Exact solutions and area bounds in the moving sofa problem

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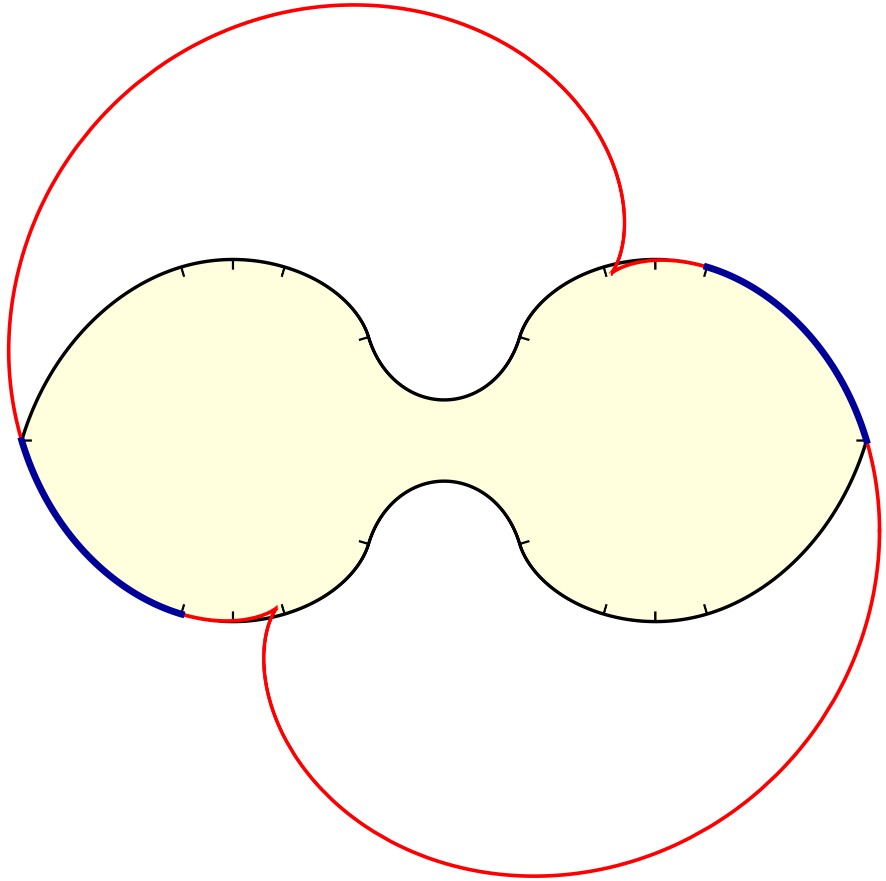


Talk outline

1. Introduction to the problem and exact solutions

   Based on the paper "Differential equations and exact solutions in the moving sofa problem" (R. 2016).

2. Improved area bounds and computer-assisted proofs

   Based on the paper "Improved upper bounds in the moving sofa problem" (joint work with Yoav Kallus, 2017)

3. New developments

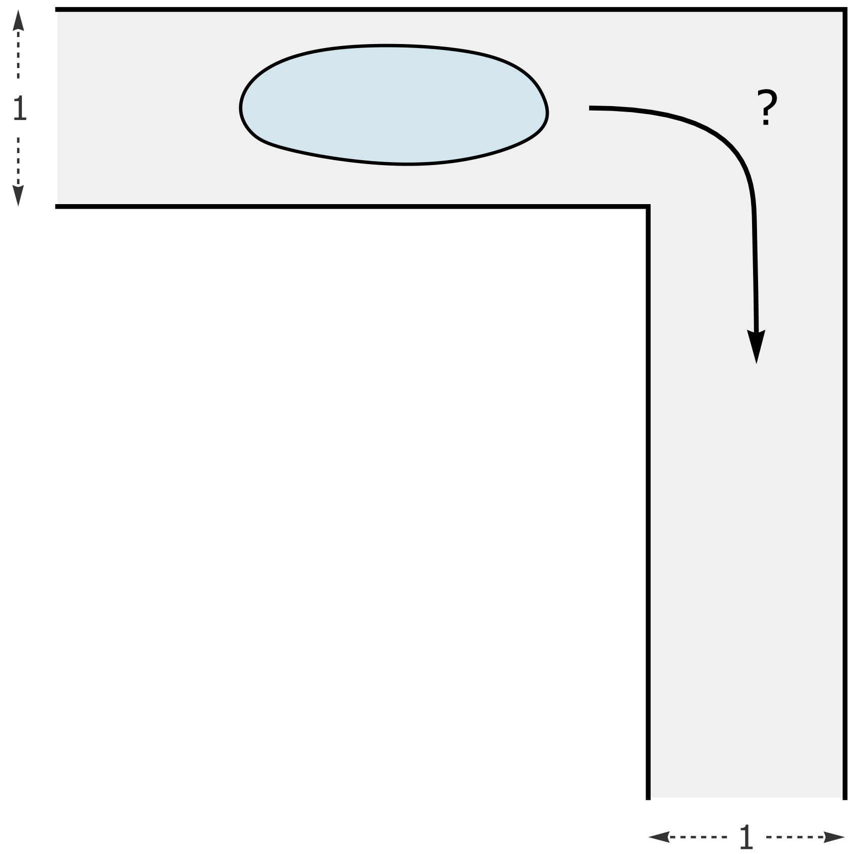
   Based on the preprint "A conditional upper bound for the moving sofa problem" (Jineon Baek, 2024)

Part 1: Introduction to the problem and exact solutions

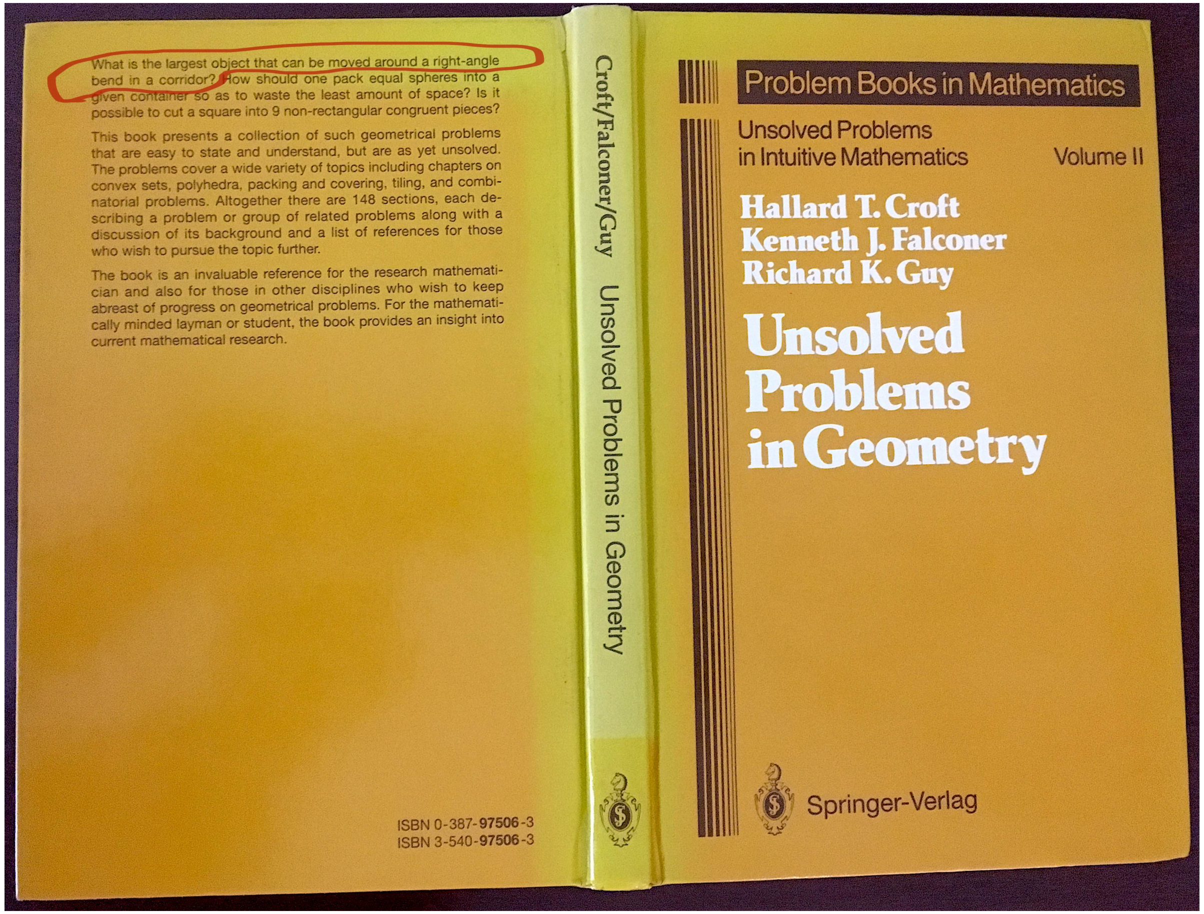
The moving sofa problem

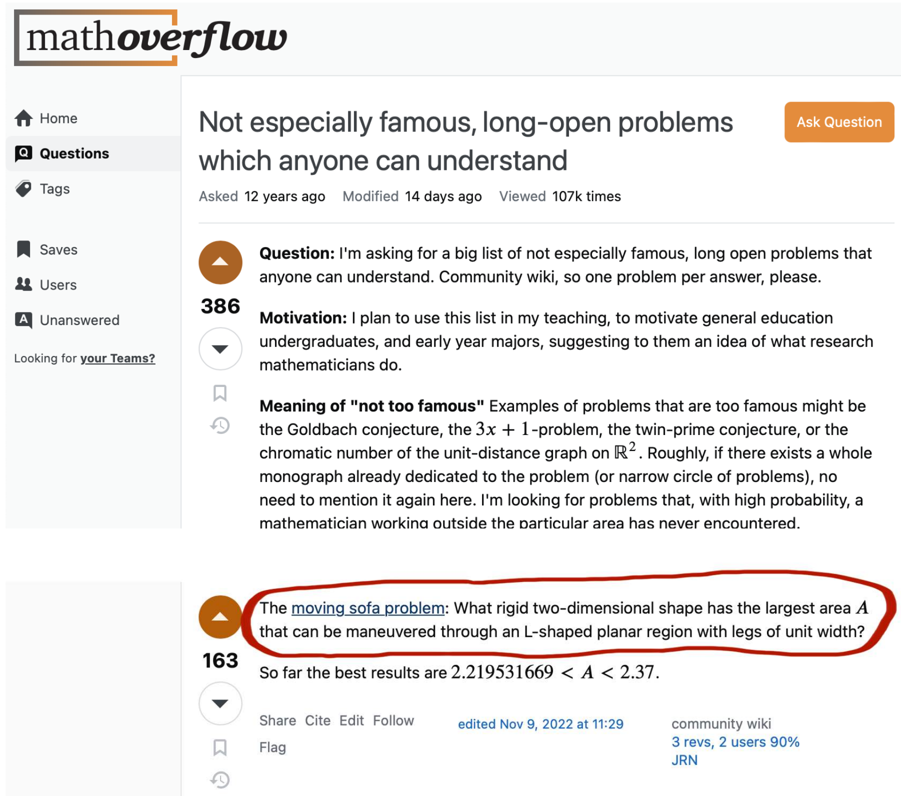
Leo Moser asked in 1966 the following question, now known as the **moving sofa problem**:

What is the largest shape in the plane that can be moved around a right-angled turn in a hallway of unit width?



The problem, which remains unsolved, has captured the imaginations of mathematicians and math enthusiasts.

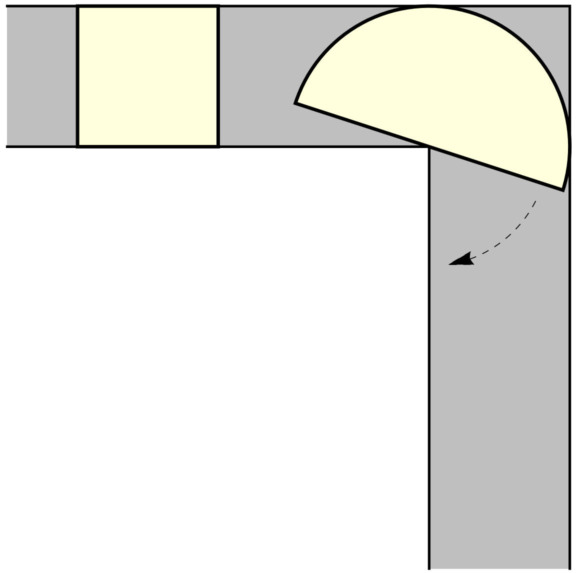




Initial observations

Some trivial examples of valid sofa shapes are:

- a unit square (area 1)
- a unit semicircle (area $\pi/2 \approx 1.57$)



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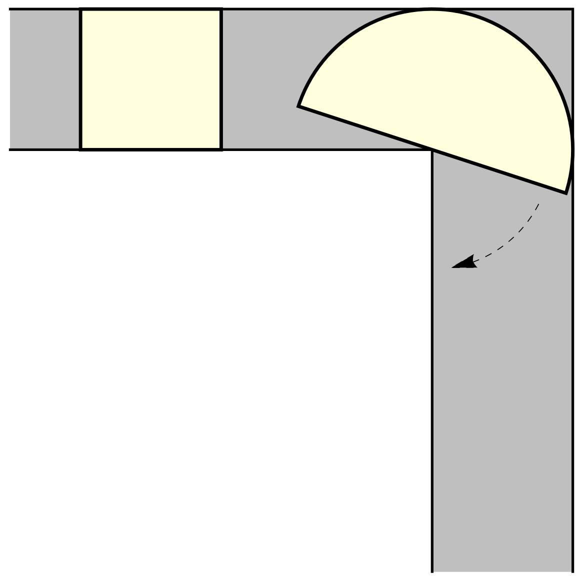

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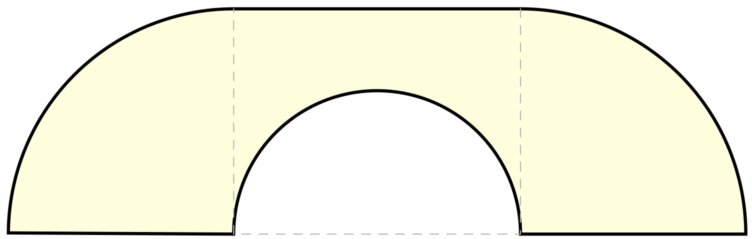
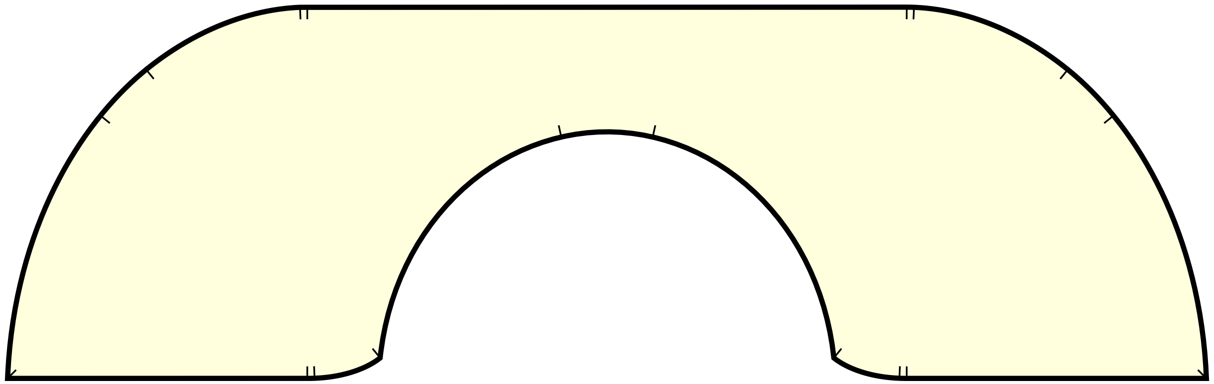
A more interesting example:



- **Hammersley's sofa**, proposed by John Hammersley in 1968. Its area is $\pi/2 + 2/\pi \approx 2.2074$.

Gerver's sofa

Joseph Gerver proposed in 1992 a construction with larger area, known as **Gerver's sofa**:



The boundary of Gerver's sofa has 18 distinct pieces, each given by a separate analytic formula. He conjectured that his shape has the largest area. That is still open.

The area of Gerver's sofa is an exotic number, **Gerver's constant**

$$\mu_{\mathrm{Gerver}} = 2.21953166\ldots.$$

Area bounds

Let μ_{MS} denote the maximal area of a moving sofa, the so-called **moving sofa constant**.

Gerver's construction establishes the lower bound

$$\mu_{\mathrm{MS}} \geq \mu_{\mathrm{Gerver}} \approx 2.2195,$$

conjectured to be sharp.

In the opposite direction, Hammersley (1968) proved the upper bound

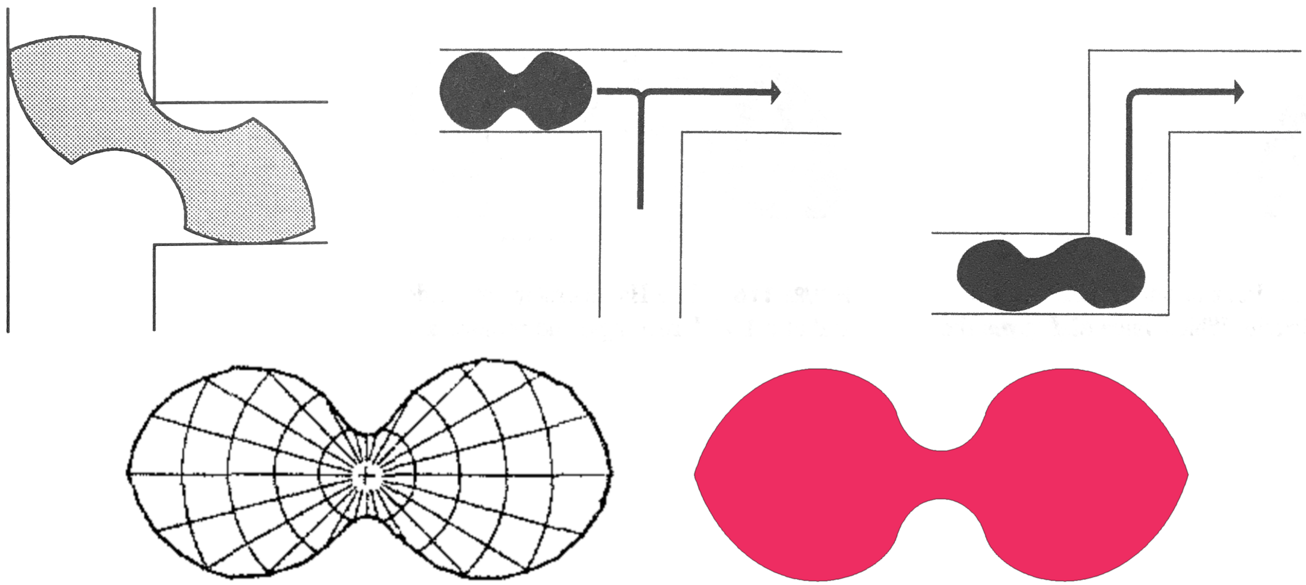
$$\mu_{\mathrm{MS}} \leq 2\sqrt{2} \approx 2.82.$$

In 2017, Yoav Kallus and I improved this to 2.37. Baek (2024) proved an upper bound of $1 + \frac{\pi^2}{8} \approx 2.234$ for a class of moving sofas satisfying a certain technical assumption.

The ambidextrous moving sofa problem

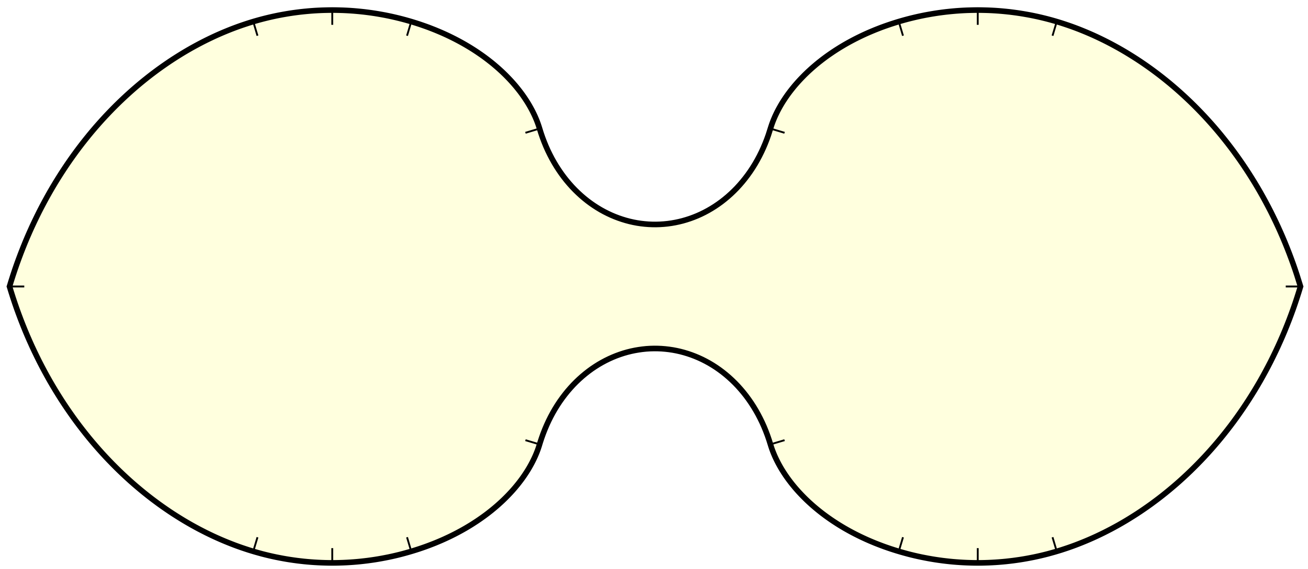
The **ambidextrous moving sofa problem** is a variant of the moving sofa problem studied by John Conway and others since the 1960s. In this variant, we ask for the shape of largest area that can turn a right-angled corner both to the right and to the left.

Some suggestions made over the years (Maruyama, Gibbs and others):



An ambidextrous moving sofa

In 2016 I derived an analytic shape that, analogously to Gerver's sofa, is a good candidate to be a solution to the ambidextrous moving sofa problem. It has area ≈ 1.64495 (which confirms a 2014 prediction of Gibbs based on numerical calculations), and its boundary has 18 distinct curved segments, each obtained by solving a differential equation.



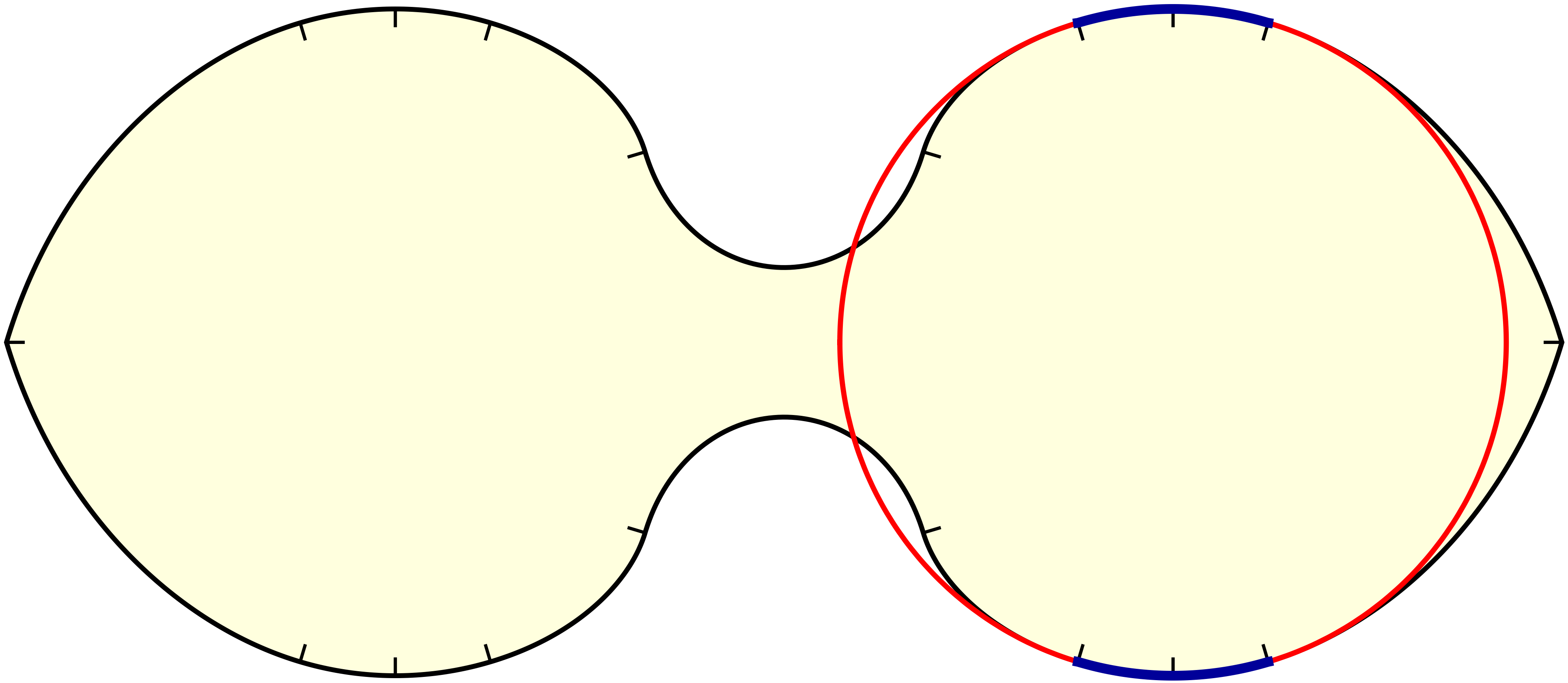
Surprisingly, unlike Gerver's sofa, the new ambidextrous shape *can be expressed in closed form*. In fact, it is piecewise algebraic![†] Its area is given by the strange constant

$$\sqrt[3]{3+2\sqrt{2}}+\sqrt[3]{3-2\sqrt{2}}-1+\arctan\left[\frac{1}{2}\left(\sqrt[3]{\sqrt{2}+1}-\sqrt[3]{\sqrt{2}-1}\right)\right]$$

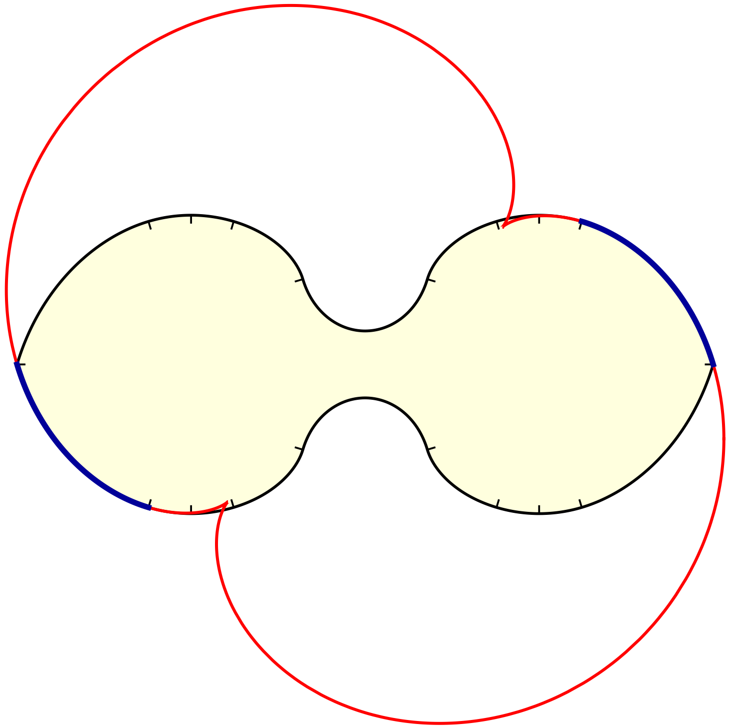
$$= 1.64495521842544 0\ldots.$$

[†] Note to self: thank Greg.

Some of its segments are circular arcs:



Others are more complicated algebraic curves:



(a *nephroid*)

$$\left(X^2 + Y^2 - 8\right)^3 - 216(Y - X)^2 = 0,$$

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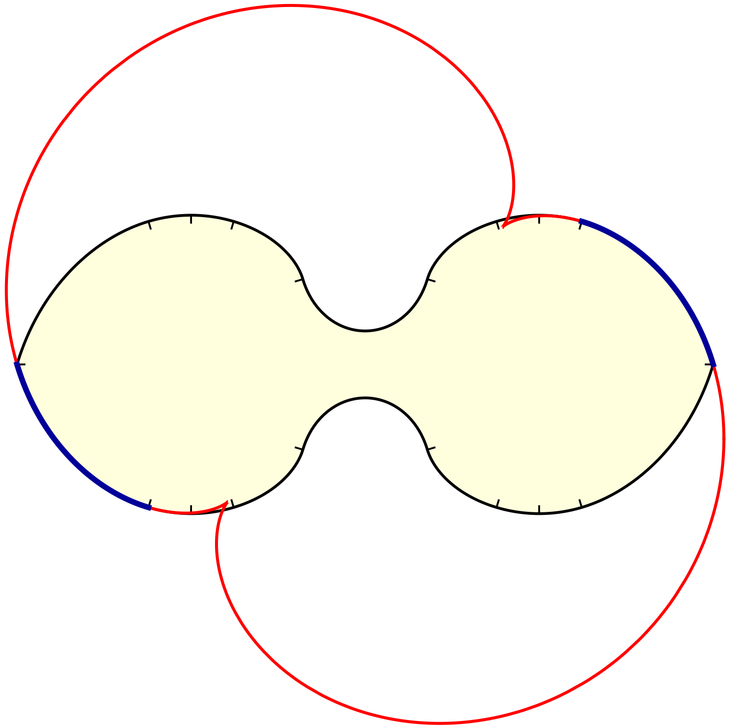

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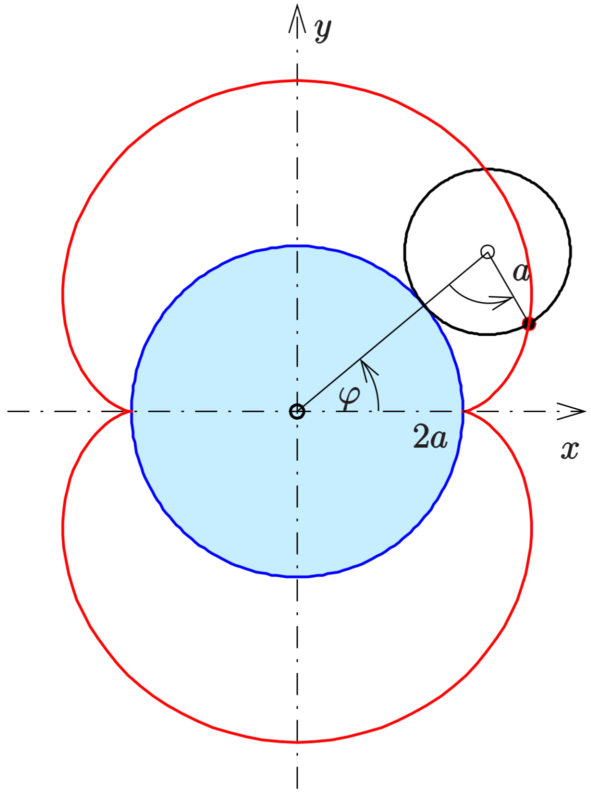
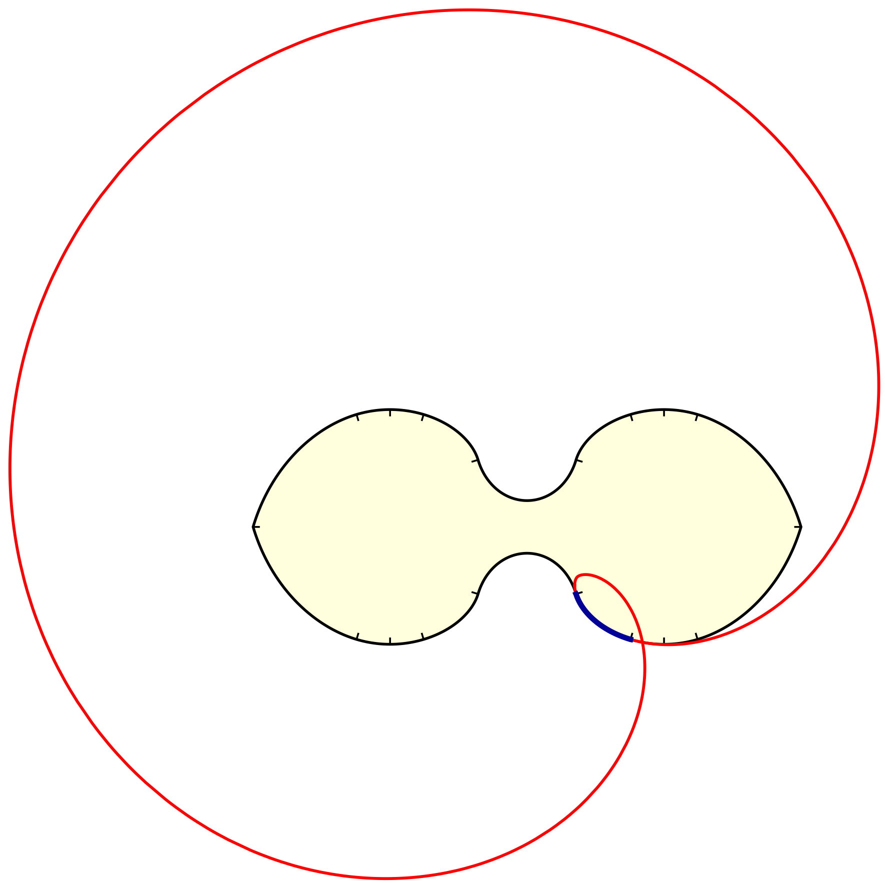
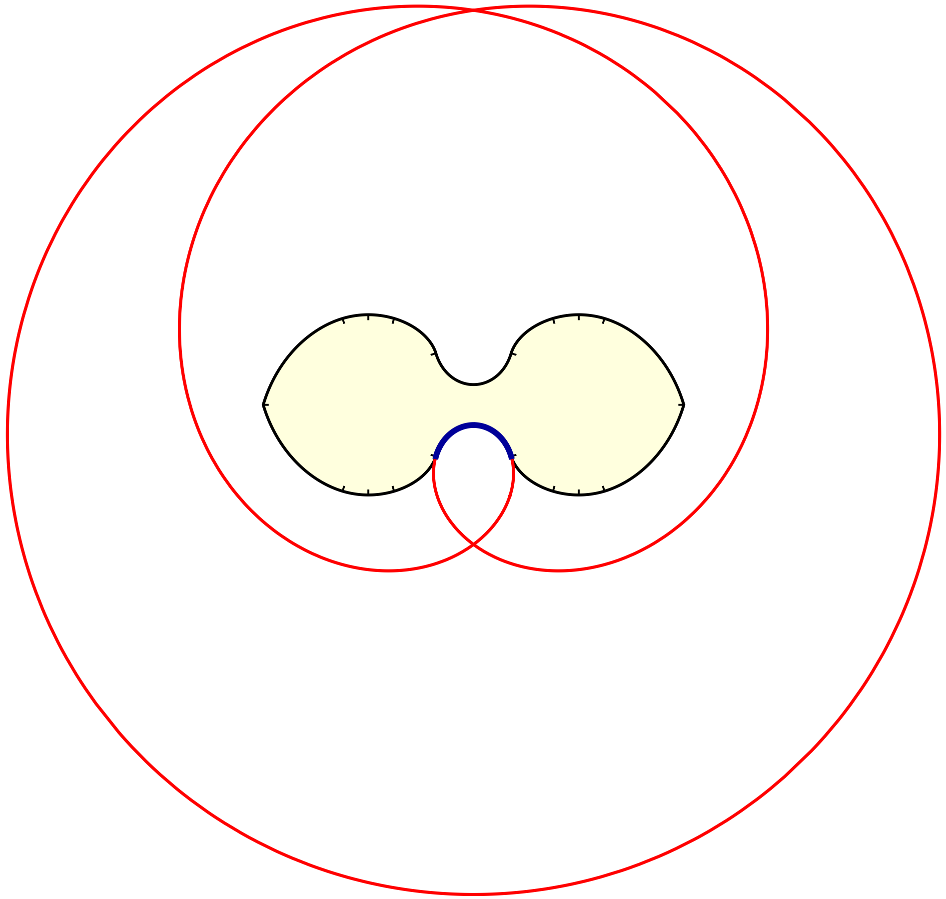


(source: Wikipedia)

$$\left(X^2+Y^2-8\right)^3-216(Y-X)^2=0,$$



$$\left(X^2+Y^2\right)^3-12\gamma_1\left(X^2+Y^2\right)^2-216\sqrt{\gamma_2}\left(X^2+Y^2\right)(Y-X)$$
$$-12\gamma_3\left(X^2+Y^2\right)-432\sqrt{\gamma_4}(Y-X)+432XY-32\gamma_5=0,$$



$$\left(X^2+Y^2\right)^3-24\alpha_1\left(X^2+Y^2\right)^2+48\alpha_2\left(X^2+Y^2\right)$$
$$+13824\sqrt{\alpha_3}\,Y+4096\alpha_4=0,$$

…where $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ are explicit algebraic numbers in $\mathbb{Q}[(4+2\sqrt{2})^{1/3} + (4-2\sqrt{2})^{1/3}]$.

Constructing the candidate solution sofas

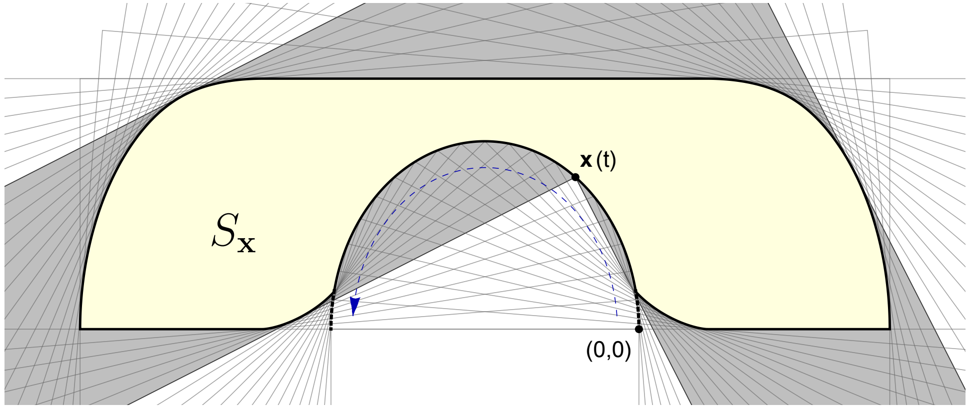
My derivation, like Gerver's, is based on the observation that a maximal area moving sofa cannot be perturbed locally near some point (or finite set of points) along the boundary to increase the area.

I showed that, under mild smoothness assumptions, this local optimality leads to a family of six ordinary differential equations its boundary must satisfy.

To make this precise, we start by parametrizing sofa shapes in terms of their **rotation path** $\mathbf{x}(t)$.

Rotation paths and contact paths

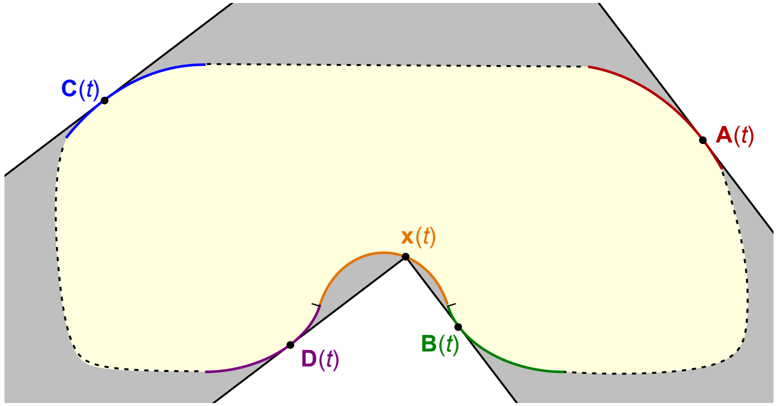
The **rotation path** $\mathbf{x}(t)$ is the path of the inner corner of the corridor as it is rotating around the sofa between the angles 0 and α (usually assumed to be $\pi/2$). Given such a path, it defines a sofa shape $S_{\mathbf{x}}$.



Can we describe precisely the mapping $\mathbf{x} \mapsto S_{\mathbf{x}}$? It seems extremely difficult to do in general. But under mild assumptions, one can give a simple description of $\mathbf{x} \mapsto S_{\mathbf{x}}$ in terms of **contact paths**. Denote

$$\boldsymbol{\mu}_t = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, \quad \boldsymbol{\nu}_t = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \quad \text{(a rotating orthonormal frame).}$$

Lemma. The four **contact paths** $\mathbf{A}(t), \mathbf{B}(t), \mathbf{C}(t), \mathbf{D}(t)$ in the figure



are given (assuming they are well-defined) by

$$\mathbf{A}(t) = \mathbf{x}(t) + \left\langle \dot{\mathbf{x}}(t), \boldsymbol{\mu}_t \right\rangle \boldsymbol{\nu}_t + \boldsymbol{\mu}_t,$$

$$\mathbf{B}(t) = \mathbf{x}(t) + \left\langle \dot{\mathbf{x}}(t), \boldsymbol{\mu}_t \right\rangle \boldsymbol{\nu}_t,$$

$$\mathbf{C}(t) = \mathbf{x}(t) - \left\langle \dot{\mathbf{x}}(t), \boldsymbol{\nu}_t \right\rangle \boldsymbol{\mu}_t + \boldsymbol{\nu}_t$$

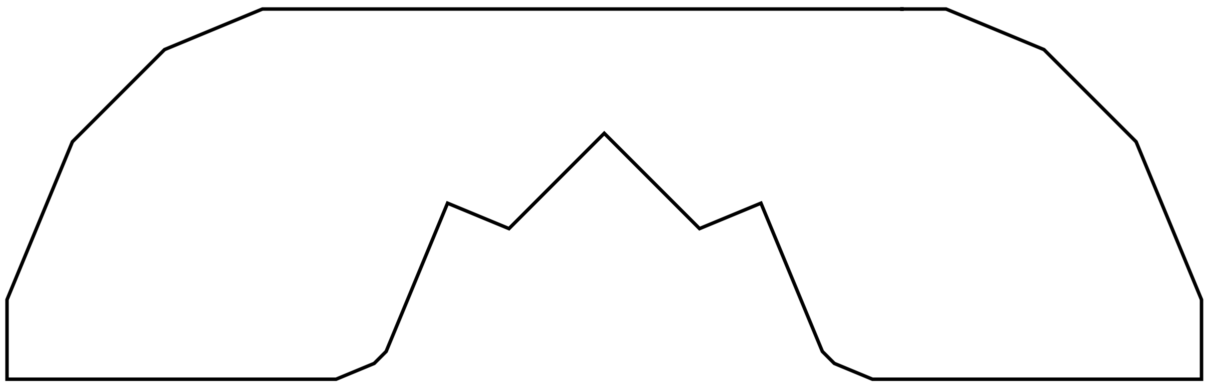
$$\mathbf{D}(t) = \mathbf{x}(t) - \left\langle \dot{\mathbf{x}}(t), \boldsymbol{\nu}_t \right\rangle \boldsymbol{\mu}_t.$$

Proof. A short, easy computation — see my paper.

The differential equations

Gerver considered a discrete-geometric approximation to the moving sofa problem where one intersects only finitely many translated and rotated copies of the corridor. This leads in a natural way to the concept of a *balanced polygon*:

Definition. A planar polygon is called **balanced** if for any side of the polygon, that side and all other sides that are parallel to it lie on one of two lines, such that the distance between the lines is 1 and the total lengths of the sides lying on each of the two lines are equal.



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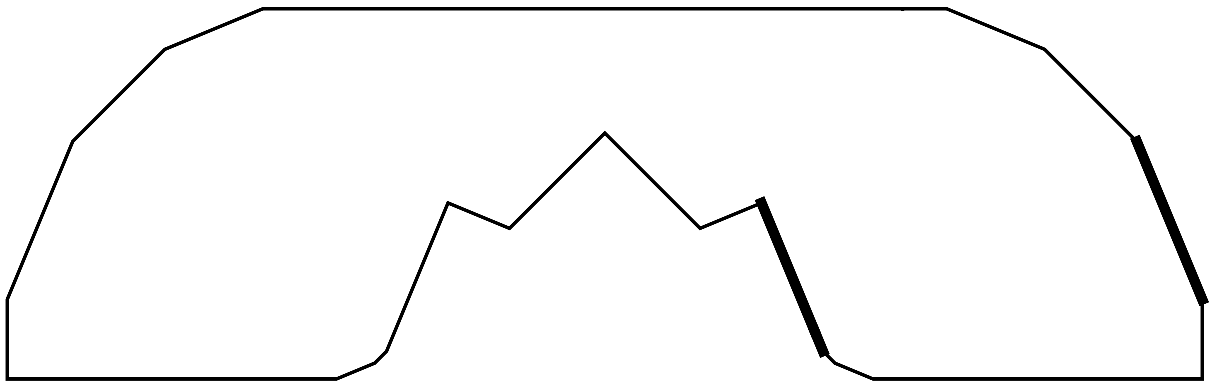

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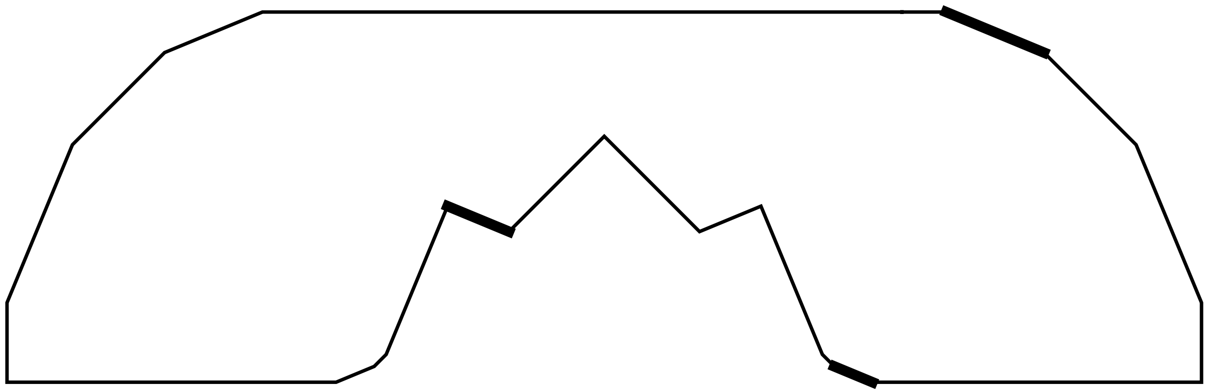
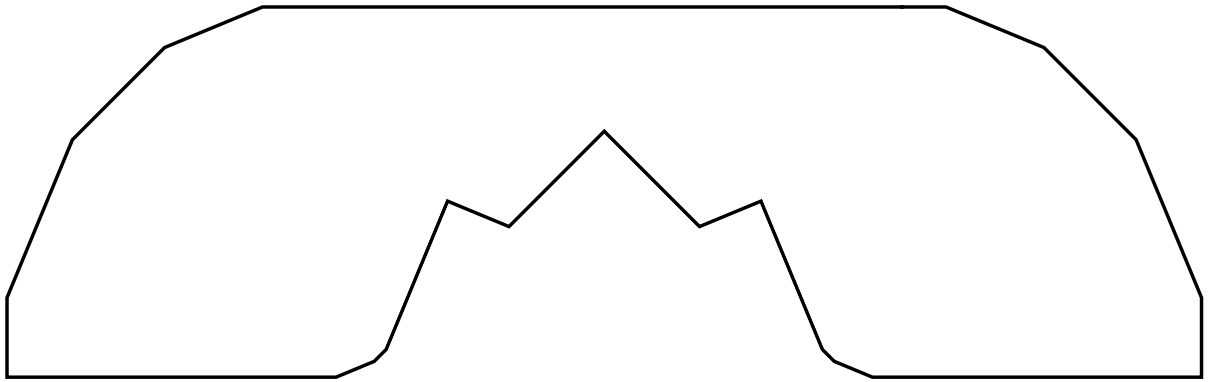


In this discrete version of the problem, it is easy to see that a sofa with locally maximal area must be balanced — otherwise, we could push one of the copies of the L-shaped hallway a small distance in a direction orthogonal to the unbalanced side lengths and increase the area slightly.

By taking a limit from the discrete to the continuum setting, following in Gerver's footsteps I derived a family of six differential equations that the rotation path $\mathbf{x}(t)$ needs to satisfy (piecewise) for a shape to be locally optimal.



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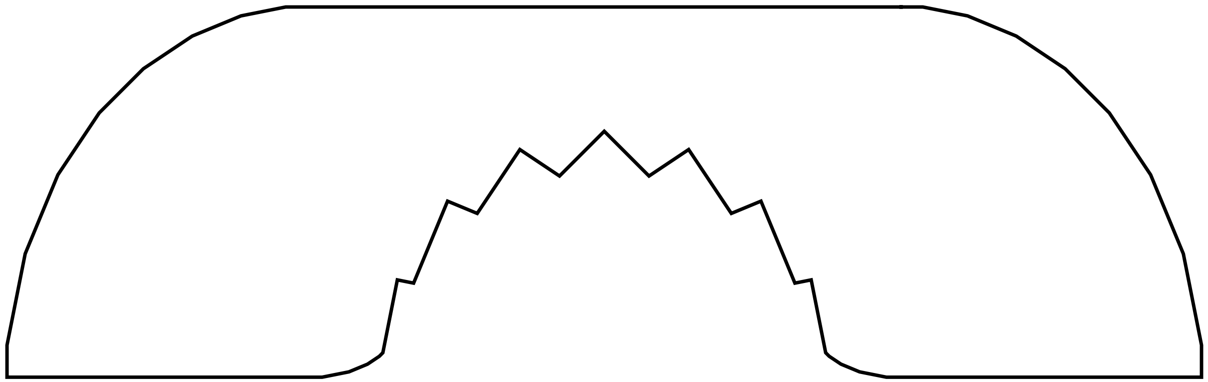

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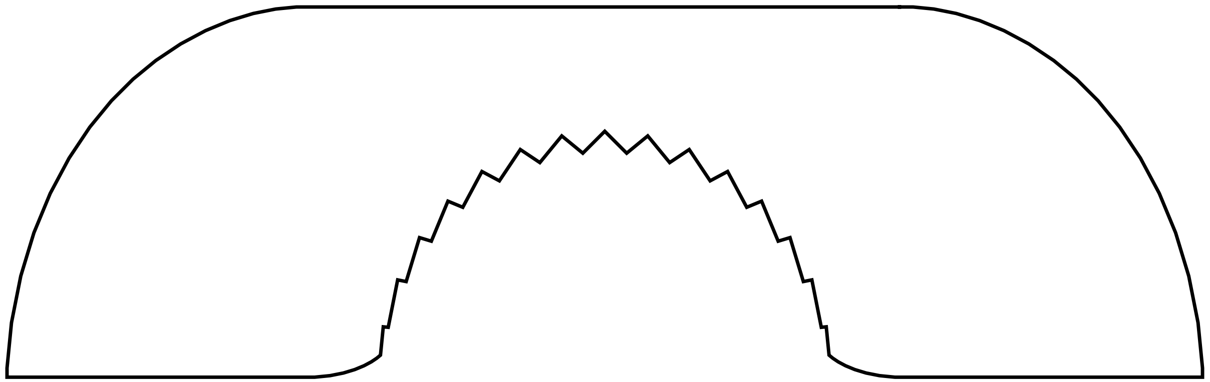

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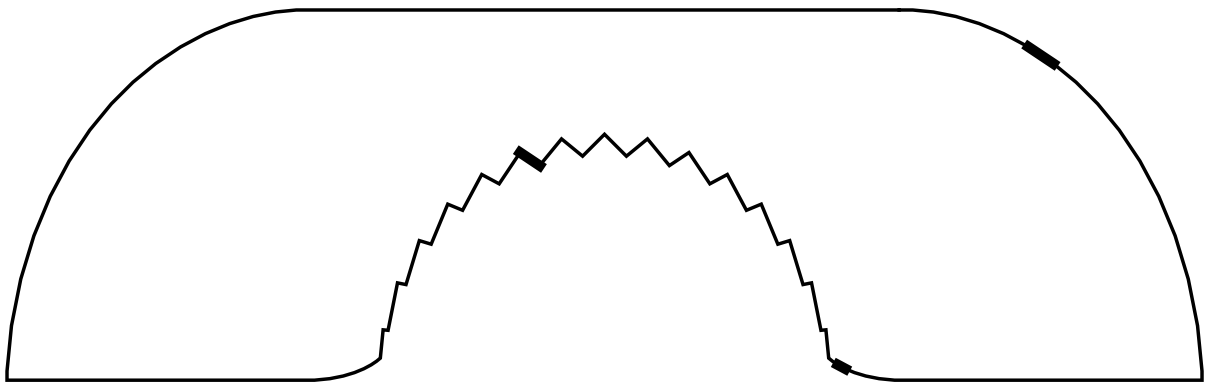


The six ODEs take the form

$$\ddot{\mathbf{x}}(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} (A_j(t)\dot{\mathbf{x}}(t) + \mathbf{b}_j),$$

where $A_j(t)$, $j = 1, \ldots, 6$ are the six matrices

$$\begin{pmatrix} 2\sin t & -2\cos t \\ 2\cos t & 2\sin t \end{pmatrix}, \quad \begin{pmatrix} \sin t & -\cos t \\ \frac{3}{2}\cos t & \frac{3}{2}\sin t \end{pmatrix}, \quad \begin{pmatrix} \sin t & -\cos t \\ \cos t & \sin t \end{pmatrix},$$

$$\begin{pmatrix} \frac{3}{2}\sin t & -\frac{3}{2}\cos t \\ \cos t & \sin t \end{pmatrix}, \quad \begin{pmatrix} 2\sin t & -2\cos t \\ 2\cos t & 2\sin t \end{pmatrix}, \quad \begin{pmatrix} \frac{3}{2}\sin t & -\frac{3}{2}\cos t \\ \frac{3}{2}\cos t & \frac{3}{2}\sin t \end{pmatrix},$$

and $\mathbf{b}_j$, $j = 1, \ldots, 6$ are the six vectors

$$\begin{pmatrix} -1 \\ -\frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} -1 \\ -\frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}, \quad \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}, \quad \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix},$$

and where the choice of ODE in each segment of the rotation path is dictated by the set Γ of contact points at that time t:

Case 1. $\Gamma = \{\mathbf{A}, \mathbf{C}, \mathbf{D}\}$.

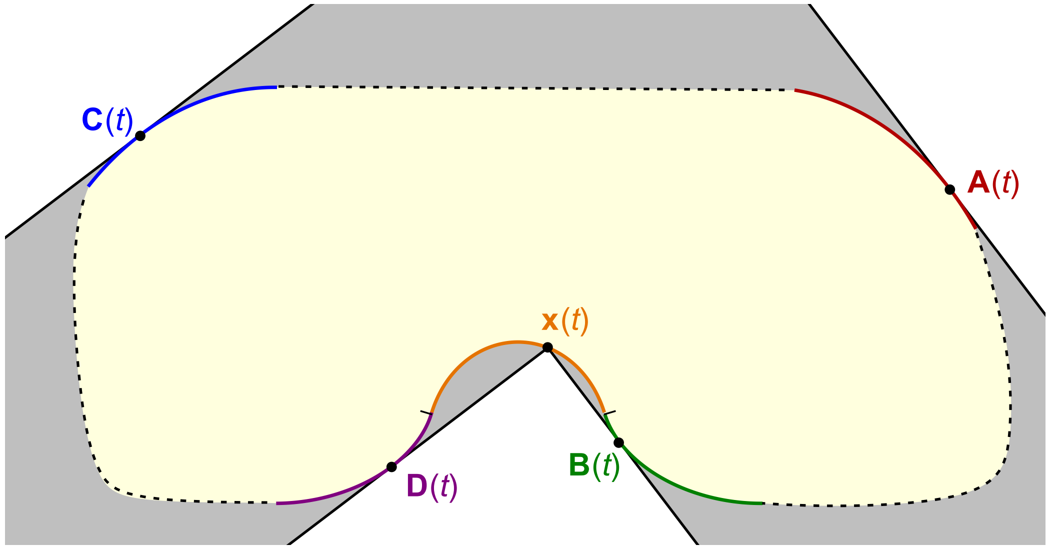
Case 2. $\Gamma = \{\mathbf{x}, \mathbf{A}, \mathbf{C}, \mathbf{D}\}$.

Case 3. $\Gamma = \{\mathbf{x}, \mathbf{A}, \mathbf{C}\}$.

Case 4. $\Gamma = \{\mathbf{x}, \mathbf{A}, \mathbf{B}, \mathbf{C}\}$.

Case 5. $\Gamma = \{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$.

Case 6. $\Gamma = \{\mathbf{x}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$.



Solutions of the ODEs: the six ODEs are easily solved, leading to the solutions

$$\mathbf{x}_1(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} a_1 \cos t + a_2 \sin t - 1 \\ -a_2 \cos t + a_1 \sin t - 1/2 \end{pmatrix} + \begin{pmatrix} \kappa_{1,1} \\ \kappa_{1,2} \end{pmatrix}$$

$$\mathbf{x}_2(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} -\frac{1}{4}t^2 + b_1 t + b_2 \\ \frac{1}{2}t - b_1 - 1 \end{pmatrix} + \begin{pmatrix} \kappa_{2,1} \\ \kappa_{2,2} \end{pmatrix}$$

$$\mathbf{x}_3(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} c_1 - t \\ c_2 + t \end{pmatrix} + \begin{pmatrix} \kappa_{3,1} \\ \kappa_{3,2} \end{pmatrix}$$

$$\mathbf{x}_4(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} -\frac{1}{2}t + d_1 - 1 \\ -\frac{1}{4}t^2 + d_1 t + d_2 \end{pmatrix} + \begin{pmatrix} \kappa_{4,1} \\ \kappa_{4,2} \end{pmatrix}$$

$$\mathbf{x}_5(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} e_1 \cos t + e_2 \sin t - 1/2 \\ -e_2 \cos t + e_1 \sin t - 1 \end{pmatrix} + \begin{pmatrix} \kappa_{5,1} \\ \kappa_{5,2} \end{pmatrix}$$

$$\mathbf{x}_6(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} f_1 \cos(t/2) + f_2 \sin(t/2) - 1 \\ -f_2 \cos(t/2) + f_1 \sin(t/2) - 1 \end{pmatrix} + \begin{pmatrix} \kappa_{6,1} \\ \kappa_{6,2} \end{pmatrix},$$

involving free parameters a_j, b_j, c_j, d_j, e_j, f_j, $\kappa_{j,1}$, $\kappa_{j,2}$, $(j = 1, \ldots, 6)$.

By gluing together solutions to these ODEs in the correct order, one can obtain Gerver's shape and the new ambidextrous shape. Specifically, we try to construct functions

$$\mathbf{x}_{\text{Gerver}}(t) = \begin{cases} \mathbf{x}_1(t) & \text{if } 0 < t < \varphi, \\ \mathbf{x}_2(t) & \text{if } \varphi < t < \theta, \\ \mathbf{x}_3(t) & \text{if } \theta < t < \pi/2 - \theta, \\ \mathbf{x}_4(t) & \text{if } \pi/2 - \theta < t < \pi/2 - \varphi, \\ \mathbf{x}_5(t) & \text{if } \pi/2 - \varphi < t < \pi/2, \end{cases} \qquad \mathbf{x}_{\text{ambi}}(t) = \begin{cases} \mathbf{x}_1(t) & \text{if } 0 < t < \beta, \\ \mathbf{x}_6(t) & \text{if } \beta < t < \pi/2 - \beta, \\ \mathbf{x}_5(t) & \text{if } \pi/2 - \beta < t < \pi/2, \end{cases}$$

that are "rotation paths" encoding valid sofa shapes, where $0 < \varphi < \theta < \pi/2$ and $0 < \beta < \pi/2$ are unknown angles.

This leads to complicated systems of equations for the unknown angles and ODE free parameters:

- 28 equations in 22 variables in the case of Gerver's shape
- 17 equations in 13 variables for the ambidextrous shape

Solving the equations in `Mathematica`

The systems of equations for Gerver's shape and the ambidextrous shape are complicated, but Mathematica can solve them after the appropriate setup. See the Mathematica package `MovingSofas` accompanying my paper, which also includes nice graphics and interactive animations.

Part 2: Improved area bounds and computer-assisted proofs

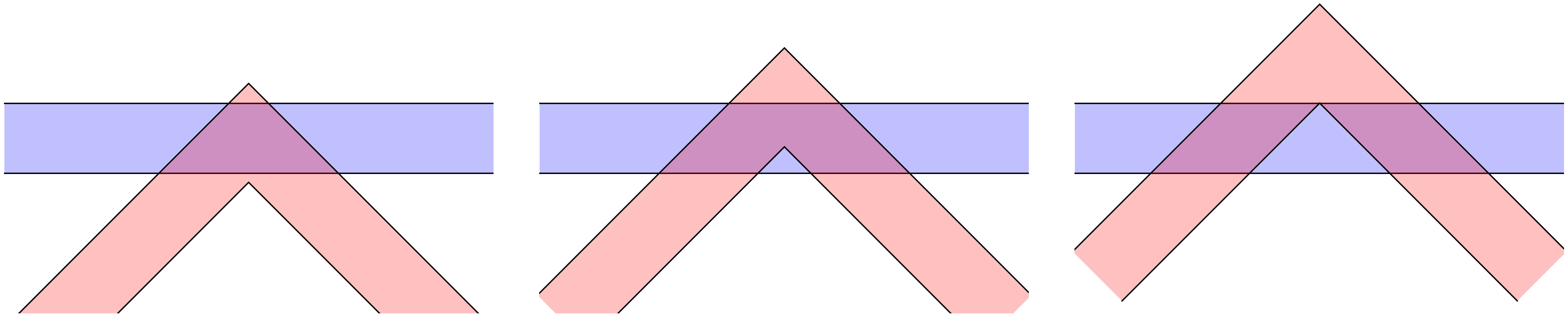
Joint work with Yoav Kallus

A computational approach to area upper bounds

Hammersley's proof of the upper bound $2\sqrt{2}$ relied on the observation that a moving sofa must fit in the corridor when it's rotated by 45 degrees. So its area is bounded by the maximal area of the intersection of

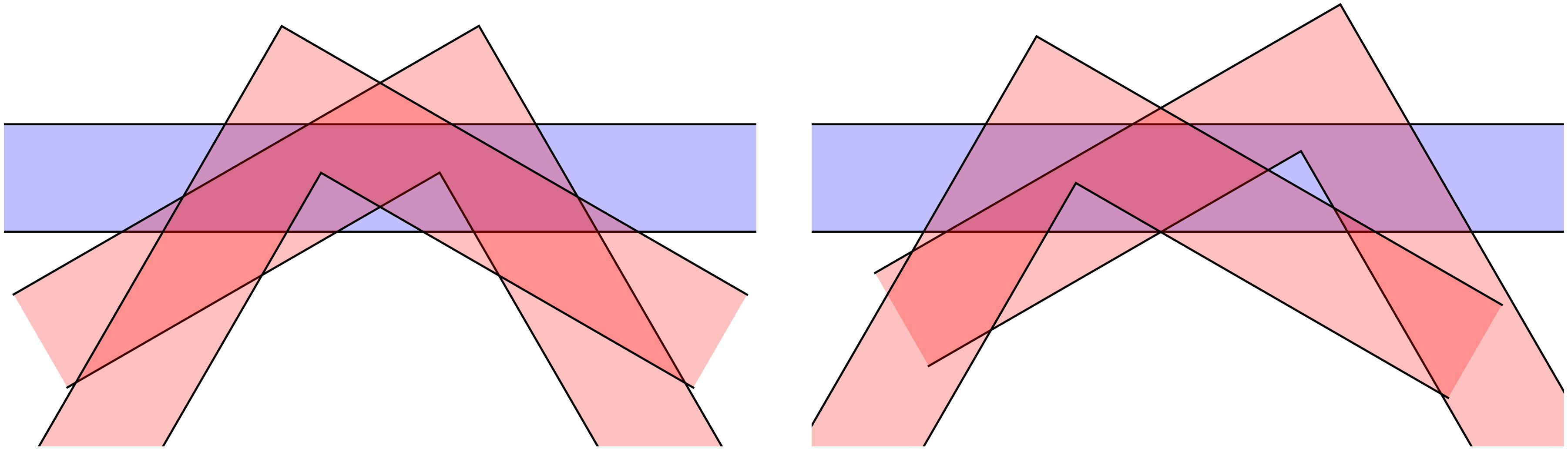
- a horizontal strip, and
- a translated copy of the corridor rotated by 45 degrees

It is a simple exercise to get $2\sqrt{2}$ as the maximal area:



(max area $= 2\sqrt{2}$)

Generalizing this idea, we can imagine intersecting a horizontal strip with multiple translated and rotated copies of the corridor at several different angles, for example, 30 degrees and 60 degrees:



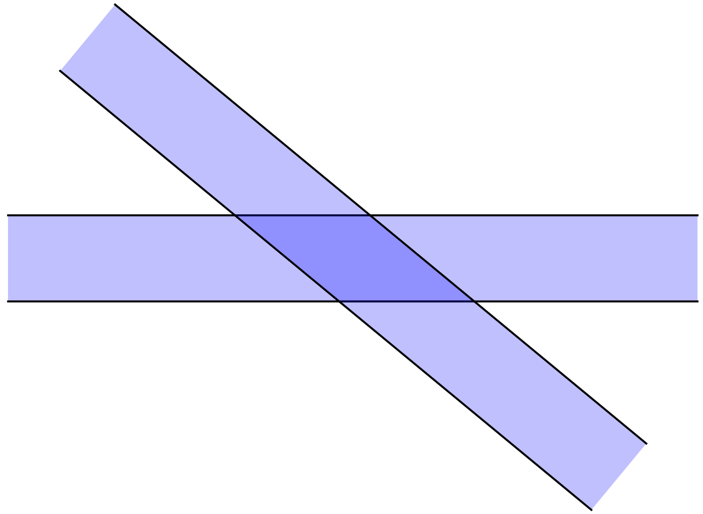
If we could find when the area of the intersection is maximized, we would get an upper bound for the area of a moving sofa. Potential problems:

- Is this a rigorous bound? That is, do we know that a maximal-area moving sofa must rotate by at least 60 degrees?
- How do we compute this maximum? Very hard to do by hand...

Angle bounds

Gerver noticed a simple way to lower-bound the angle of rotation of an optimal moving sofa: if the sofa rotates through an angle of α, it must be contained in the intersection of

- a horizontal strip, and
- a copy of the vertical strip rotated by an angle of α



The area is $\sec(\alpha)$, which is $< \mu_{\text{Gerver}} = 2.2195\ldots$ iff $0 \leq \alpha < \sec^{-1}(\mu_{\text{Gerver}}) \approx 63^\circ$. This proves that an optimal moving sofa must rotate through at least 63°, which partially resolves the first potential problem from the previous slide.

To overcome the second problem, we designed and implemented an algorithm to numerically bound the maximum area of intersection of a horizontal strip with translates of a finite set of corridors rotated by angles

$$0 < \alpha_1 < \ldots < \alpha_k < \pi/2.$$

Formally, we are trying to maximize the function

$$g(\mathbf{u}_1, \ldots, \mathbf{u}_k) = \operatorname{area}\left(H \cap \bigcap_{j=1}^{k} \operatorname{Rot}_{\alpha_j}(L + \mathbf{u}_j) \right) \qquad \Big(\mathbf{u}_1, \ldots, \mathbf{u}_k \in \mathbb{R}^2 \Big),$$

where H is the horizontal strip, L is the L-shaped corridor, and $\operatorname{Rot}_\alpha(\cdot)$ denotes rotation by an angle of α around the origin.

A sketch of the main ideas behind the algorithm:

- First, show that we can restrict the search to a compact subset $\Omega = [-A, A]^{2k}$ of $\mathbb{R}^{2k}$. (Tricky detail: need to change the definition of g a bit, but that's okay.)
- Next, observe that given a smaller box $B \subset \Omega$ in the configuration space, we can compute a numerical bound for the objective function that holds *uniformly on the box* B.
- Now apply a **geometric branch and bound** search scheme: start with the initial box Ω, and successively compute uniform upper bounds, keeping track of the best upper and lower bounds obtained so far; a box whose upper bound is lower than the best lower bound can be discarded, and other boxes are successively subdivided along different axes into smaller sub-boxes for which the same procedure is repeated.
- Boxes still under consideration are kept in a **priority queue**, where the priority of a box is the upper bound associated with it, and boxes are considered in order of their priority so as to optimize performance.

```
box_queue ← an empty priority queue of boxes
initial_box ← box representing Ω
push initial_box into box_queue with priority Π(initial_box).
best_lower_bound_so_far ← 0
while true do
    pop highest priority element of box_queue into current_box
    current_box_lower_bound ← g(P_mid(current_box))
    best_upper_bound_so_far ← Π(current_box).
    if current_box_lower_bound > best_lower_bound_so_far then
        best_lower_bound_so_far ← current_box_lower_bound
    end if
    i ← ind(current_box)
    for j = 1, 2 do
        new_box ← split_i,j(current_box)
        if Π(new_box) ≥ best_lower_bound_so_far  then
            push new_box into box_queue with priority Π(new_box)
        end if
    end for
    Reporting point: print the values of best_upper_bound_so_far
                     and best_lower_bound_so_far.
end while
```

The algorithm in pseudocode

Notation:

$g(\cdot)$	intersection area function
$\Pi(B)$	priority of a box B
$\mathrm{ind}(B)$	splitting index of a box B
$P_{\mathrm{mid}}(B)$	midpoint of a box B
$\mathrm{split}_{\mathtt{i},\mathtt{j}}(B)$	jth subbox of B split along coordinate i

The `SofaBounds` software package

`SofaBounds` is our software implementation of the algorithm.

- Written in `C++`
- Uses CGAL (Computational Geometry Algorithms Library) open source library
- Uses exact rational arithmetic to ensure results are mathematically rigorous
- ≈ 1000 lines of code
- Source code available from `github`:

`https://github.com/ykallus/SofaBounds`

New moving sofa bounds proved using `SofaBounds`

Theorem.

1. **New area bound.** Any moving sofa has area at most 2.37.
2. **New angle bound.** A moving sofa of maximal area has to rotate by an angle of at least $\sin^{-1}(84/85) = 81.203°$.

Proof idea. Define angles

$$\alpha_1 = \sin^{-1}\tfrac{7}{25} \approx 16.26°, \qquad \alpha_5 = \sin^{-1}\tfrac{24}{25} \approx 73.74°,$$
$$\alpha_2 = \sin^{-1}\tfrac{33}{65} \approx 30.51°, \qquad \alpha_6 = \sin^{-1}\tfrac{60}{61} \approx 79.61°,$$
$$\alpha_3 = \sin^{-1}\tfrac{119}{169} \approx 44.76°, \qquad \alpha_7 = \sin^{-1}\tfrac{84}{85} \approx 81.2°.$$
$$\alpha_4 = \sin^{-1}\tfrac{56}{65} \approx 59.59°,$$

We used `SofaBounds` to prove the numerical bounds

$$G_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)} \leq 2.37, \tag{1}$$
$$G^{\alpha_4,\alpha_5}_{(\alpha_1,\alpha_2,\alpha_3)} \leq 2.21, \tag{2}$$
$$G^{\alpha_5,\alpha_6}_{(\alpha_1,\alpha_2,\alpha_3)} \leq 2.21, \tag{3}$$
$$G^{\alpha_6,\alpha_7}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} \leq 2.21, \tag{4}$$

where

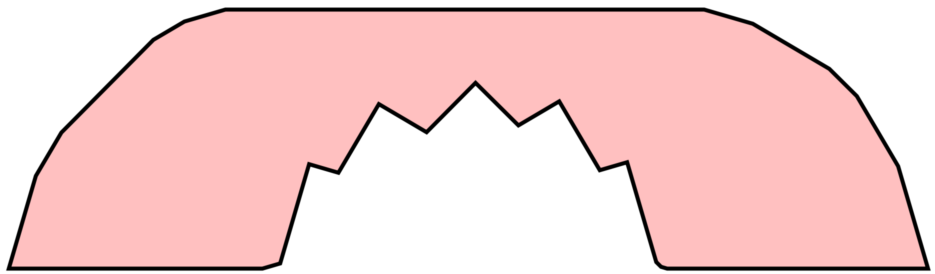
- $G_{(\alpha_1,\ldots,\alpha_k)}$ denotes the maximum of the g area-of-intersection function associated with the angles $\alpha_1 < \ldots < \alpha_k$, and
- $G^{\beta_1,\beta_2}_{(\alpha_1,\ldots,\alpha_k)}$ is a more complicated variant of $G_{(\alpha_1,\ldots,\alpha_k)}$ that bounds the area for shapes that rotate by any final angle between β_1 and β_2. (New idea: include an additional “butterfly set” in the intersection)

These four bounds are sufficient to imply the result.

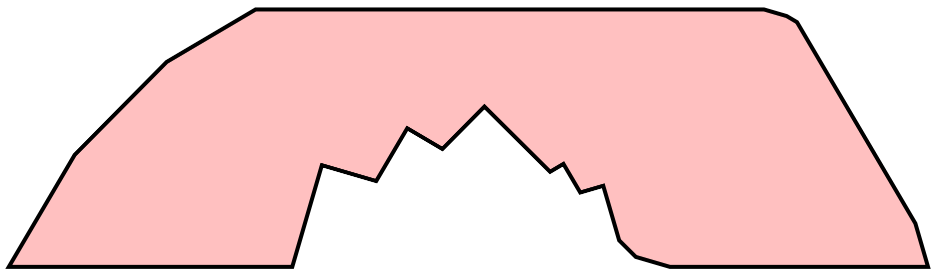
Running times to prove the new bounds

Bound	Saved profile file	Number of iterations	Computation time
(1)	`thm9-bound1.txt`	7,724,162	480 hours
(2)	`thm9-bound2.txt`	917	2 minutes
(3)	`thm9-bound3.txt`	26,576	1:05 hours
(4)	`thm9-bound4.txt`	140,467	6:23 hours

`SofaBounds` can generate "bound polygons" associated with the best upper and lower bounds achieved so far, which are useful for visualization purposes. Here are sample bound polygons produced during the "$16°$–$30°$–$45°$–$60°$–$74°$" computation corresponding to the theorem above.



A lower bound polygon with area ≈ 2.3274



An upper bound polygon with area ≈ 2.4276

Part 3: Jineon Baek's new moving sofa bounds

Baek's new bounds

In a recent preprint (`arXiv:2406.10725`), Jineon Baek introduced an intriguing new approach to upper-bounding the area of a moving sofa, and proved the following result:

Theorem (Baek, 2024).

1. **New conditional area bound.** Any *injective* moving sofa has area at most

$$\mu_{\mathrm{Baek}} = 1 + \frac{\pi^2}{8} = 2.2337\ldots \approx 1.006 \times \mu_{\mathrm{Gerver}}.$$

2. **New conditional angle bound.** Any injective moving sofa of maximal area has to rotate by an angle of at least $89.48°$.

For the definition of an injective moving sofa, refer to Baek's paper.

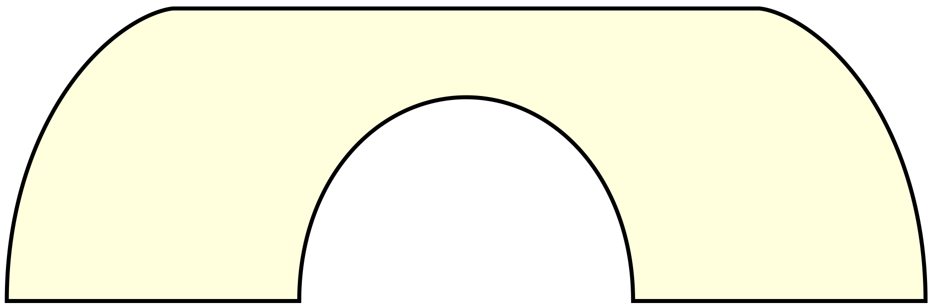
Rough idea of Baek's approach

(Disclaimer: this represents my own limited understanding of the paper, and may contain inaccuracies)

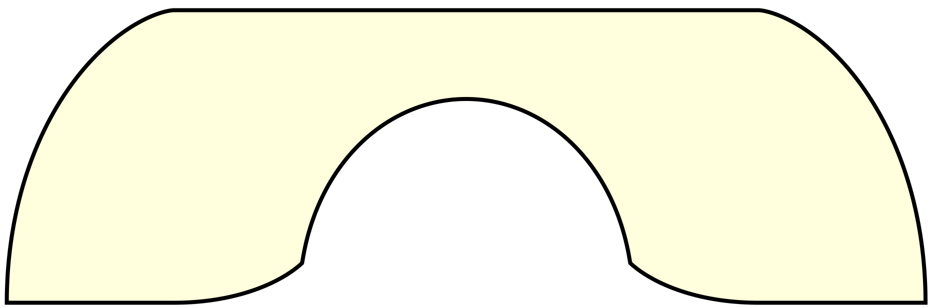
1. As in earlier works, parametrize sofas in terms of rotation paths.
2. Imagine a notion of "partially routed area" $\mathcal{A}_1(\mathbf{x})$ for rotation paths that corresponds to the area of the "presofa" associated with the rotation path instead of the associated sofa (I'll show a figure). Under the injectivity condition, this is an upper bound for ordinary sofa area.
3. The partially routed area functional $\mathcal{A}_1(\cdot)$ is a much nicer functional of rotation paths than ordinary area. It can be shown to be (when expressed in the correct coordinates) quadratic and strictly concave.
4. $\mathcal{A}_1$ can be maximized explicitly using calculus of variations techniques. This leads to the upper bound of $1 + \frac{\pi^2}{8}$.
5. ... and lots of clever details.

Baek's sofa and presofa

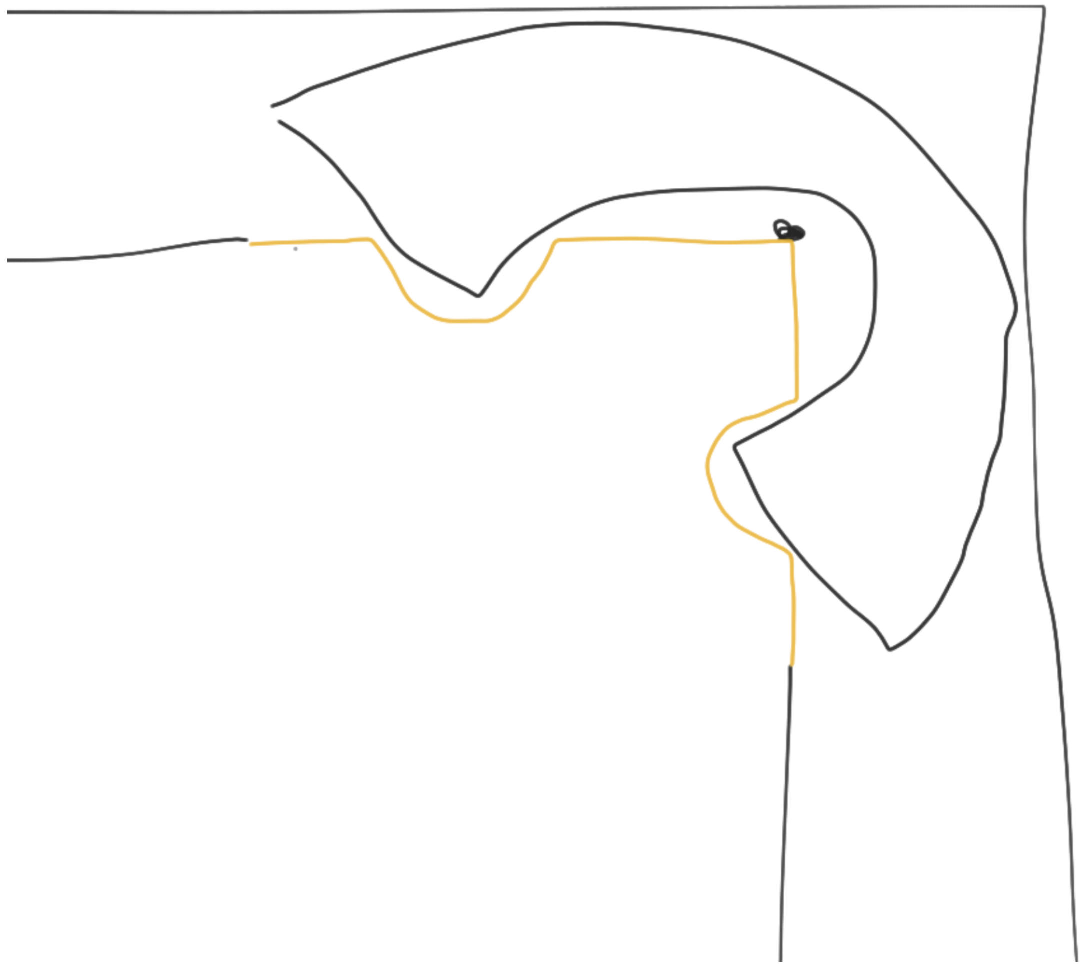
Baek's calculus of variations maximization leads to a shape, **Baek's presofa**, with area $1 + \frac{\pi^2}{8} \approx 2.2337$:



The presofa has an associated sofa shape, **Baek's sofa**, with area ≈ 2.2009:



More on presofas



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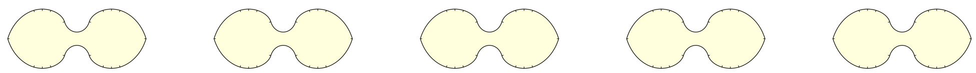
Propposition 1. The moving sofa problem is intrinsically beautiful.

Propposition 2. The moving sofa problem has inspired, and continues to inspire, beautiful mathematics.

Propposition 3. The moving sofa problem is very difficult!

That's all. Thank you! And happy birthday, Jim!



References

1. D. Romik. Differential equations and exact solutions in the moving sofa problem. 2016. *Experimental Math.* 27 (2018), 316–330.

2. Y. Kallus, D. Romik. Improved upper bounds in the moving sofa problem. *Adv. Math.* 340 (2018), 960–982.

3. J. Baek. A conditional upper bound for the moving sofa problem. Preprint, `arXiv:2406.10725`.