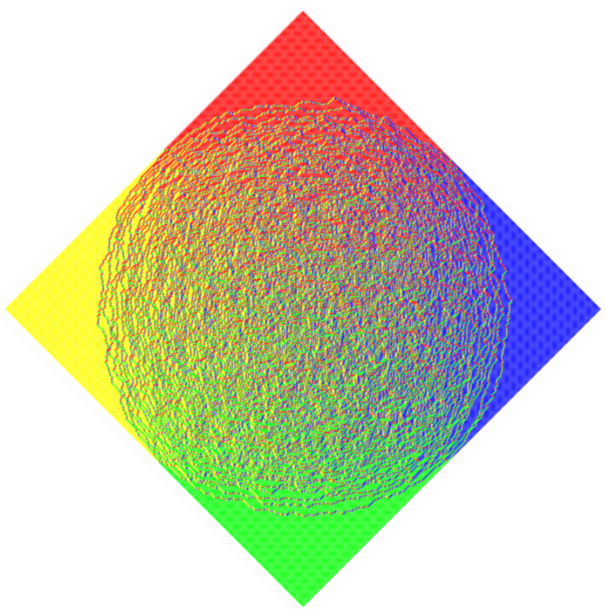
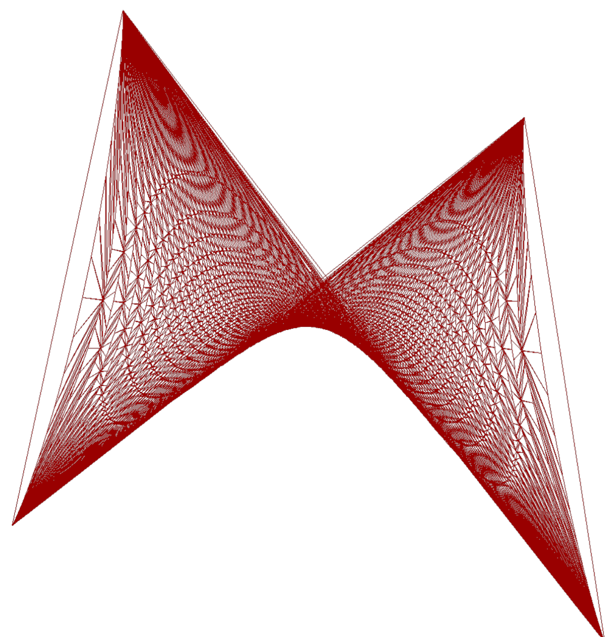
Perfect t-embeddings of Aztec diamond

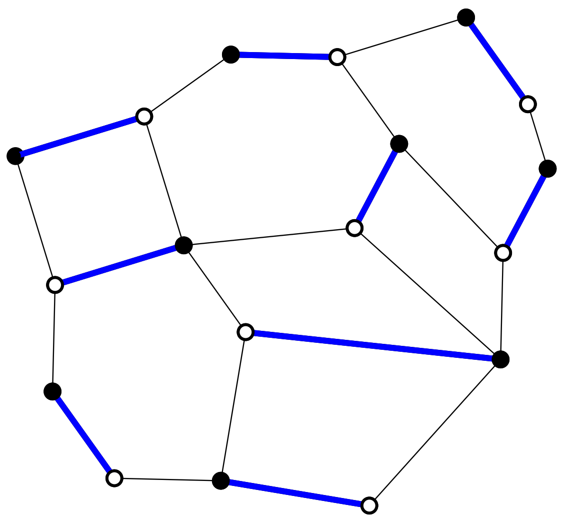
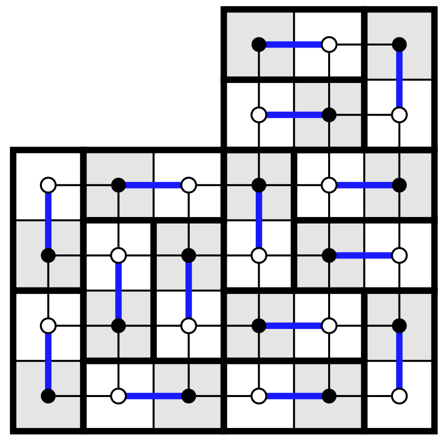
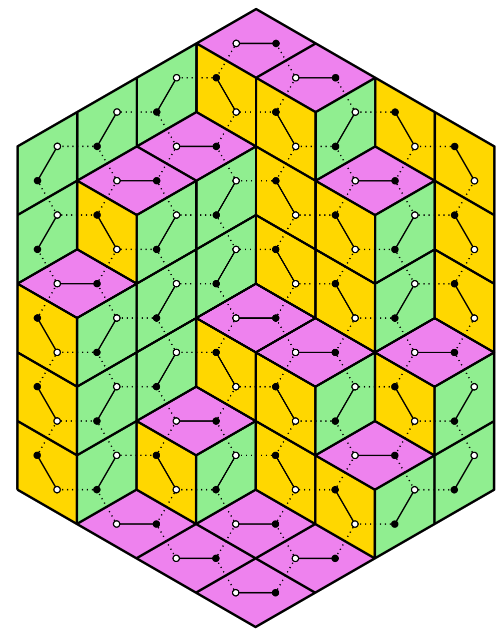
Marianna Russkikh

University of Notre Dame





Dimer model

A dimer cover of a planar bipartite graph is a set of edges with the property: every vertex is contained in exactly one edge of the set.

(On the square lattice / honeycomb lattice it can be viewed as a tiling of a domain on the dual lattice by dominos / lozenges.)

Weighted dimers

Weight function on edges:

$$\nu : E \to \mathbb{R}_{>0}$$

Associated weight of a dimer cover:

$$\nu(m) = \prod_{e \in m} \nu(e)$$

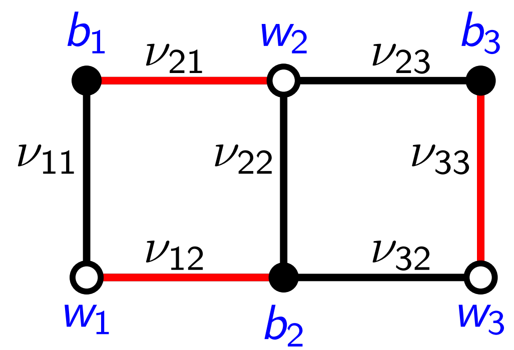
Partition function:

$$Z = \sum_{m \in M} \nu(m)$$

Probability measure on dimer coverings:

$$\mu(m) = \frac{1}{Z}\nu(m)$$

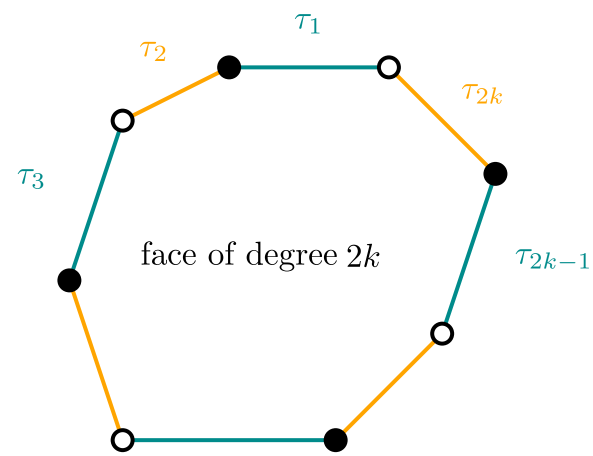
An example for 2×3 graph:



$$\nu(m) = \nu_{12} \cdot \nu_{21} \cdot \nu_{33}$$

$$\begin{aligned} Z = & \nu_{12} \cdot \nu_{21} \cdot \nu_{33} \\ & + \nu_{23} \cdot \nu_{32} \cdot \nu_{11} \\ & + \nu_{11} \cdot \nu_{22} \cdot \nu_{33} \end{aligned}$$

Kasteleyn matrix



Complex Kasteleyn signs:
$\tau_i \in \mathbb{C}$, $|\tau_i| = 1$,

$$\frac{\tau_1}{\tau_2} \cdot \frac{\tau_3}{\tau_4} \cdot \ldots \cdot \frac{\tau_{2k-1}}{\tau_{2k}} = (-1)^{(k+1)}$$

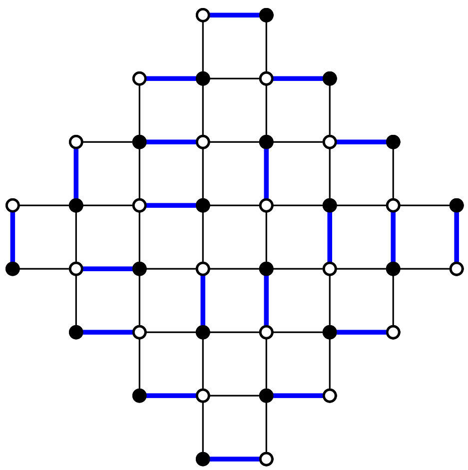
A **(Percus–)Kasteleyn matrix** K is a weighted, **signed** adjacency matrix whose rows index the white vertices and columns index the black vertices: $K(w,b) = \tau_{wb} \cdot \nu(wb)$.

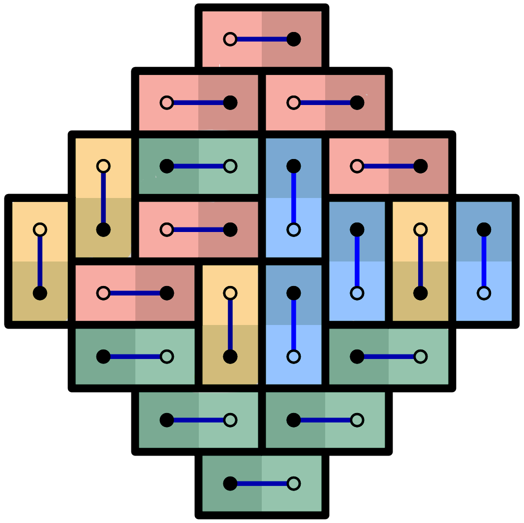
- [Percus'69, Kasteleyn'61]: $Z = |\det K| = \sum_{m \in M} \nu(m)$
- The local statistics for the measure μ on dimer configurations can be computed using **the inverse Kasteleyn matrix**.

Aztec diamond

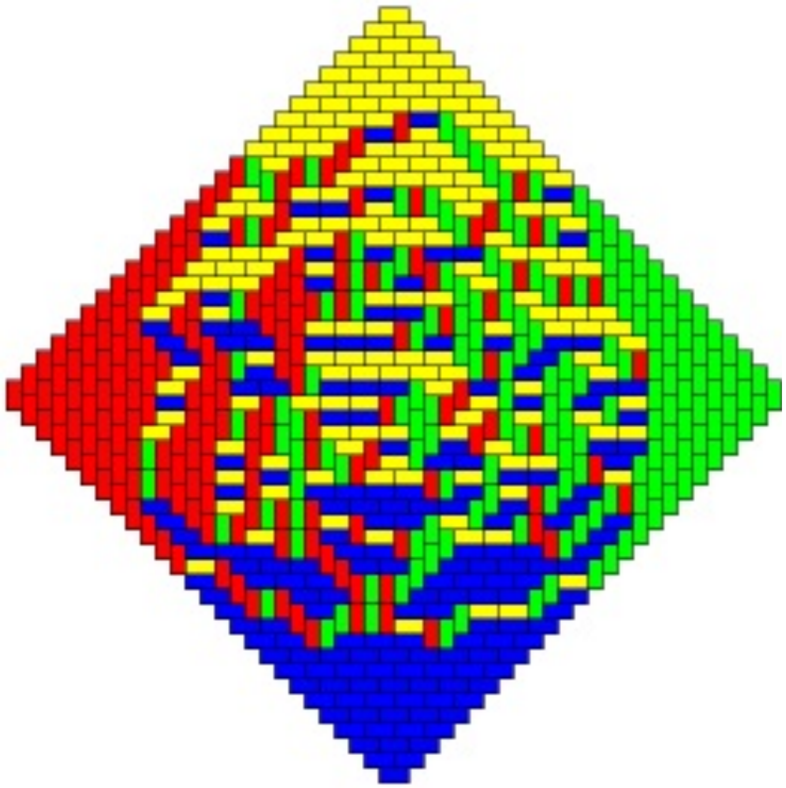
(uniformly weighted)

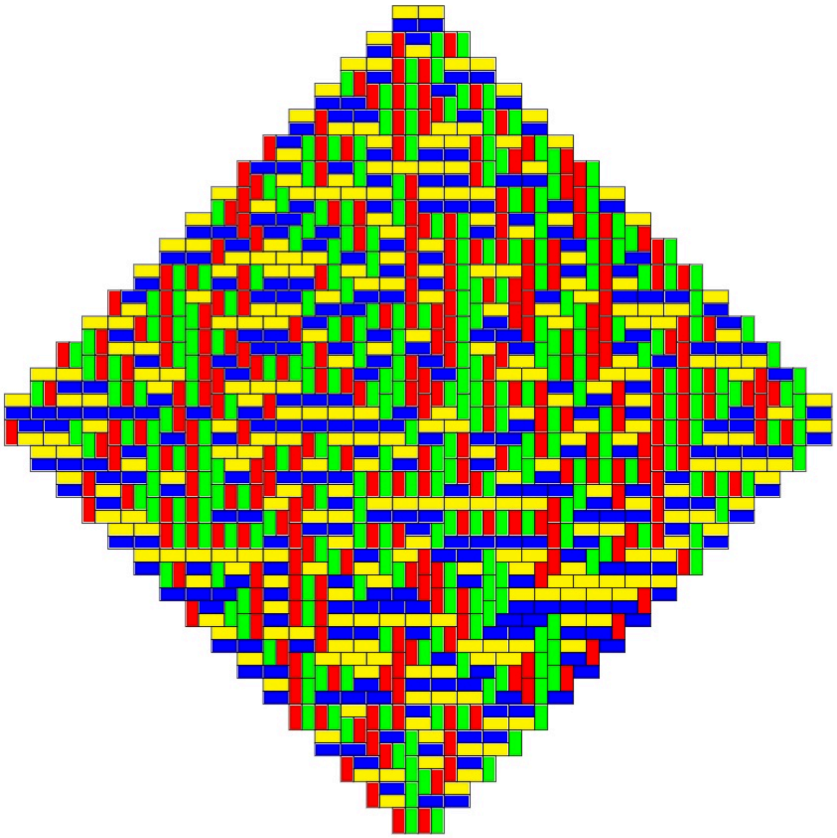
Domino tilings of Aztec diamond



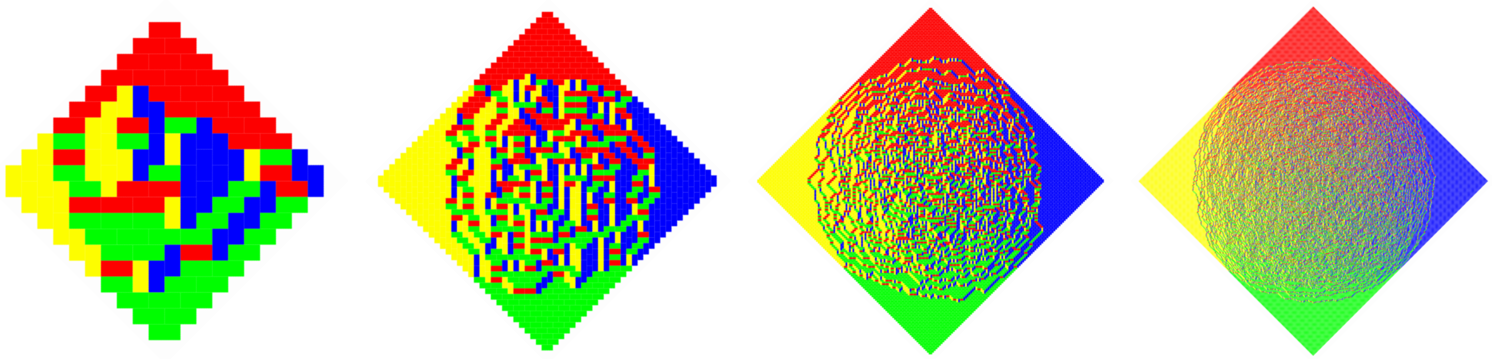


Uniform domino tilings





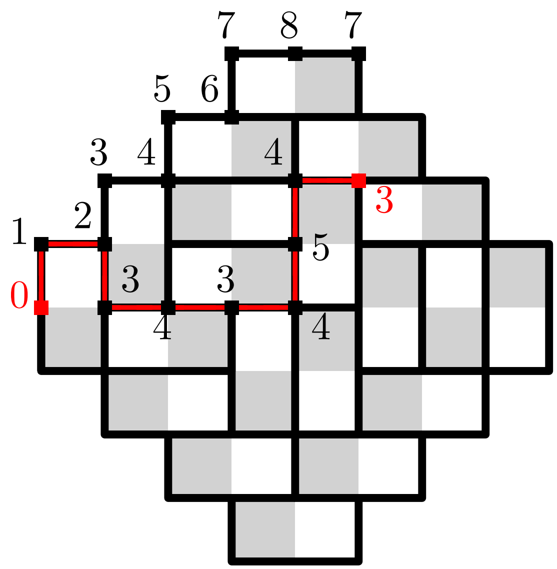
Uniformly distributed domino tilings of the Aztec diamond

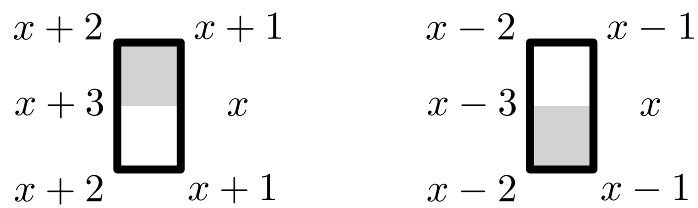


The tiling pictures are generated using a program that was kindly provided by S. Chhita

Height function

Thurston introduced the height function of a tiling which uniquely assigns integer values to all vertices.





Dimer height function on vertices: along each edge not covered by a domino the height changes by ± 1, increases by 1 if this edge has a black face on its left, and decreases by 1 otherwise.

Height function

The key questions: the large-scale behavior of

(a) the limit shape of the height function,
(b) fluctuations of the height function.

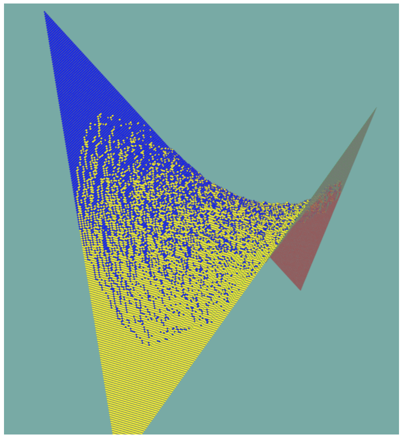
Intuition:

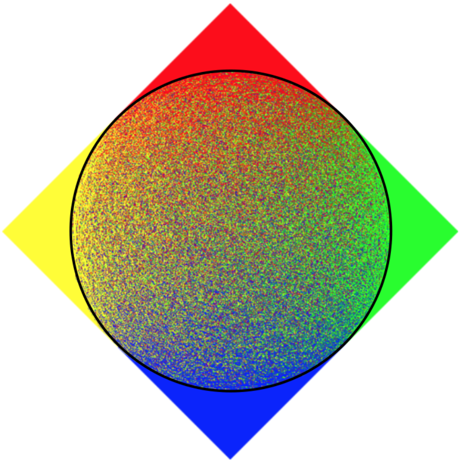
(a) Law of Large Numbers
(b) Central Limit Theorem

Dimer model on Aztec diamond: limit shape

[Cohn, Kenyon, Propp '00], [Cohn, Elkies, Propp '96]

1. **Theorem:** The local density of each type of edge converges to a deterministic limit. Equivalently: The normalized random height function converges to a deterministic limit shape.
2. The region where these densities are strictly between 0 and 1 is called the Liquid Region and is given by $\{x^2 + y^2 < 1/2\}$.

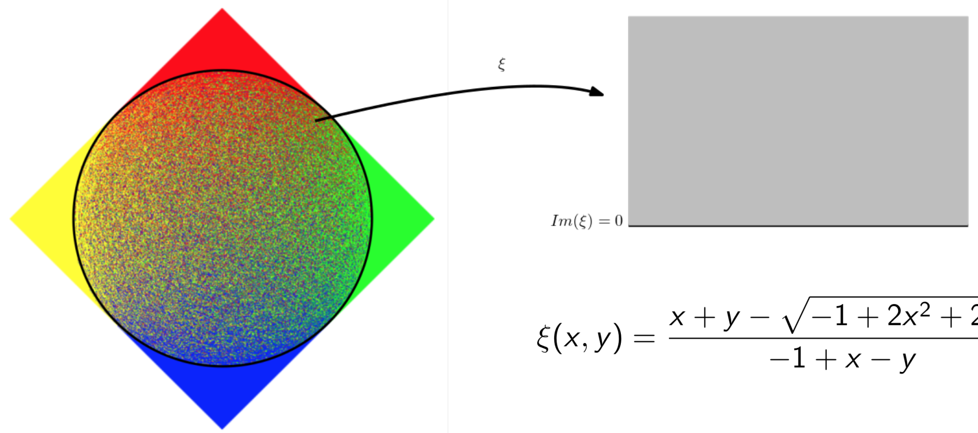




figures from A. & M. Borodin

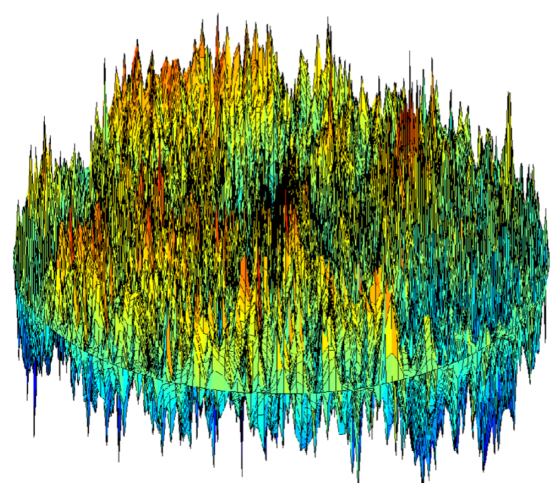
Fluctuations: Gaussian free field

Theorem (Bufetov, Gorin '18) The fluctuations $\tilde{h}_n = h_n - \mathbb{E}[h_n]$ converge to the GFF in the complex structure defined by $\xi(x,y)$: $\tilde{h}_n \circ \xi^{-1} \to F$, where $F = F_{\mathbb{H}}$ is the Dirichlet GFF in the upper half plane.



$$\xi(x,y) = \frac{x + y - \sqrt{-1 + 2x^2 + 2y^2}}{-1 + x - y}$$

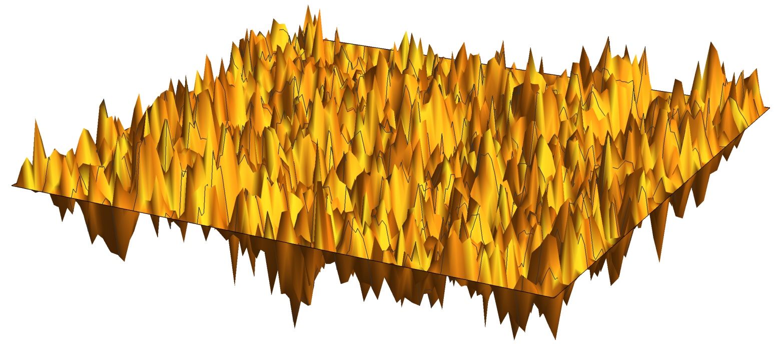
- The GFF is a random generalized function (distribution) on a domain $D \in \mathbb{C}$, a 2-dimensional analogue of a Brownian bridge.
- [Kenyon-Okounkov '05] conjectured it to appear universally in tiling models.



Gaussian Free Field

The Gaussian Free Field is not a random function, but a random distribution.

[1d analog: Brownian Bridge]



A. Kassel

The Gaussian free field Φ on $\mathcal{D}$ is the random distribution such that pairings with test functions $\int_{\mathcal{D}} f\Phi$ are jointly Gaussian with covariance

$$\mathrm{Cov}\left(\int_{\mathcal{D}} f_1\Phi, \int_{\mathcal{D}} f_2\Phi\right) = \int_{\mathcal{D}\times\mathcal{D}} f_1(z)G(z,w)f_2(w).$$

Results

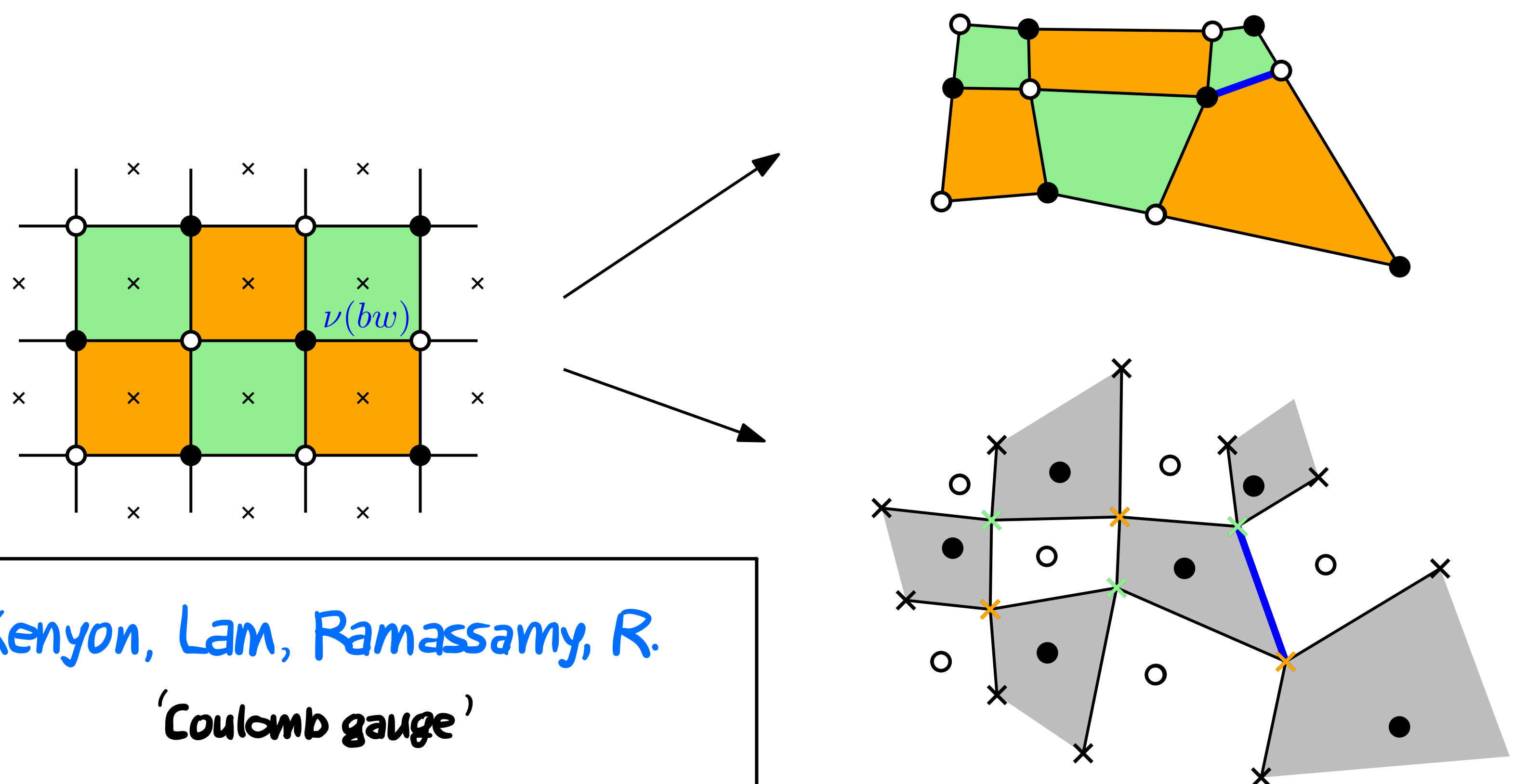
Theorem (Berggren, Nicoletti, R. '23)

Perfect t-embeddings of the uniformly weighted Aztec diamond converge to a Lorentz-minimal surface.

$\Rightarrow$ *convergence of height fluctuations to the Gaussian free field in the conformal parametrization of this surface.*

Perfect t-embeddings

Embeddings of a dimer graph



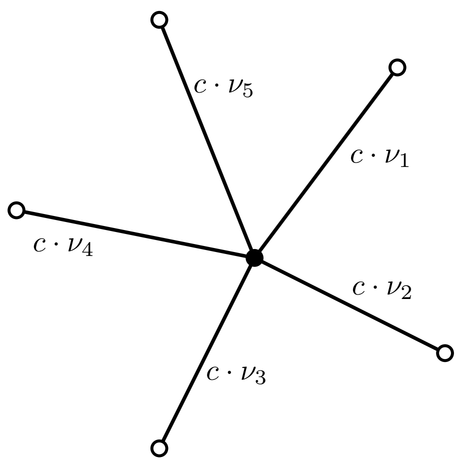
- Kenyon, Lam, Ramassamy, R.

  'Coulomb gauge'

- Chelkak, Laslier, R.

  't-embedding'

Weighted dimers and gauge equivalence



Weight function $\nu : E(\mathcal{G}) \to \mathbb{R}_{>0}$

Probability measure on dimer covers:

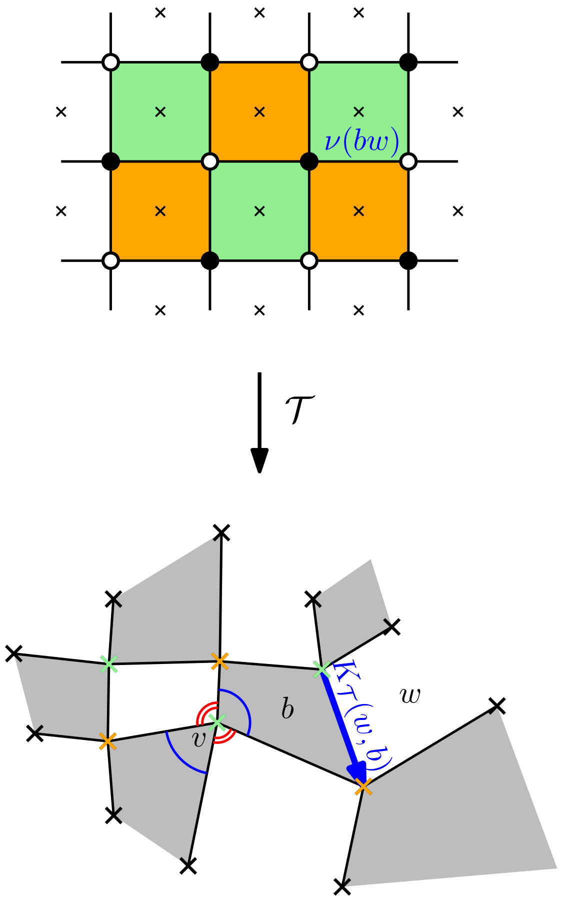
$$\mu(m) = \frac{1}{Z} \prod_{e \in m} \nu(e)$$

Definition

Two weight functions ν_1, ν_2 are said to be *gauge equivalent* if there are two functions $F : B \to \mathbb{R}$ and $G : W \to \mathbb{R}$ such that for any edge bw, $\nu_1(bw) = F(b)G(w)\nu_2(bw)$.

Gauge equivalent weights define the same probability measure μ.

Definition: t-embedding



$K_{\mathcal{T}}$ is a Kasteleyn matrix.

[Chelkak, Laslier, R.]

$\mathcal{T}$ is embedding of $\mathcal{G}^*$ such that

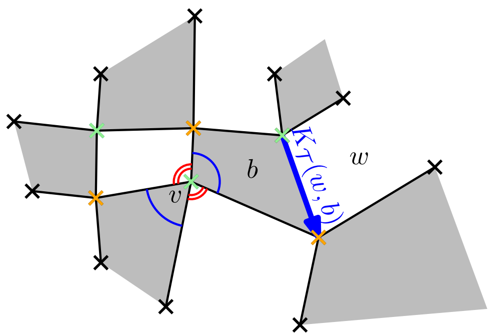
1) **lengths** are gauge equivalent to (given) dimer weights
2) **angles** at (inner) vertices are balanced:

$$\sum_{f \text{ white}} \theta(f,v) = \sum_{f \text{ black}} \theta(f,v) = \pi.$$

Rmk: (2) $\implies$ Kasteleyn sign condition.

Origami map

t-embedding $\mathcal{T}(\mathcal{G}^*)$:

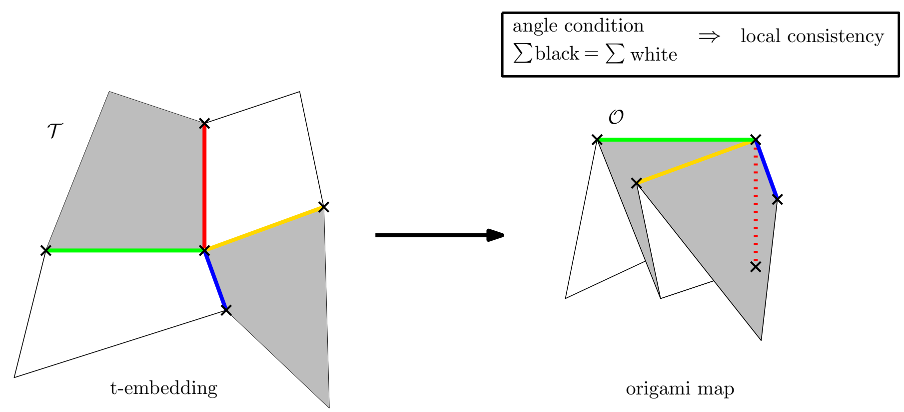


1) **lengths** are gauge equivalent to (given) dimer weights
2) **angles** at vertices are balanced:

$$\sum_{f \text{ white}} \theta(f,v) = \sum_{f \text{ black}} \theta(f,v) = \pi.$$

[Chelkak, Laslier, R.]

To get an origami map $\mathcal{O}(\mathcal{G}^*)$ from $\mathcal{T}(\mathcal{G}^*)$ one can fold the plane along every edge of the embedding.



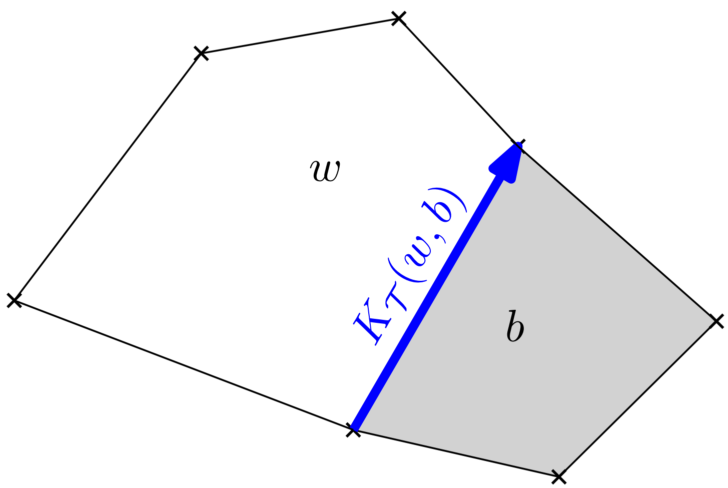
t-embeddings: $(\mathcal{T},\mathcal{O}) \subset \mathbb{R}^{2+2}$

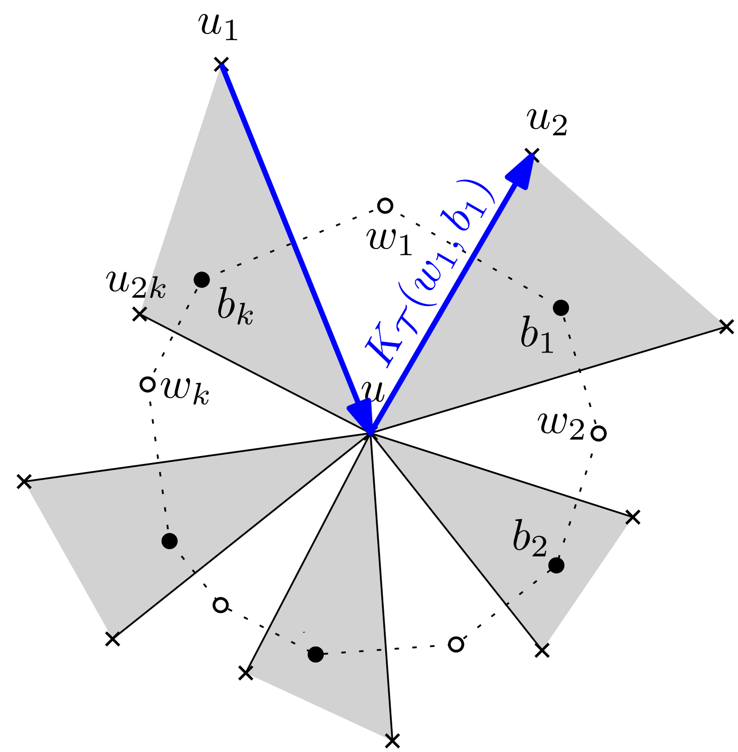
$|\mathcal{O}(z)-\mathcal{O}(z')| \leq |\mathcal{T}(z)-\mathcal{T}(z')|$

discrete space-like surfaces in Minkowski space $\mathbb{R}^{2+2}$

Kasteleyn weights

$$\mathcal{T} \to (\mathcal{G}, K_{\mathcal{T}}), \quad \text{where} \quad \sum_b K_{\mathcal{T}}(w,b) = \sum_w K_{\mathcal{T}}(w,b) = 0$$





Then $K_{\mathcal{T}}$ is a Kasteleyn matrix.

Kasteleyn sign condition ⟷ angle condition

$$\prod \frac{K_{\mathcal{T}}(w_i,b_i)}{K_{\mathcal{T}}(w_{i+1},b_i)} \in (-1)^{k+1}\mathbb{R}_+ \qquad \sum \text{white} = \pi \mod 2\pi$$

General setup

Theorem (Kenyon, Lam, Ramassamy, R. '19)

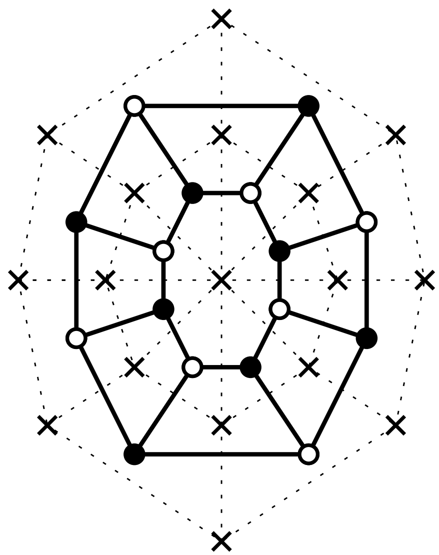
t-embeddings exist at least in the following cases:

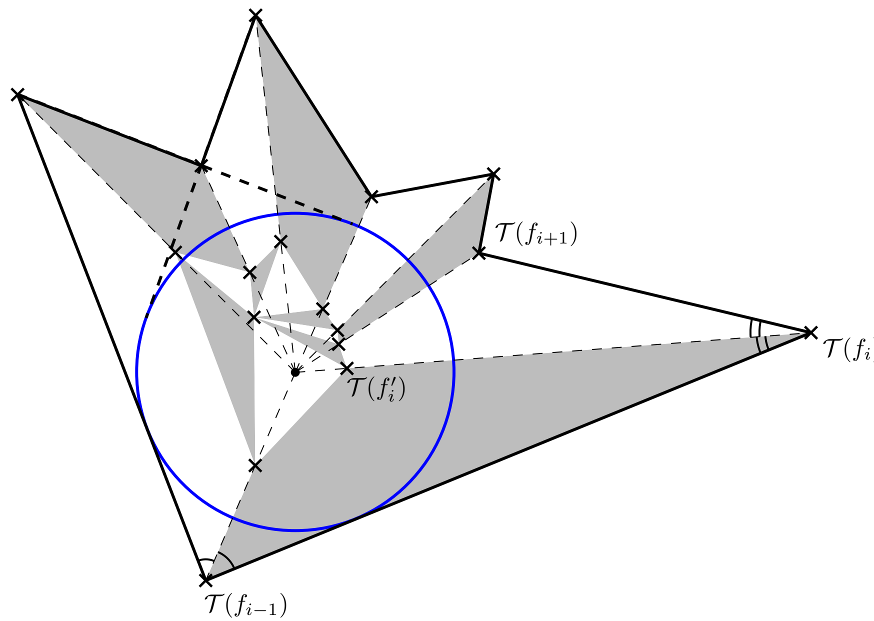
- *If $\mathcal{G}^\delta$ is a bipartite finite graph with outer face of degree 4.*
- *If $\mathcal{G}^\delta$ is a biperiodic bipartite graph.*

Scaling limit results: [Chelkak, Laslier, R. '20-21]

- Develop new discrete complex analysis techniques on t-embeddings
- Perfect t-embeddings reveal the relevant conformal structure of the Dimer model

Perfect t-embeddings

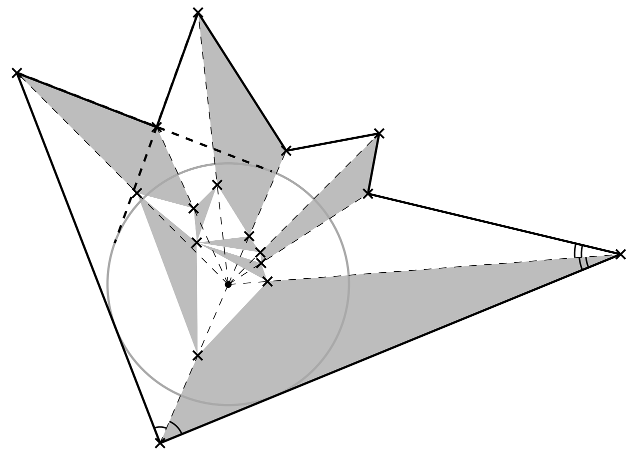


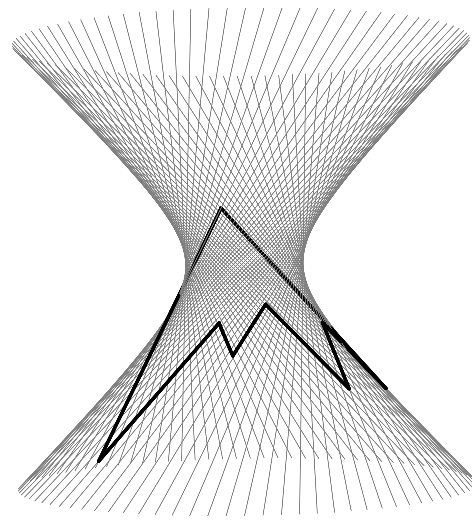


Definition [Chelkak, Laslier, R.] Perfect t-embeddings:

- outer face is tangental (not necessary convex)
- outgoing edges = bisectors

General setup





Theorem (Chelkak, Laslier, R. '21)

Assume $\mathcal{G}^\delta$ are perfectly t-embedded.

a) *Technical assumptions on faces*

b) *The origami maps converge to a maximal surface in the Minkowski space $\mathbb{R}^{2,1}$*

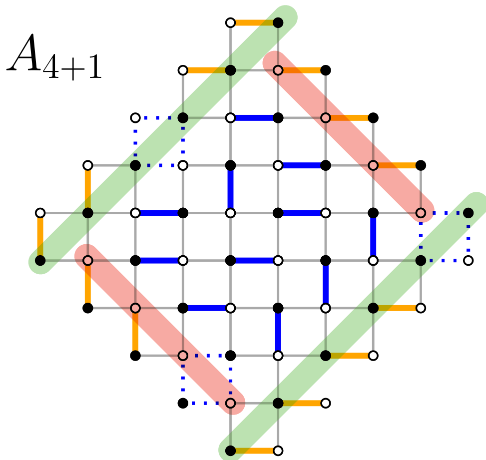
$\Rightarrow$ *convergence to the Gaussian free field in the conformal parametrization of this surface.*

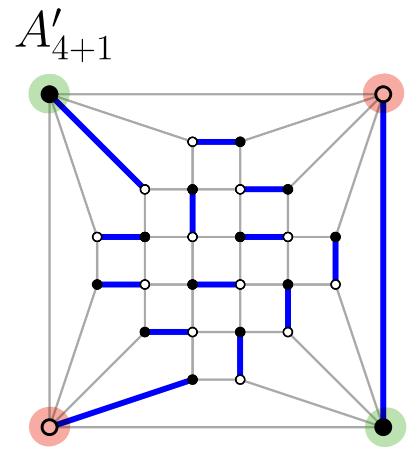
Rmk: Existence of perfect t-embeddings remains an open question.

Perfect t-embedding

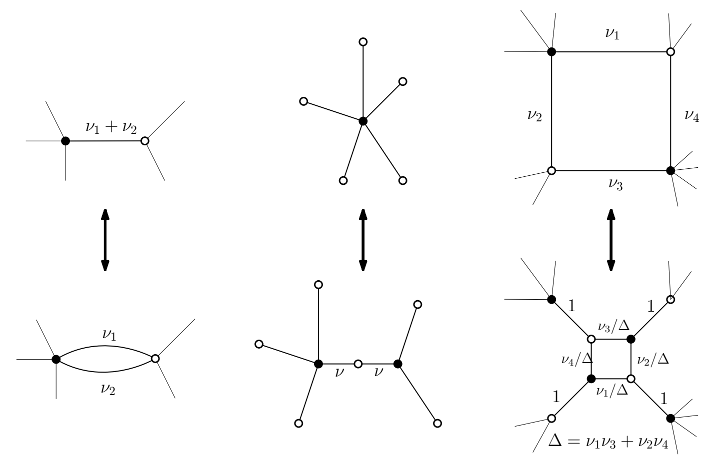
of Aztec diamond

Reduced Aztec diamond





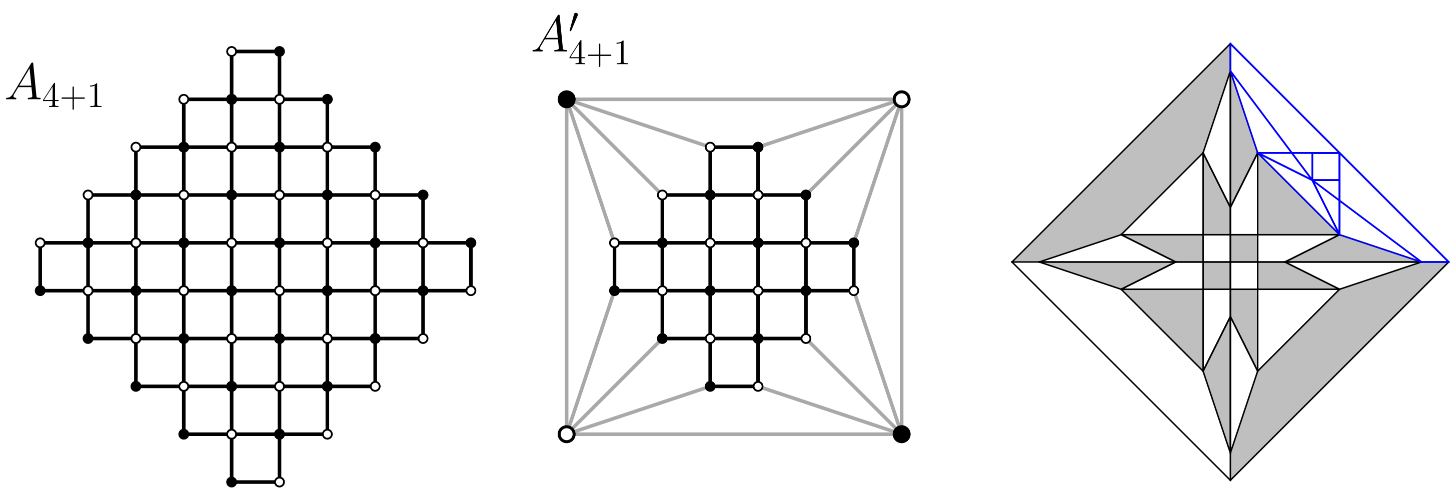
Elementary transformations preserving the dimer measure



[Kenyon, Lam, Ramassamy, R. '19]:

T-embeddings of $\mathcal{G}^*$ are preserved under elementary transformations of $\mathcal{G}$.

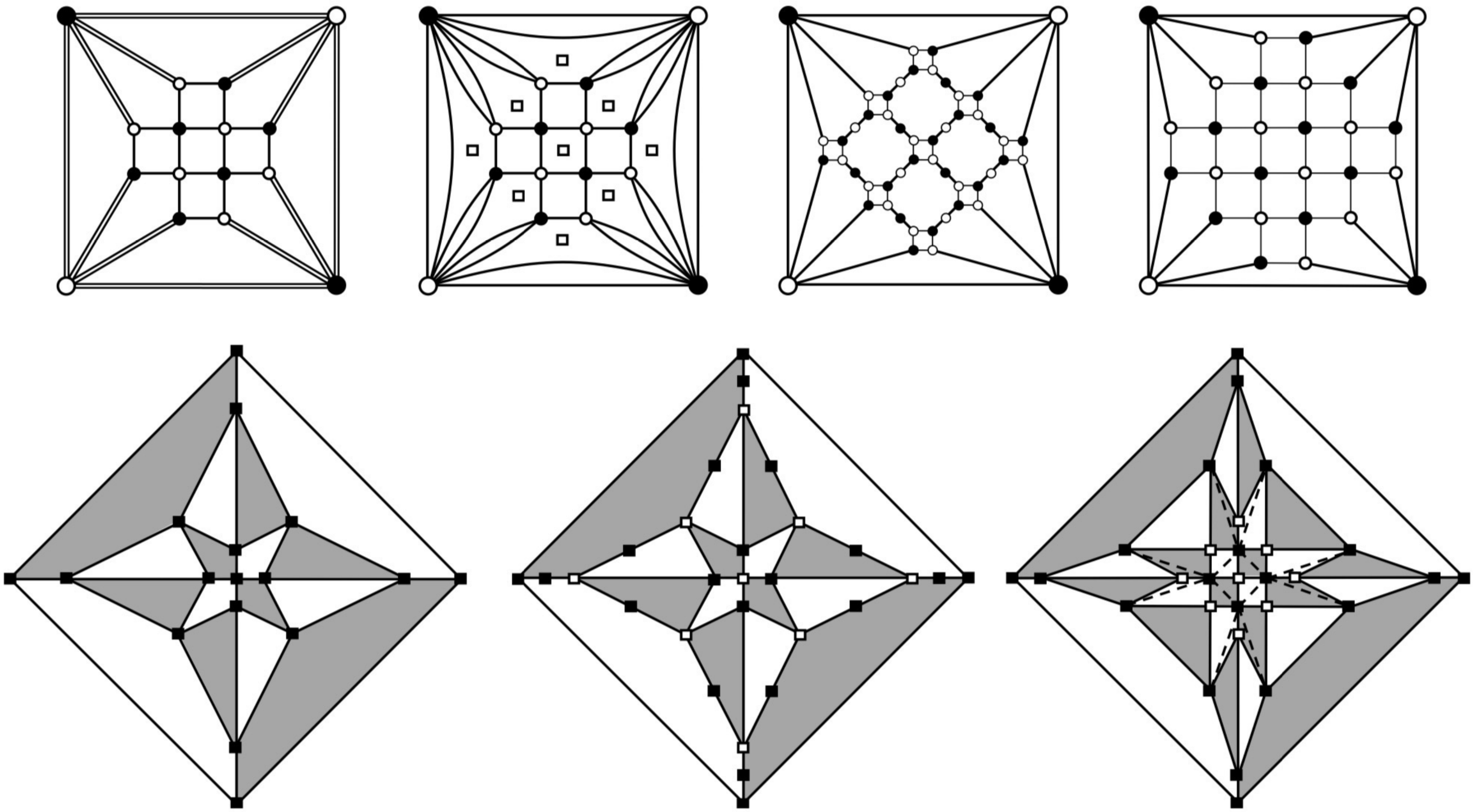
Perfect t-embedding of Aztec diamond



[Chelkak, Ramassamy '21] introduced a construction of perfect t-embeddings $\mathcal{T}_n(\mathcal{G}^*)$ of homogeneous Aztec diamond:

$$\mathcal{T}_{n+1}(j,k)+\mathcal{T}_{n-1}(j,k) = \frac{1}{2}\Big(\mathcal{T}_n(j-1,k)+\mathcal{T}_n(j+1,k)+\mathcal{T}_n(j,k+1)+\mathcal{T}_n(j,k-1)\Big).$$

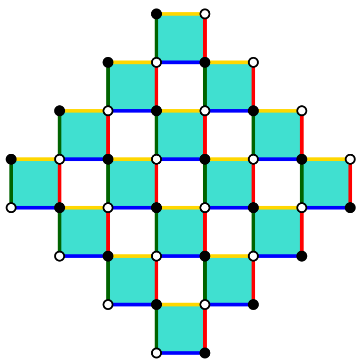
Perfect t-embedding of Aztec diamond



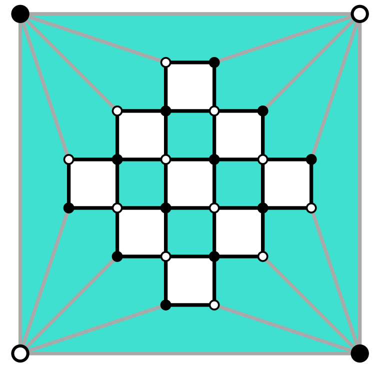
[Chelkak, Ramassamy '21]

Perfect t-embedding of Aztec diamond

A_4



A'_{4+1}



Theorem (Berggren, Nicoletti, R. '23)

For $|j| + |k| < n$ and $j + k + n$ odd, let $p_E(j,k,n)$ (p_N, p_W, p_S) denote the probability that the edge on the East (North, West, South, resp.) boundary of the face (j,k) is present in a uniformly random dimer cover of A_n. Then

$$\mathcal{T}_n(j,k) = \underline{p_E(j,k,n)} + \underline{ip_N(j,k,n)} - \underline{p_W(j,k,n)} - \underline{ip_S(j,k,n)}.$$

Perfect t-embedding of Aztec diamond

$$\mathcal{T}_n(j,k) = p_E(j,k,n) + ip_N(j,k,n) - p_W(j,k,n) - ip_S(j,k,n).$$

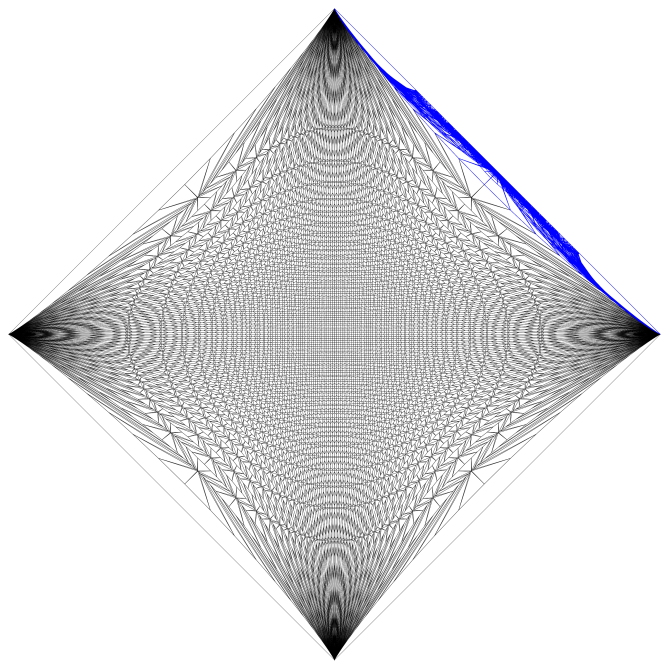
$$\mathcal{O}_n(j,k) = p_E(j,k,n) + ip_N(j,k,n) + p_W(j,k,n) + ip_S(j,k,n).$$

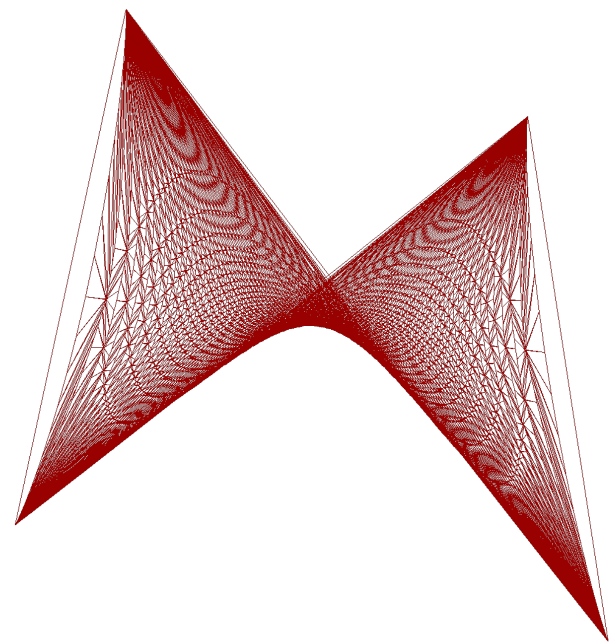
Define $\mathcal{O}'_n := e^{i\frac{\pi}{4}}\left(\mathcal{O}_n - \frac{1+i}{2}\right)$, which is a composition of a translation and rotation of the initial origami map.

- The edge probabilities can be expressed in terms of *inverse Kasteleyn matrix*, which is known to admit a double integral formula.
- This provides us with expressions of $\mathcal{T}_n$ and $\mathcal{O}'_n$ in terms of double integrals.
- The integral expression allows for asymptotic analysis using a classical stepest descent analysis

The **first order** term in the asymptotic expansion of $\mathcal{T}_n$ and $\mathcal{O}'_n$ is enough to prove the following theorem.

Perfect t-embedding of Aztec diamond





Theorem (Berggren, Nicoletti, R. '23)

The pair $(\mathcal{T}_n, \mathcal{O}_n) \to (z, \vartheta(z))$*, as* $n \to \infty$*, where* $(z, \vartheta(z)) \in \mathbb{R}^2 \times \mathbb{R}$ *is the graph of a Lorentz-minimal surface.*

Remark: This confirms the prediction of [Chelkak, Ramassamy].

Perfect t-embedding of Aztec diamond

The main theorem of [CLR'21] assumes the existence of a sequence of perfect t-embeddings $\mathcal{T}_n$ satisfying the following three properties

I) The pair $(\mathcal{T}_n, \mathcal{O}'_n) \to (z, \vartheta(z))$, as $n \to \infty$, where $(z, \vartheta(z)) \in \mathbb{R}^2 \times \mathbb{R}$ is the graph of a Lorentz-minimal surface;

II) At the discrete level the origami map is Lipschitz continuous with constant strictly less then one;

III) For almost every face, the radius of the largest circle which can be inscribed in the face cannot decay exponentially fast as $n \to \infty$.

Rmk: Assumptions (II) and (III) are conditions on the discrete level, therefore the leading term in the asymptotic expansion of $\mathcal{T}_n$ is not sufficient to prove these assumptions (need to go to the **2nd order** term).

Rigidity condition

Theorem (Berggren, Nicoletti, R. '23)

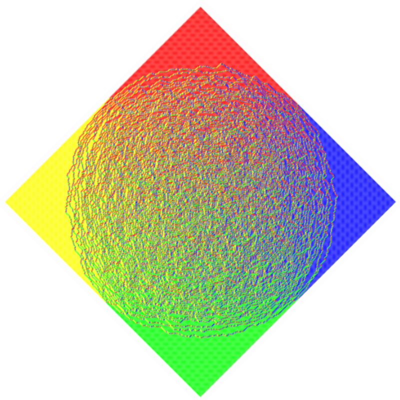
Given a compact set $\mathcal{K} \subset \Omega$, there exist positive $N_{\mathcal{K}}, C_{\mathcal{K}}$ and $\varepsilon_{\mathcal{K}}$ which only depend on $\mathcal{K}$, such that for all pairs of vertices $v \sim v'$ of the dual graph $(A'_n)^*$ such that both $\mathcal{T}_n(v), \mathcal{T}_n(v') \in \mathcal{K}$ we have

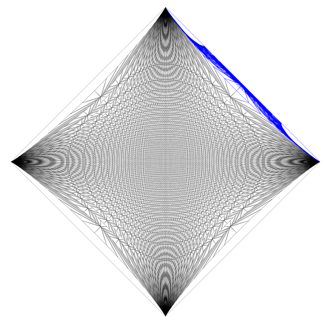
$$\frac{1}{nC_{\mathcal{K}}} \leq |\mathcal{T}_n(v') - \mathcal{T}_n(v)| \leq \frac{C_{\mathcal{K}}}{n}$$

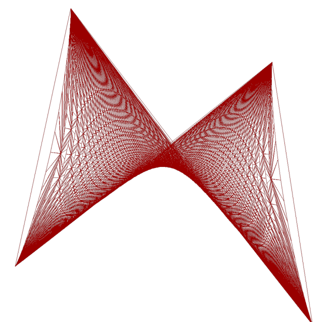
for all $n > N_{\mathcal{K}}$.

In addition the angles of the faces of the perfect t-embedding inside $\mathcal{K}$ are contained in $(\varepsilon_{\mathcal{K}}, \pi - \varepsilon_{\mathcal{K}})$ for all $n > N_{\mathcal{K}}$.

The scaling limit of dimer fluctuations in homogeneous Aztec diamonds via the intrinsic conformal structure of a Lorentz-minimal surface.



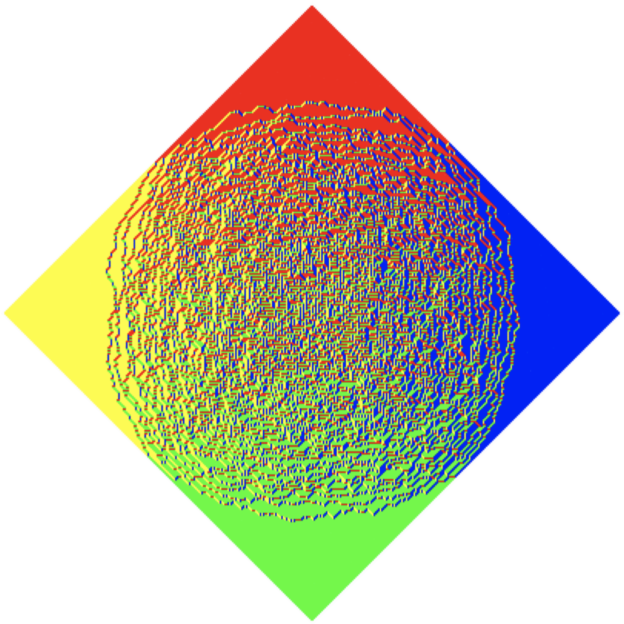


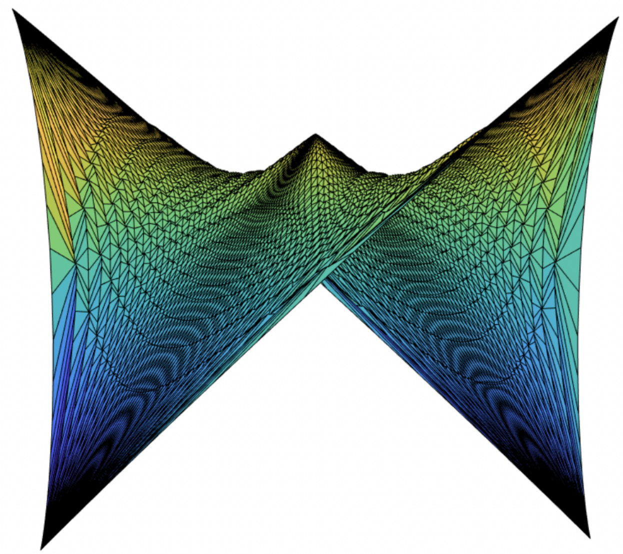


Theorem (Berggren, Nicoletti, R. '23)

Let $\mathcal{T}_n$ be the sequence of perfect t-embeddings of the reduced uniformly weighted Aztec diamonds A'_{n+1}, with corresponding origami maps $\mathcal{O}_n$. All assumptions of the main theorem of [CLR'21] hold for the sequence $\mathcal{T}_n$.

Thank you!





Aztec diamond with gaseous regions