2$.

CRONSSING $\rightsquigarrow$





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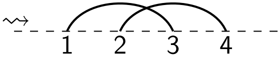
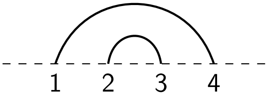
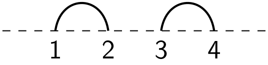
CROSSING $\rightsquigarrow$ 



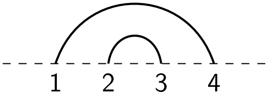
Definition (The bijection)

Arc starts go on the top row. Arc ends go on the bottom row.

1	3
2	4



1	2
3	4



Back to 2-row Springer fibers

Kuperberg described noncrossing matchings in the context of webs, which encode representations that can be used to create another version of this bijection. Khovanov and Kuperberg do important work on webs together.

Fung and Khovanov both knew the components of 2-row Springer fibers were iterated $\mathbb{P}^1$-bundles and used arcs to represent these $\mathbb{P}^1$s.

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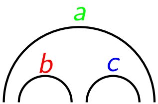
The Issue (Problem?): We didn't have a cell decomposition in terms of noncrossing matchings.

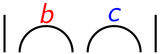
Matchings and 2-row Springer fibers

Theorem (Goldwasser, Sun, T)

The Springer Schubert cells $C_w \cap \mathcal{S}_X$ for 2-row Springer fibers are naturally indexed by noncrossing matchings.



$$\begin{pmatrix} a & b & 1 & 0 & 0 & 0 \\ 0 & a & 0 & c & 1 & 0 \\ 0 & 0 & 0 & a & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$



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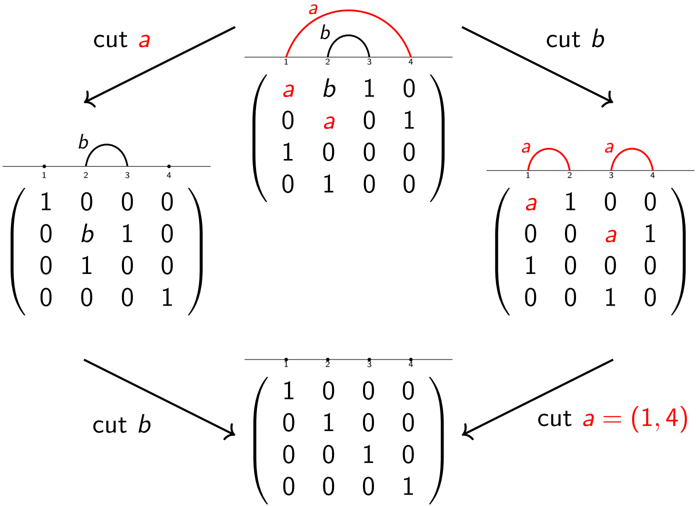
Matchings and 2-row Springer fibers

Theorem (Goldwasser, Sun, T)

Let M be a noncrossing matching with Springer Schubert cell $\mathcal{C}_M$. The Springer Schubert cells in the closure of $\mathcal{C}_M$ are obtained by unnesting each subset of arcs, relabeling with variable of the top arc after each unnesting.

These cells form a paving by affines, not (eg) a CW-complex. Sometimes only *part* of a cell is in the closure of another cell.

Matchings and 2-row Springer fibers



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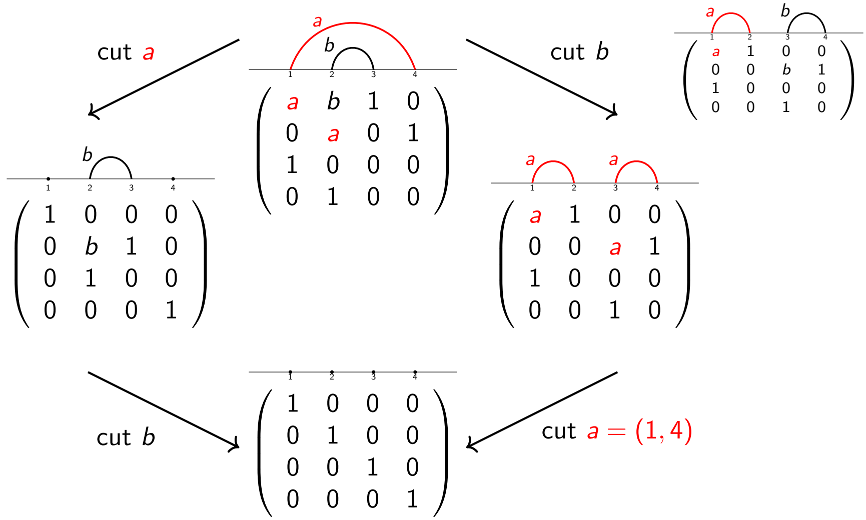
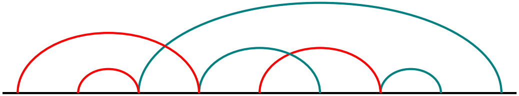


Does this extend to 3-row Springer fibers?

Fact: There are natural bijections between the *reduced web basis* for $\mathfrak{sl}_3$ webs, 3-row standard tableaux, Yamanouchi words on 3 symbols, and certain *multicolored noncrossing matchings* (arcs colored so that same-colored arcs don't cross).

1	2	5
3	4	7
6	8	9



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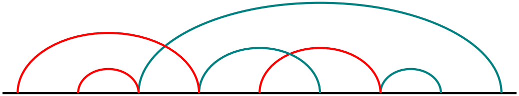
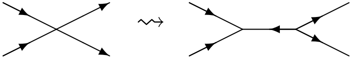
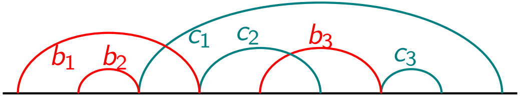


To obtain a reduced web from a 2-colored noncrossing matching, replace the point at each crossing with a small edge:



3-row Springer fibers

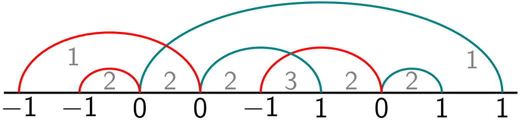


$$\begin{pmatrix} a_1 & a_2 & c_1 & c_2 & a_3 & 1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & c_1 & a_2+* & 0 & c_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & a_1+\dagger & 0 & c_1 & 0 & 1 \\ \hline b_1 & b_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_3 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$* = -b_2 c_1 + (b_3 - b_1) c_2$ and $\dagger = +(b_3 - b_1) c_1$

3-row Springer fibers

The extra variables are indexed by the *depths* of the regions in the multicolored noncrossing matching. Khovanov and Kuperberg realized depth is important to understanding webs for $\mathfrak{sl}_3$.



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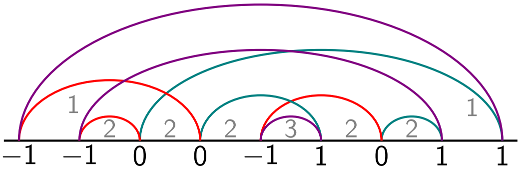

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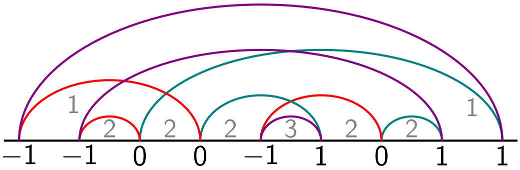

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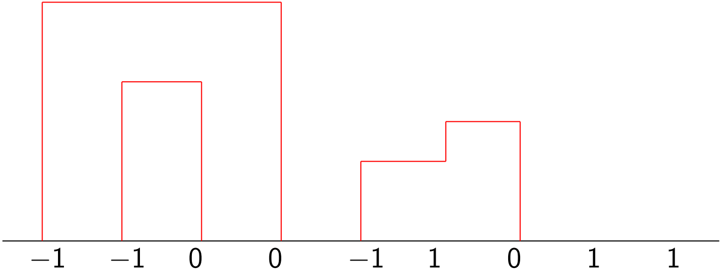
The extra entries: Springer webs versus reduced webs

Square off the arc construction to get (reduced) webs:



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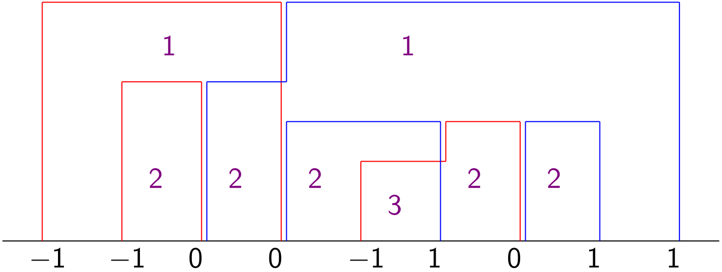
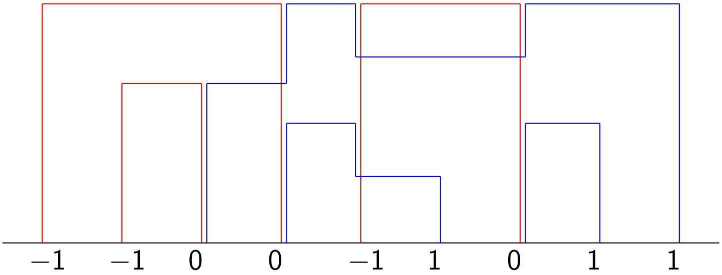


The extra entries: Springer webs versus reduced webs

Pull reds above blues to get entries of Springer cells:



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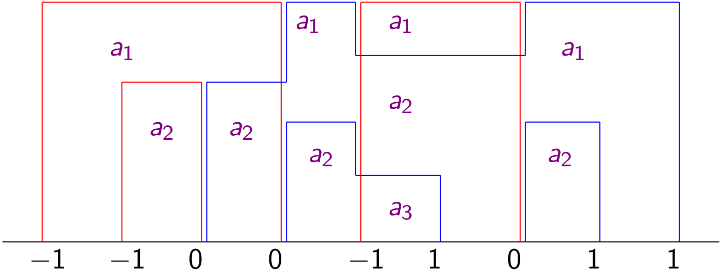

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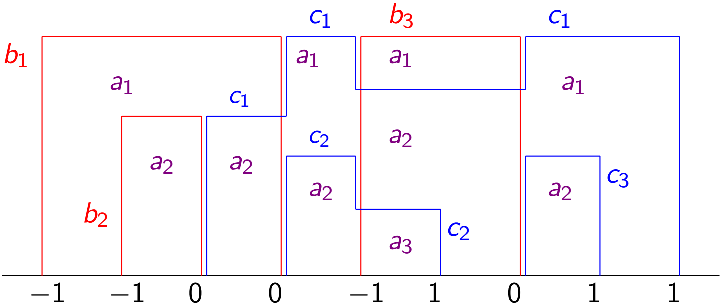

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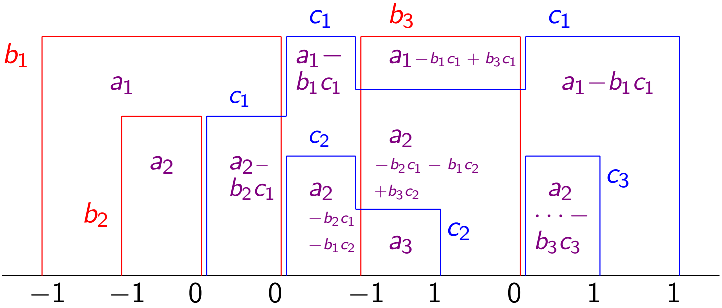
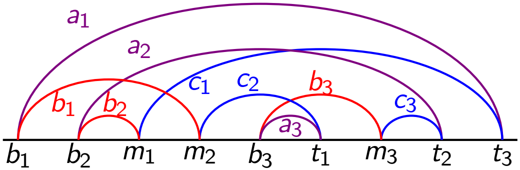


3-row Springer fibers



$$\left(\begin{array}{ccccccccc}
a_1 & a_2 & c_1 & c_2 & a_3 & 1 & 0 & 0 & 0 \\
0 & a_1 & 0 & c_1 & a_2+* & 0 & c_3 & 1 & 0 \\
0 & 0 & 0 & 0 & a_1+\dagger & 0 & c_1 & 0 & 1 \\
\hline
b_1 & b_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b_3 & 0 & 1 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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\end{array}\right)$$

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3-row Springer fibers

Theorem (Hafken, Lang, Tashman, T, et al.)

Create the Springer web $\mathcal{M}_T$ *obtained from a balanced Yamanouchi word* T *on* $\{-1, 0, 1\}$ *as above, and color it with red and blue arcs.*

The blue and red entries of the Springer Schubert cells appear according to nesting exactly as in the 2-row case.

The purple entries are the labels in the nested depth bands just to the right of each 1.

Webs and Springer fibers

Takeaways:

- Each Springer fiber permutation gives us a web graph and a multicolored noncrossing matching.
- The way that the multicolored noncrossing matching passes through the web graph encodes the entries of the corresponding Springer fiber cell.
- Nesting and depth encodes the bundle structure.

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- Nesting and depth encodes the bundle structure.
- This should extend to all Schubert cells, not just top-dimensional. Lower-dimensional cells *might not be affine*.
- BUT ALSO: webs seem to have tricolored, well-behaved paths.

Webs: the idea

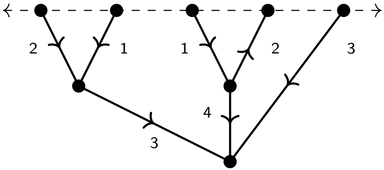
Webs encode representations of $U_q(\mathfrak{sl}_n)$ using graph theory. Each web can be viewed as an invariant vector. Skein-theoretic relations on webs encode algebraic relations on the corresponding representations.

Our web conventions

An $\mathfrak{sl}_n$ web is a plane graph with

- univalent boundary vertices along a top axis and
- trivalent internal vertices.
- Each edge is oriented and weighted with an element of $\{1, \ldots, n-1\}$.
- At each trivalent vertex, *flow is conserved mod n*: the sum of incoming edge weights minus the sum of outgoing edge weights is divisible by n.

An $\mathfrak{sl}_5$ web:



Constructing a web vector

Idea: (1) For each trivalent vertex, specify an invariant vector in a three-fold tensor product.

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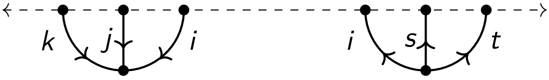

$$\sum_{I \sqcup J \sqcup K = [n]} (-q)^{\ell(I,J)+\ell(I \cup J,K)} x_K^* \otimes x_J^* \otimes x_I^* \quad \bigotimes \quad \sum_{I \sqcup S \sqcup T = [n]} (-q)^{-\ell(T,S)-\ell(S \cup T,I)} x_I \otimes x_S \otimes x_T$$

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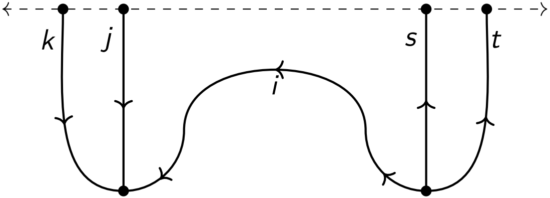
Idea: (1) For each trivalent vertex, specify an invariant vector in a three-fold tensor product. (2) Combine these vectors using a contraction process where only compatible terms persist.



$$\sum_{J \sqcup K = S \sqcup T} (-q)^{\ell([n]-(J\cup K),J)+\ell([n]-K,K)-\ell(T,S)-\ell(S\cup T,[n]-(T\cup S))} x_K^* \otimes x_J^* \otimes x_S \otimes x_T$$

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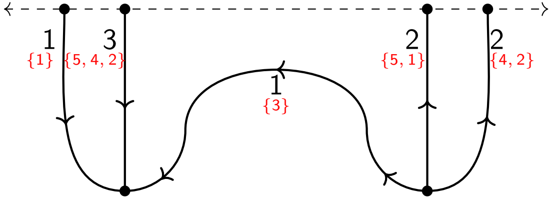
Notice: Each term in the web vector corresponds to a binary labeling of the web.



$$q^{-2}\left(x_{\{1\}}\right)^{*}\otimes\left(x_{\{5,4,2\}}\right)^{*}\otimes x_{\{5,1\}}\otimes x_{\{4,2\}}$$

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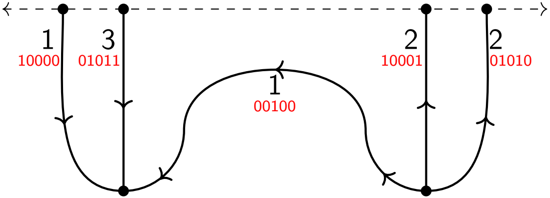

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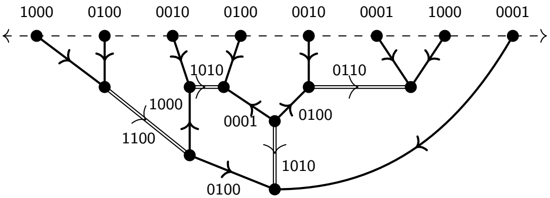
Binary labeling and web vectors

Shorthand for these vectors: We record the summands in each invariant vectors via a binary labeling of an $\mathfrak{sl}_n$ web. This is a choice of n-bit binary string for each edge such that

- the string for an edge labeled k has exactly k ones, and
- the sum of the incoming binary strings minus the sum of the outgoing binary strings is constant.



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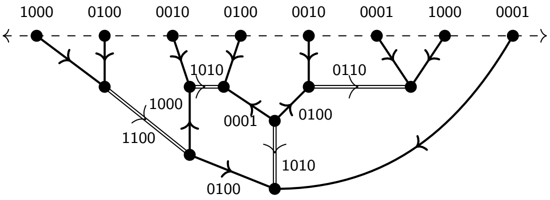


Key Fact: A web vector is a state sum over all binary labelings with coefficients that are a product of the local contributions coming from trivalent vertices and contractions.

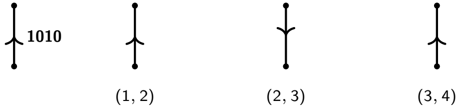
Amazing fact: these binary labelings actually have a global structure

Definition

Given a binary labeling of an $\mathfrak{sl}_n$ web, the λ_i stranding is the following directed graph. For each web edge uv with binary label b include

- edge uv if $\mathbf{b}$ has i^{th} bit 1 and $i+1^{th}$ bit 0,
- edge vu if $\mathbf{b}$ has i^{th} bit 0 and $i+1^{th}$ bit 1, and
- no edge otherwise.

Ex:



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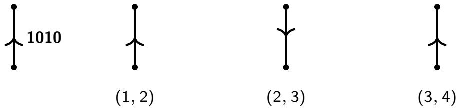
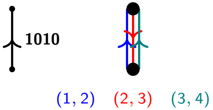


We often draw all $(i, i+1)$ strandings together using colors to distinguish them.

Ex:



Properties of stranding

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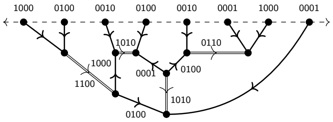
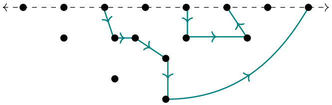
This means a λ_i stranding is a directed noncrossing matching on a subset of the web's boundary vertices (possibly with closed, oriented loops in the interior of the web).





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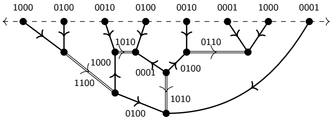
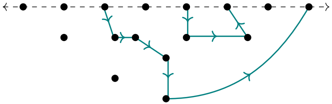
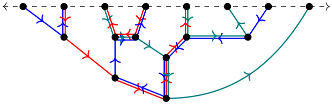
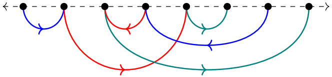




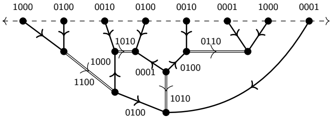
Together, the λ_i strandings form a multicolored, directed noncrossing matching covering the entire web.

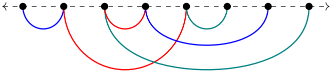




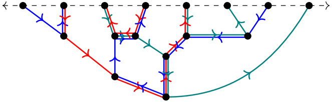
Stranding webs merges two combinatorial models: balanced Yamanouchi words and multicolored noncrossing matchings

The boundary word for a web lists the vectors in order — for instance $\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_2, \vec{x}_3, \vec{x}_4, \vec{x}_1, \vec{x}_4$ or just 12323414. The matching tells you which boundary points are connected.





Putting them together in directed paths through the web graph gives a stranding.



Features of stranding: depth and complexity of webs

Strandings give several different measures of depth (directed/undirected, monochrome/weighted by color, etc.). Essentially, they all coincide when $n = 3$ and also coincide with ordinary graph depth.

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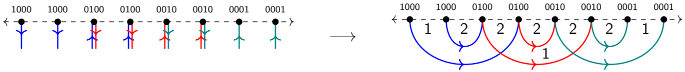
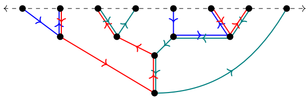
Terms in a web vector are read just from the boundary word. Whether a web supports a particular boundary word is a measurement of the complexity of a web.



Features of stranding: Base stranding

We can use stranding to generate a base stranding. It has several nice properties:

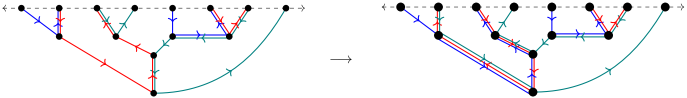
- Each strand wraps clockwise around one (bounded) face in the web graph.
- Each edge has at most two strands on it, and if two then are directed opposite to each other.
- We can explicitly write down the boundary word for the base stranding.



Features of strandings: Flipping strands

We can act by transpositions $(i, i+1)$ on all of the stranding or individual strands. This flips the direction of strands of type i between clockwise and counterclockwise. But it can also pull strands of type $i-1$ and $i+1$ out of position, disrupting the global picture.

For instance, flipping the leftmost red strand connects the green strands differently as follows:



Main message: Stranding extracts global information from local information, has many remarkable properties, and develops a new set of tools to analyze webs (and Springer fibers).

Thank you!

Thanks for the invitation, and to NSF, AWM-MERP, and Budapest Semesters in Mathematics for supporting this work.

Student participants: Felicia Flores (Wesleyan, soon), Talia Goldwasser, Emily Hafken (Virginia, soon), Annabelle Hendrickson (Virginia*, soon), Veronica Lang (Michigan), Meera Nadeem, Garcia Sun (Washington), Orit Tashman (Hawaii, soon), and others.

Diagrammatic representation theory

The $\mathfrak{sl}_n$ spider category diagrammatically models the representation theory of $\mathfrak{sl}_n$ and its associated quantum group. The morphisms in this category are called $\mathfrak{sl}_n$ webs.

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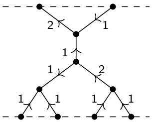
- Edge decorations at the boundary dictate source and target spaces.
- The web interior describes a specific homomorphism between representations.



$$\begin{array}{c} \mathcal{U}_q(\mathfrak{sl}_3) \curvearrowright \Lambda^2_q(\mathbb{C}^3_q) \otimes \left(\mathbb{C}^3_q\right)^* \\ \uparrow \\ \mathcal{U}_q(\mathfrak{sl}_3) \curvearrowright \mathbb{C}^3_q \otimes \mathbb{C}^3_q \otimes \mathbb{C}^3_q \otimes \mathbb{C}^3_q \end{array}$$

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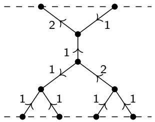

$$\mathcal{U}_q(\mathfrak{sl}_3) \curvearrowright \Lambda_q^2(\mathbb{C}_q^3) \otimes \left(\mathbb{C}_q^3\right)^*$$

$$\uparrow$$

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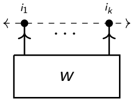
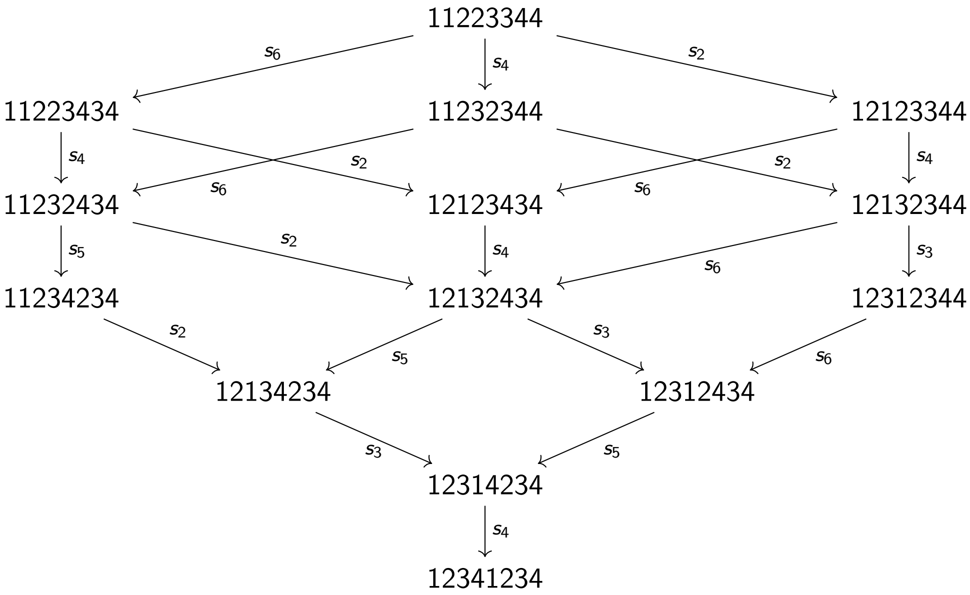
- For webs with vertices on one axis, this setup specifies a $\mathcal{U}_q(\mathfrak{sl}_n)$-invariant web vector.



$$\mapsto v_w \in \mathrm{Inv}(\Lambda_q^{i_1}(\mathbb{C}_q^n) \otimes \cdots \otimes \Lambda_q^{i_k}(\mathbb{C}_q^n))$$

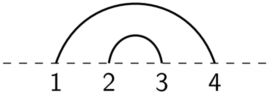
The poset of balanced Yamanouchi words



Nesting and unnesting NCMs

Definition

Two arcs $(i, j), (i', j')$ in an NCM are unnested if $j < i'$ and nested if $i < i' < j' < j$. Suppose M is a matching in which (i, j) is nested over (i', j'). Then the operation that leaves all arcs the same except exchanges endpoints $(i, j'), (j, i')$ is called unnesting.



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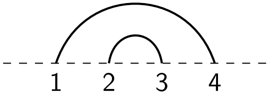
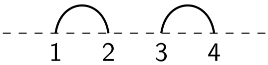


$\rightsquigarrow$



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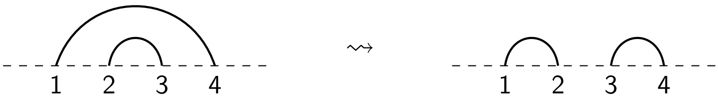


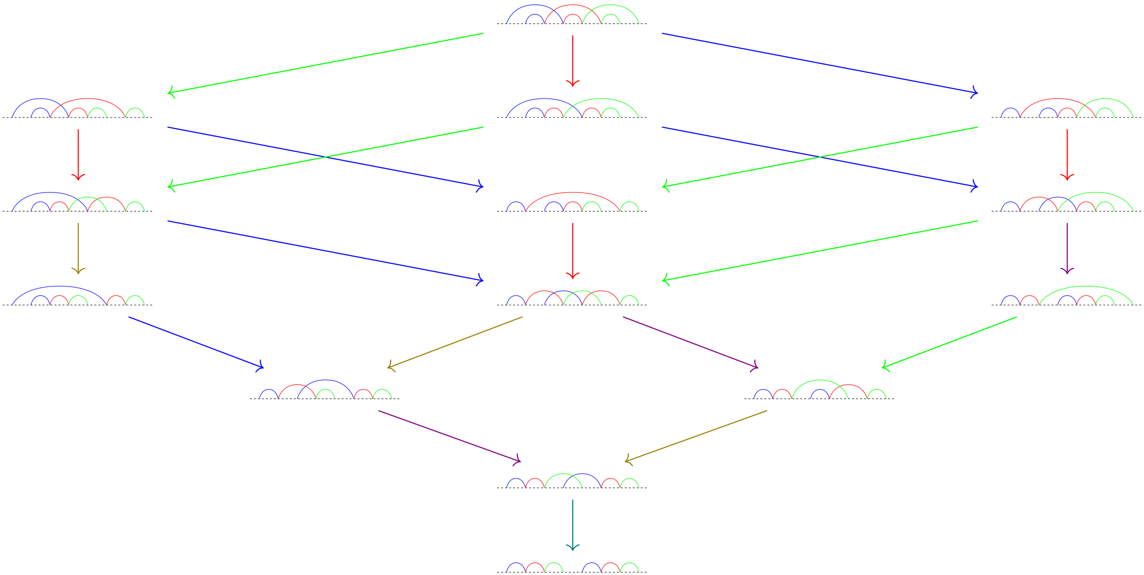
Fact

If M is an NCM with two nested arcs that have no arc nested properly in between, then unnesting those arcs creates another NCM.

Nesting and unnesting gives a poset on multicolored NCMs.

The poset of (all $\binom{n}{2}$) multicolored NCMs

Each edge is a single unnesting that doesn't disturb the relative position of arcs of other colors.



Theorem (Tashman): the posets are the same

The first poset comes from the permutation action on *positions* in the word, while the second poset comes from the permutation actions on *values* in the word.