Propperties and Propppositions

Statistical and Dynamical Combinatorics

A celebration of Jim Propp's $2^{\binom{2^2}{2}}$th birthday

June 2024

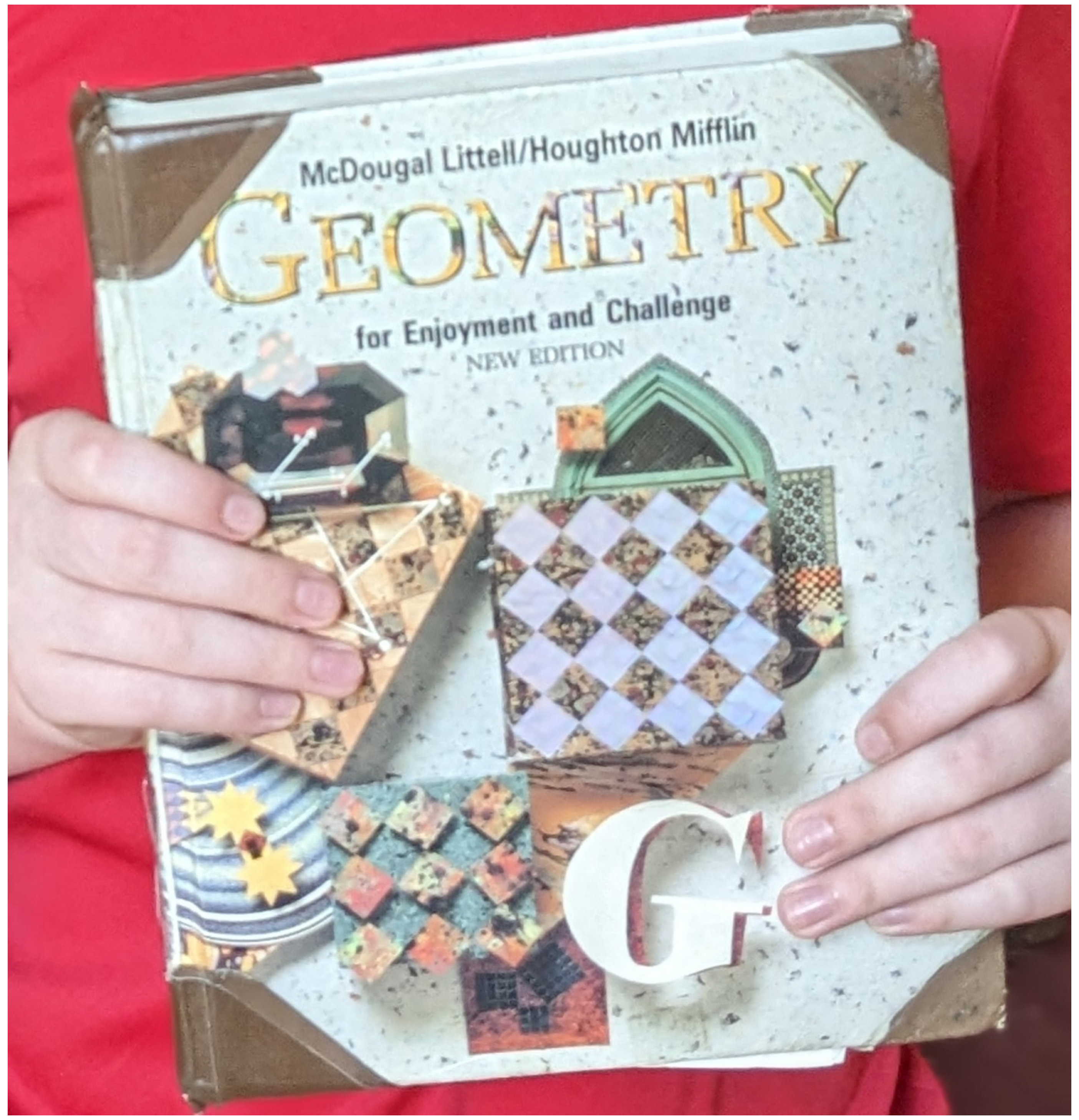
Bridget Tenner - DePaul University

Combinatorics near Chicago circa 1992

Combinatorics near Chicago circa 1992

Geometry for Enjoyment and Challenge

with Richard Rhoad



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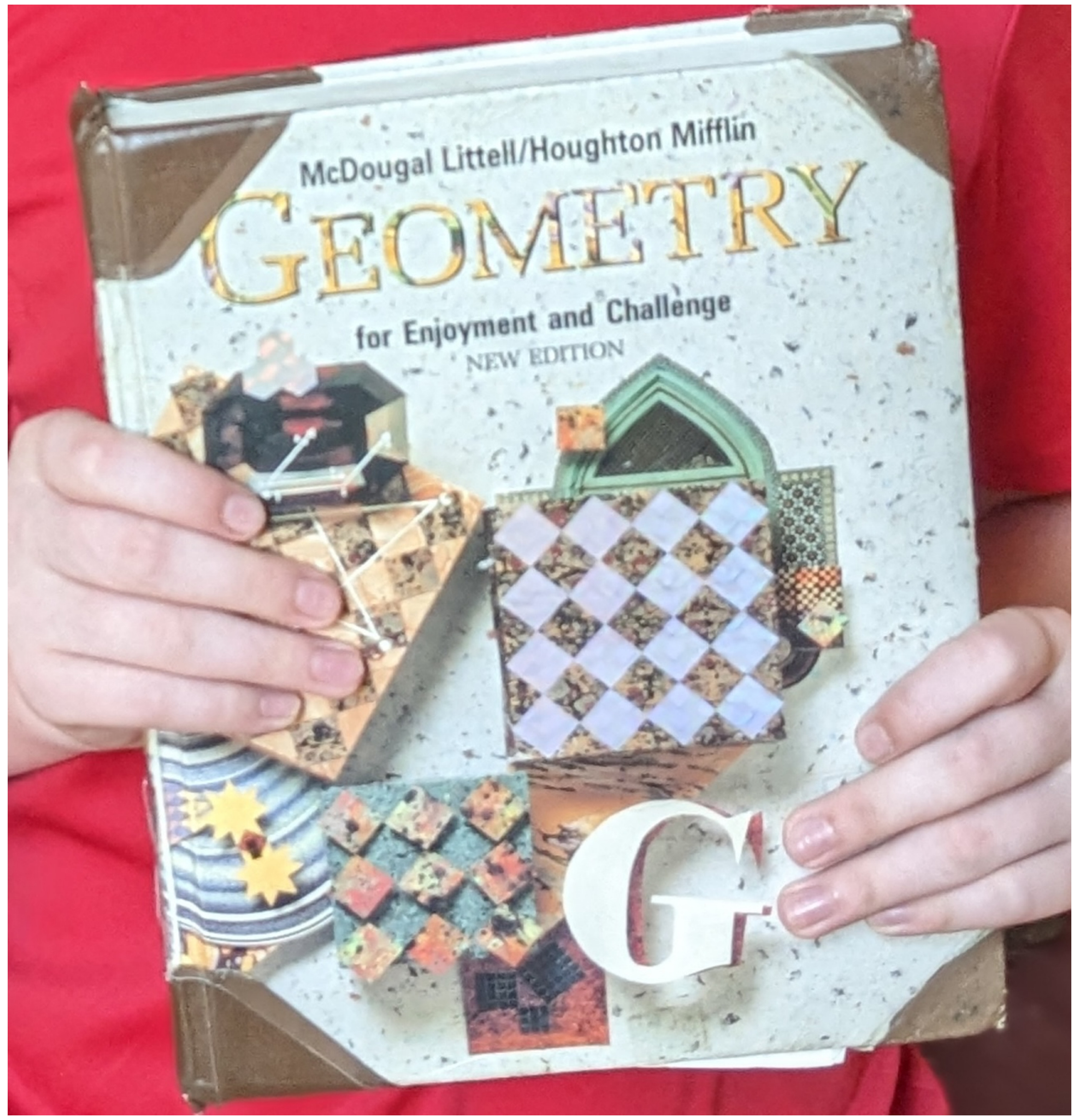
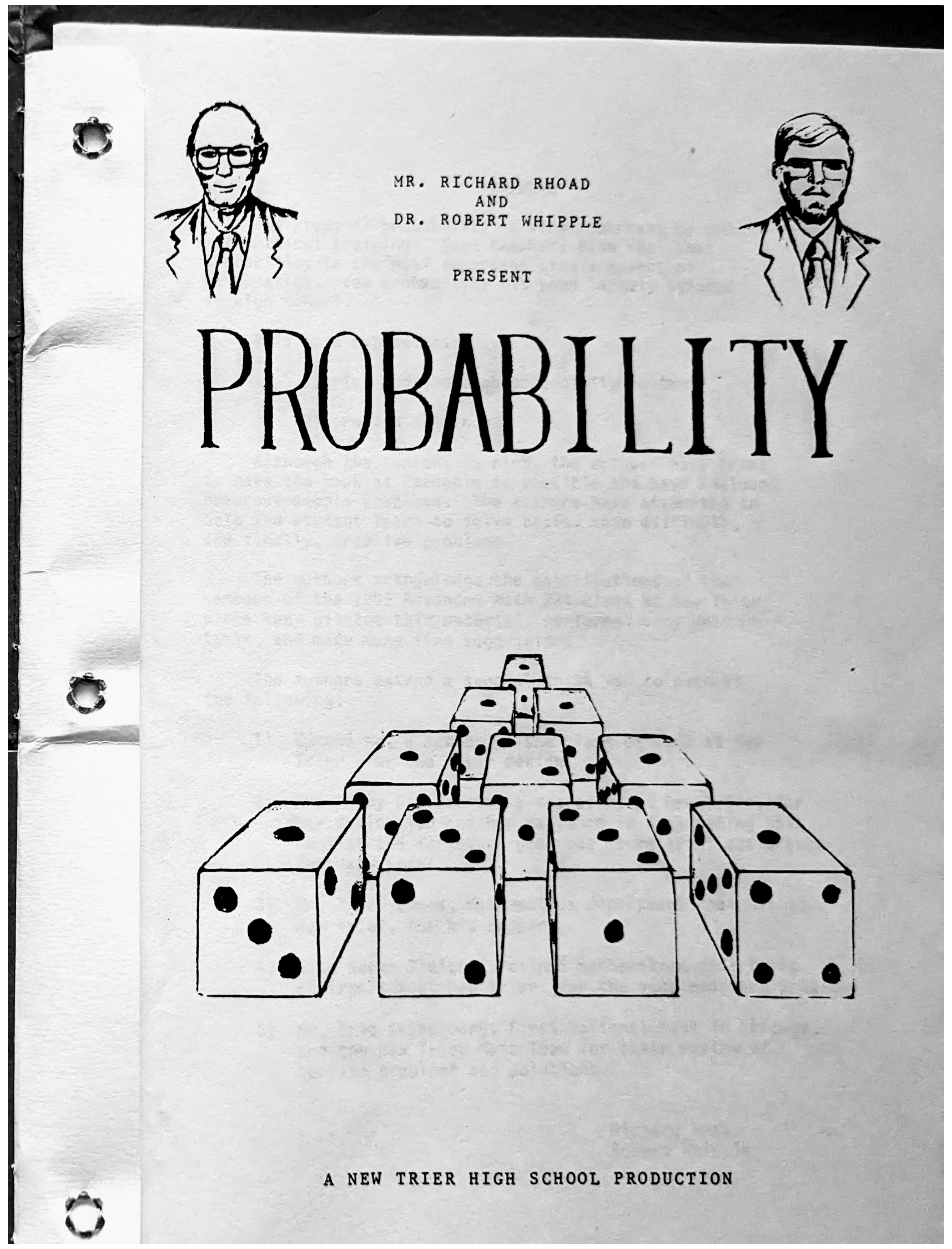


Probability

with Richard Rhoad



Combinatorics at Harvard circa 2000

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[crickets]

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Making the big trek from MIT/02139 to Harvard/02138…

Richard Stanley and Math 192

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Including Gregg Musiker and David Speyer!

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Research **E**xperiences in **A**lgebraic **C**ombinatorics at **H**arvard

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Research **E**xperiences in **A**lgebraic **C**ombinatorics at **H**arvard

Including Lionel Levine and Kyle Petersen!

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reach (*verb*):

come to, arrive

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reach (*noun*):

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within reach (*adverb*):

in play, attainable

How to learn without ever going to class

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How to play with math

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How valuable a good question can be, even well before it's answered

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This lets us look for patterns!

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Pattern questions

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Check out https://math.depaul.edu/~bridget/patterns.html ☺

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In fact, this is true for avoiding any $p \in S_3$!

More precisely, the number of permutations in S_n avoiding $p \in S_3$ is C_n, the nth Catalan number.

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- 1234, 4321, 1243, 3421, 4312, 2134, 2143, 3412, 4123, 3214, 1432, 2341
- 4132, 2314, 1423, 3241, 3142, 2413, 1342, 4213, 2431, 3124
- 1324, 4231

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Theorem [T]. A permutation avoids 3412 and 321 if and only if it has a boolean order ideal in the Bruhat order.

Equivalently, its reduced words contain no repeated letters.

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Theorem [Gutierres, Mamede, & Santos]. The diameter of the commutation graph $C(w)$ is equal to the number of 321-patterns in w.

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$[321](45312) = 4$ and $[123](45312) = 0$

Pattern-functions

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The results cited earlier refer to pattern-functions like $[321]$ and $[321] + [3412]$:

E.g., The diameter of $C(w)$ is equal to $[321](w)$.

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$$\text{length}(w) = [21](w)$$

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Theorem [Berman & T].

$$V(w) = 2([21] + [231] + [312] + [321])(w)$$

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Why is this useful? What does this approach give us, that we might not have had otherwise?

Now different statistics, which may have seemed incomparable, can be written as linear combinations of the same set of objects. So maybe they can be compared in this language!

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Theorem [Petersen & T]. $\text{dis}(w) \geq \text{length}(w) + \text{refl length}(w)$

When do we have equality?

Pattern-functions

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Theorem [Berman & T].

$\text{dis}(w) = \text{length}(w) + \text{refl length}(w)$ if and only if [pattern property]

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An (affine) (signed) permutation w globally contains p if there exist unfrozen integers $i_1 < \cdots < i_k$ such that $w(i_1)\cdots w(i_k)$ is order-isomorphic to p.

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$\overline{1}\,\overline{2} \in S_2^B$ globally contains 321:

$$\boxed{w(-1) = 1} > \boxed{w(1) = -1} > \boxed{w(2) = -2}$$

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Note: This differs from "classical" signed pattern containment!

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"unbranched George groups" = Coxeter groups of…

- finite types A and B
- affine types A and C

Global patterns

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Theorem [Lewis & T]. For w in any nondegenerate unbranched George group of window size n, the following are equivalent:

- dis$(w) = 2 \cdot$ length(w)
- No reduced word for w contains consecutive letters $i(i \pm 1)i$ for $i \in [n-1]$
- w globally avoids the pattern 321

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How to connect people and share ideas

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How to spend 30 years on a revision

Closing the loop

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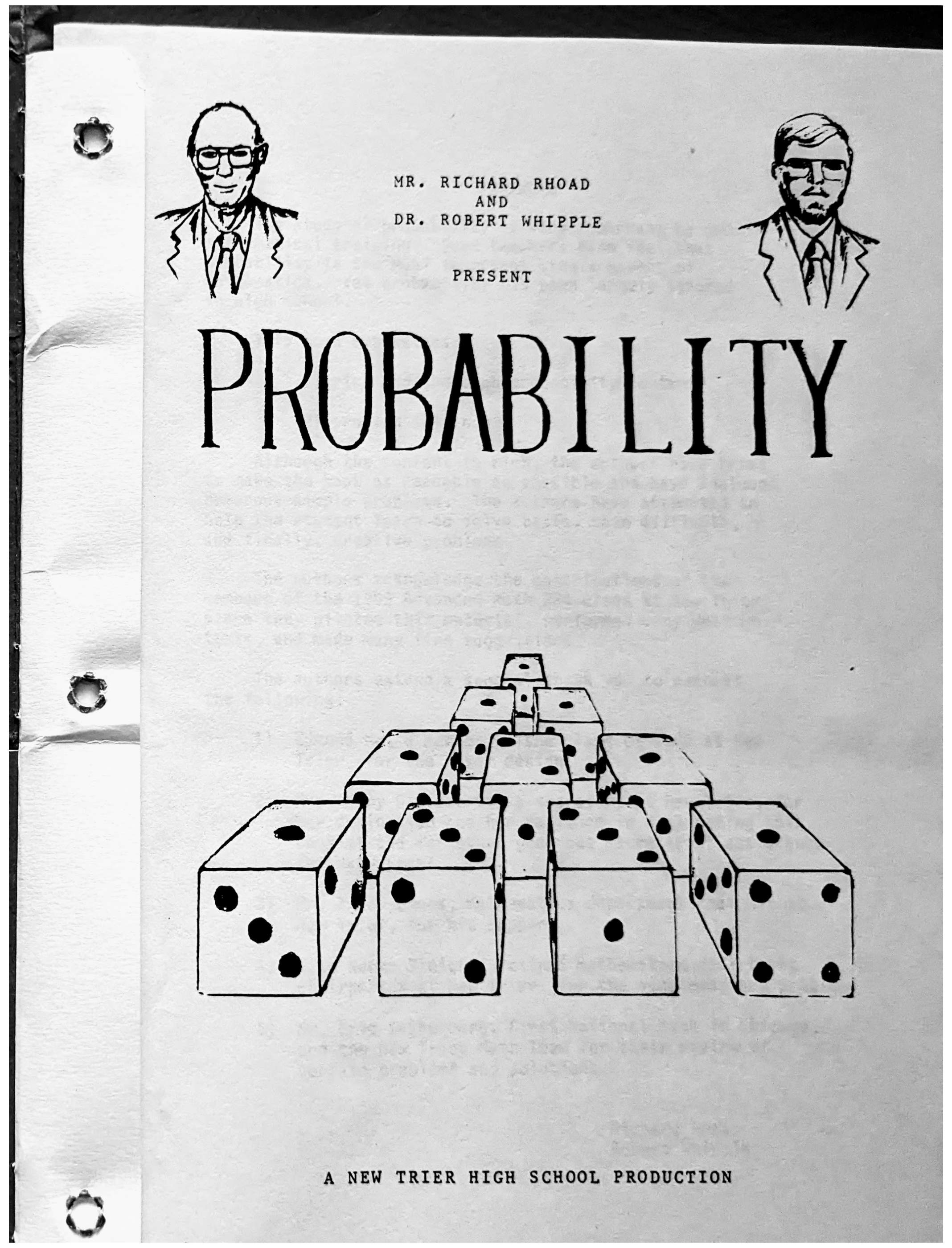
§ Geometric Probability Tuffies

#3. If a stick is broken at random into 3 pieces, what is the probability the 3 pieces can be used to form a triangle?



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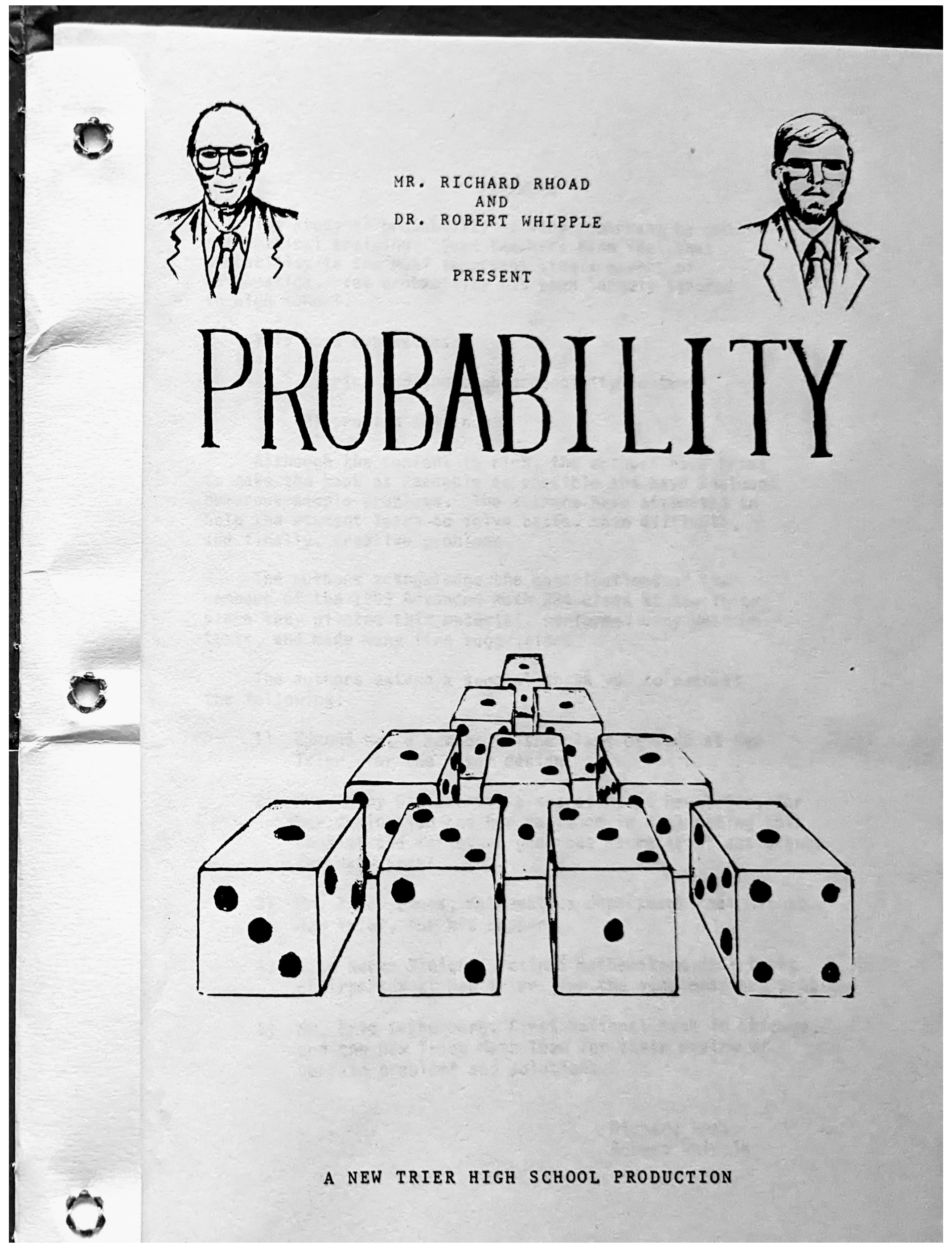

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"Broken bricks and the pick-up sticks problem" with Kyle Petersen

Mathematics Magazine (2020)

We generalize the well-known broken stick problem in several ways, including a discrete "brick" analogue and a sequential "pick-up sticks/ bricks" version…

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Happy $2^{\binom{2^2}{2}}$th birthday, Jim!