Combinatorics of Macdonald polynomials through the ASEP and TAZRP

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Happy Birthday Jim!

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How I first met the ASEP

- In Spring 2012, Jim taught "Topics in Applied Mathematics" at Berkeley.
- He showed us the Aztec diamond.

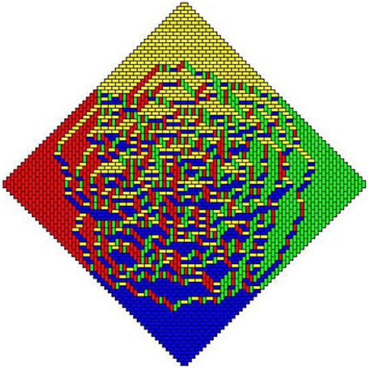
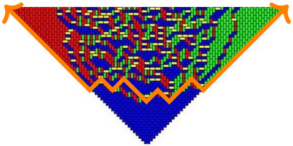


Image credit: Wikipedia

- Consider a random tiling of the Aztec diamond of size N. There is a "frozen region" at each corner: the boundary of the frozen region is the arctic circle.

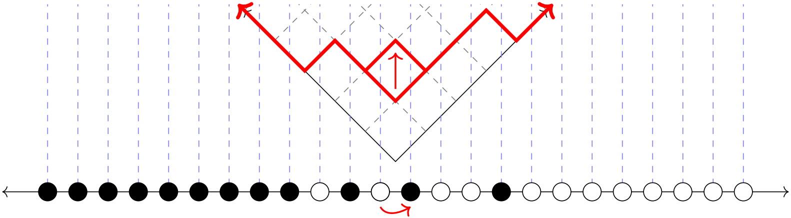
How I first met the ASEP



Theorem (Jockush–Propp–Shor '98)

As $N \to \infty$, the arctic circle has a circular limit shape.

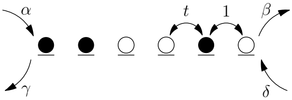
Proof sketch.

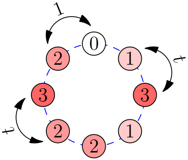


Boundary of the arctic circle as size $N \to \infty$ can be described through the behavior of the TASEP (totally asymmetric simple exclusion process) on $\mathbb{Z}$ as time $N \to \infty$

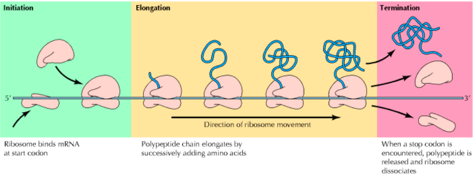
asymmetric simple exclusion process (ASEP)

- the ASEP is a statistical mechanics model describing particles hopping on a 1D lattice (1 particle per site)

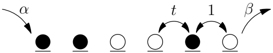




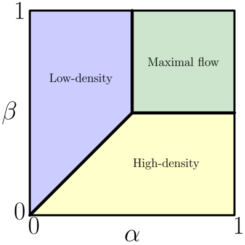
- introduced in the 1960's by Spitzer and Macdonald-Gibbs-Pipkin
- model for transport processes: translation in protein synthesis, traffic flow, molecular transport



asymmetric simple exclusion process (ASEP)

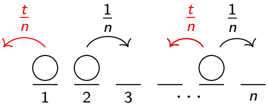


- canonical example of a non-equilibrium process that exhibits boundary-induced phase transitions:



- ASEP has beautiful combinatorial structure, deep connections to orthogonal polynomials (Askey-Wilson, Macdonald, Koornwinder). Also connected to random matrix theory, total positivity on the Grassmanian, other statistical mechanics models such as the six-vertex model and the XXZ model

Asymmetric simple exclusion process (ASEP)



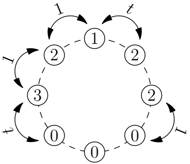
- **exclusion process:** ≤ 1 particle per site (sites labeled $1, 2, \dots, n$)
- **boundary conditions:** infinite lattice, open boundaries (particles can enter and exit at the boundaries), periodic boundary (on a circle)
- **particle types:** particle "species" labeled by integers, larger integers have higher "priority"

1 1 0 0 1 0	2 1 3 0 1 2
single species ASEP	multispecies ASEP

- **dynamics:** swaps between adjacent particles (in our case, fixed by a parameter $0 \leq t \leq 1$):

$$XABY \xrightarrow{1} XBAY \quad \text{and} \quad XBAY \xrightarrow{t} XABY \quad \text{for} \quad A > B$$

our setting: ASEP on a circle



$n = 8, \quad \lambda = (3, 2, 2, 2, 1, 0, 0, 0)$

$\alpha = (1, 2, 2, 0, 0, 0, 3, 2) \in \mathrm{ASEP}(\lambda)$

- Fix a circular lattice on n sites, and choose nonnegative integer weights recorded as a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$.
- ASEP(λ, n) is a Markov chain whose states are the compositions $\alpha \in S_n \cdot \lambda$ that are rearrangements of λ (on a circle: $\alpha_{n+1} = \alpha_1$)
- Fix $0 \leq t \leq 1$.The transitions are swaps of adjacent particles such that the **larger** particle hops
  - clockwise at rate 1
  - counterclockwise at rate t
- For example, ASEP$((2, 2, 1), n)$ has 12 states:

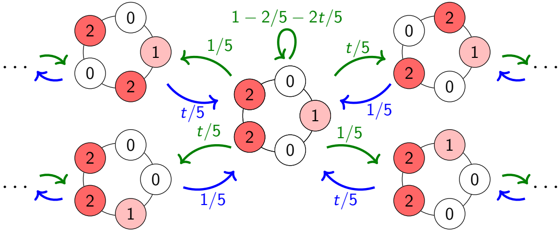
$$(2,2,1,0),(2,1,2,0),(2,1,0,2),(2,2,0,1),(2,0,2,1),(2,0,1,2),(0,2,2,1),\cdots$$

The transitions from state $(2, 1, 2, 0)$ are:

- $(1, 2, 2, 0)$ with probability $1/4$
- $(2, 1, 0, 2)$ with probability $1/4$
- $(2, 2, 1, 0)$ with probability $t/4$
- $(0, 1, 2, 2)$ with probability $t/4$

Computing the stationary probabilities

Example for $\lambda = (2,2,1)$, $n = 5$



$$\Pr(2,0,1,0,2) = \frac{1}{\mathcal{Z}_{\lambda,n}}(3+7t+7t^2+3t^3) \qquad \Pr(0,2,1,0,2) = \frac{1}{\mathcal{Z}_{\lambda,n}}(5+6t+7t^2+2t^3)$$

$$\Pr(2,1,0,0,2) = \frac{1}{\mathcal{Z}_{\lambda,n}}(6+7t+6t^2+t^3) \qquad \Pr(2,0,0,1,2) = \frac{1}{\mathcal{Z}_{\lambda,n}}(1+6t+7t^2+6t^3)$$

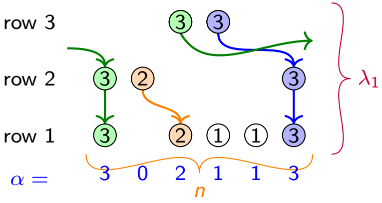
$$\Pr(2,1,2,0,0) = \frac{1}{\mathcal{Z}_{\lambda,n}}(3+7t+7t^2+3t^3) \qquad \Pr(2,0,1,2,0) = \frac{1}{\mathcal{Z}_{\lambda,n}}(2+7t+6t^2+5t^3)$$

The partition function is the sum of the unnormalized probabilities $\widetilde{\Pr}(\alpha)$:

$$\mathcal{Z}_{\lambda,n} = \sum_{\alpha \in S_n \cdot \lambda} \widetilde{Pr}(\alpha) = 5(20+40t+40t^2+20t^3) = 100(t^2+t+1)(t+1)$$

Multiline queues at $t = 0$ for TASEP (Ferrari-Martin '07)

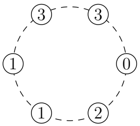
- When $t = 0$, particles only hop clockwise
- A **multiline queue** on an ASEP of type $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ on n sites is an **arrangement** of balls on a cylindric lattice of λ_1 rows and n columns, with λ'_j balls in row j
- F–M projection map: from top to bottom, each ball, in order of priority, pairs with **the first available ball** weakly to its right in the row below
- The state of the multiline queue is read off Row 1



$n = 6$

$\lambda = (3, 3, 2, 1, 1)$

$\lambda' = (5, 3, 2)$



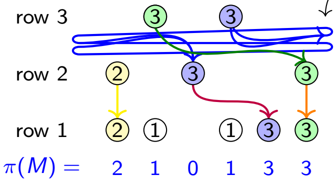
Theorem (Ferrari-Martin '07)

$$\Pr(\alpha)(t = 0) \sim |\,\mathrm{MLQ}(\alpha)|$$

multiline queues for general t: Martin '18

- Combine a **ball system** with a **queueing algorithm**.
- Each ball chooses an available ball to pair with in the row below. t counts the number of available balls skipped: assign weight $t^{\text{total skipped}}(1-t)$.
- The weight of each non-trivial pairing is $t^{\text{skipped}}\frac{(1-t)}{1-t^{\text{free}}}$

j times: total skipped $= 3j+2$



$\frac{t^2(1-t)}{1-t^3}$ $(1-t)\sum_j t^{3j+2}$ $\cdot \frac{(1-t)}{1-t^2}$ $\cdot 1$ $\cdot \frac{t(1-t)}{1-t^4}$ $= \frac{t^2(1-t)}{1-t^3}$ $\cdot 1$

skipped

free

$$\mathrm{wt}(M)(t) = \frac{t^3(1-t)^4}{(1-t^4)(1-t^3)(1-t^2)}$$

$$\mathrm{wt}(M)(t) = \prod_{\text{pairing}} t^{\text{skipped}} \frac{(1-t)}{1-t^{\text{free}}}$$

Theorem (Martin '18)

$$\widetilde{\Pr}(\alpha)(t) = \sum_{\substack{M \in \mathrm{MLQ}(\lambda) \\ \pi(M)=\alpha}} \mathrm{wt}(M)(t).$$

From ASEP to Macdonald polynomials

Recall the partition function of ASEP(λ, n):

$$\mathcal{Z}_{\lambda,n}(t) = \sum_{\alpha \in S_n \cdot \lambda} \widetilde{\Pr}(\alpha)(t) = \sum_{M \in \mathrm{MLQ}(\lambda,n)} \mathrm{wt}(M)(t).$$

Theorem (Cantini–de Gier–Wheeler '15)

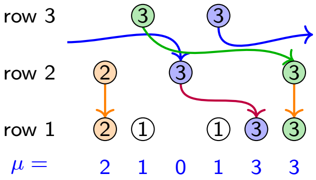
$\mathcal{Z}_{\lambda,n}(t)$ is a specialization of the Macdonald polynomial $P_\lambda(x_1, \ldots, x_n; q, t)$:

$$P_\lambda(1, \ldots, 1; 1, t) = \mathcal{Z}_{\lambda,n}(t)$$

this motivated us (Corteel–M–Williams) to search for a multiline queue formula for P_λ.

Macdonald polynomials from multiline queues

- Each pairing (of type ℓ, from row r) that wraps contributes $q^{\ell-r+1} = q^{\text{leg}+1}$
- Weight for each pairing is $t^{\text{skipped}} q^{(\text{leg}+1)\delta_{\text{wrap}}} \frac{1-t}{1-q^{\text{leg}+1}t^{\text{free}}}$
- Define the x-weight of a queue M to be $x^M = \prod_j x_j^{\#\text{ balls in col } j}$



$$x^M = x_1^2 x_2^2 x_3 x_4^2 x_5 x_6^2$$

$$\frac{qt^2(1-t)}{1-qt^3} \cdot \frac{(1-t)}{1-qt^2} \cdot 1 \cdot \frac{t(1-t)}{1-q^2t^4} \cdot 1$$

$$= \frac{qt^3(1-t)^4}{(1-q^2t^4)(1-qt^3)(1-qt^2)}$$

$$\text{wt}(M)(X;q,t) = x^M t^{\text{skipped}} \prod_{\text{pairings}} q^{(\text{leg}+1)\delta_{\text{wrap}}} \frac{1-t}{1-q^{\text{leg}+1}t^{\text{free}}}$$

Theorem (Corteel-M-Williams '18)

$$P_\lambda(X;q,t) = \sum_{M \in \text{MLQ}(\lambda,n)} \text{wt}(M)(X;q,t)$$

Interlude I: symmetric functions

- Let $X = x_1, x_2, \ldots$ be a family of commuting variables and let $\Lambda = \Lambda_{\mathbb{Q}}$ be the ring of symmetric functions in these variables with rational coefficients with an inner product $\langle \cdot, \cdot \rangle$
  (say $f(x_1, \ldots, x_n) \in \Lambda$ is symmetric if for any permutation $\pi \in S_n$, $f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$)

  E.g. $x_1 + x_2 + x_3 + \cdots$ and $x_1 x_2^2 + x_1^2 x_2 + x_1 x_3^2$ are both in Λ

- There are several natural bases for Λ, indexed by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0)$:
  - Monomial symmetric functions: $m_\lambda = \sum x_{i_1}^{\lambda_1} \cdots x_{i_k}^{\lambda_k}$
  - Power sum symmetric functions: p_λ

$$p_k = \sum_i x_i^k, \qquad p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$$

  - Schur functions: s_λ

    unique family of polynomials that are:

    i. upper triangular with respect to the m_λ basis and
    ii. orthogonal with respect to the standard inner product

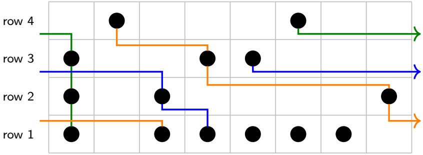
Interlude I: Macdonald polynomials

- Now let $\Lambda \cong \Lambda_{\mathbb{Q}}(q,t)$ be the ring of symmetric polynomials with parameters q, t
- In 1988, Macdonald introduced a remarkable new family of homogeneous symmetric polynomials $P_\lambda(X;q,t)$ in Λ, uniquely determined by:
  i. upper triangular with respect to $\{m_\lambda\}$:
  $$P_\lambda(X;q,t) = m_\lambda(X) + \sum_{\mu<\lambda} c_{\mu\lambda}(q,t) m_\mu(X)$$
  ii. orthogonal basis for Λ: $\langle P_\lambda, P_\mu\rangle = 0$ if $\lambda \neq \mu$

  (Macdonald's triangularity and normalization axioms)
- Example:
  $$P_{(2,1)}(X;q,t) = m_{(2,1)} + \frac{(1-t)(2+q+t+2qt)}{1-qt^2} m_{(1,1,1)}.$$
- Macdonald polynomials simultaneously generalize Schur functions (at $q=t$), Hall-Littlewood polynomials (at $q=0$), q-Whittaker polynomials (at $t=0$), and Jack polynomials (at $t=q^\alpha$ and $q \to 1$)

multiline queues at $t = 0$: q-Whittaker

- At $t = 0$, pairing lines are uniquely determined by the configuration of balls.
- Thus $\mathrm{MLQ}(\lambda, n)$ is simply a **$n \times \lambda_1$ binary matrix** with row sums λ'
- We want $P_\lambda(X; q, 0) = \sum_{M \in \mathrm{MLQ}(\lambda)} x^M q^{\mathrm{maj}(M)}$.



$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$\mathrm{cw}(M) = 3\ 2\ 1 \mid 4 \mid 2\ 1 \mid 3\ 1 \mid 3\ 1 \mid 4\ 1 \mid 1 \mid 2$

$\mathrm{charge}(M) = (4-3) + (3-2) + (4-1) = 5$

$\mathrm{wt}(M) = x_1^3 x_2 x_3^2 x_4^2 x_5^2 x_6^2 x_7 x_8 q^5$

- Define $\mathrm{cw}(M)$ to be the **column reading word** of $M = (a_{ij})$, recording an i whenever $a_{ij} = 1$ when reading M from left to right and top to bottom.

Theorem (M.–Valencia '23)

$$P_\lambda(X; q, 0) = \sum_{M \in MLQ(\lambda)} x^M q^{\mathrm{charge}(cw(M))}.$$

Interlude II: modified Macdonald polynomials

- modified Macdonald polynomials $\widetilde{H}_\lambda(X;q,t)$ (Garsia–Haiman '96) are a combinatorial form of $P_\lambda(X;q,t)$, obtained via plethystic substitution:

$$\widetilde{H}_\lambda(X;q,t) = t^{n(\lambda)} J_\lambda \left[\frac{X}{1-t^{-1}}; q, t^{-1} \right]$$

(J_λ is a scalar multiple of P_λ, we'll write $J_\lambda = f(q,t)P_\lambda$)

$$\widetilde{H}_{(2,1)} = qt\, s_{(1,1,1)} + (q+t)s_{(2,1)} + s_3$$
$$\widetilde{H}_{(2,1,1)} = q^2t^2 s_{(1,1,1,1)} + (q^2t + qt^2 + qt)s_{(2,1,1)} + (q^2+t^2)s_{(2,2)} + (qt+q+t)s_{(3,1)} + s_{(4)}$$

- these are classical objects of representation theory and combinatorics! They are polynomials with nonnegative integer coefficients in q, t, and moreover,

$$\tilde{H}_\lambda(X;q,t) = \sum_\mu K_{\mu\lambda}(q,t) s_\mu(X)$$

where $K_{\mu\lambda}(q,t) \in \mathbb{N}[q,t]$.
(positivity was proved by Haiman in '01, but a combinatorial interpretation of the Kostka–Macdonald coefficients $K_{\mu\lambda}$ is still a major open question in general)

From multiline queues to $\widetilde{H}_\lambda$

$$\widetilde{H}_\lambda(X;q,t) = f_\lambda(q,t)\ P_\lambda\left[\frac{X}{1-t^{-1}};q,t^{-1}\right] \qquad \mathbf{x_i \mapsto x_i, x_it^{-1}, x_it^{-2}, \cdots,} \quad \text{for } \mathbf{1 \le i \le n}$$

$$= f_\lambda(q,t)\ P_\lambda\left(x_1, x_1t^{-1}, x_1t^{-2}, \ldots, x_2, x_2t^{-1}, x_2t^{-2}, \ldots, x_3, x_3t^{-1}, x_3t^{-2}, \ldots;\ q, t^{-1}\right)$$

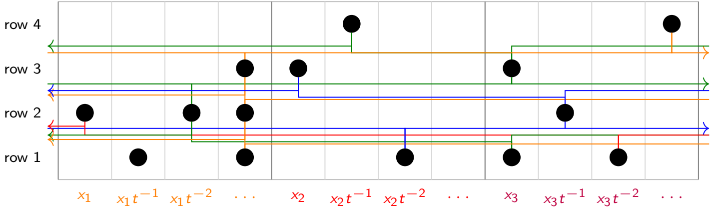
- $P_\lambda(y_1,\ldots,y_n;q,t)$ can be computed from a multiline queue with n columns labelled $y_1,\ldots,y_n$. Thus we can guess that

$$P_\lambda\left(x_1, x_1t^{-1}, x_1t^{-2}, \ldots, x_2, x_2t^{-1}, x_2t^{-2}, \ldots, x_3, x_3t^{-1}, x_3t^{-2}, \ldots;\ q, t^{-1}\right)$$

  corresponds to a multiline queue with infinitely many columns labeled

$$x_1, x_1t^{-1}, x_1t^{-2}, \ldots, x_2, x_2t^{-1}, x_2t^{-2}, \ldots, x_3, x_3t^{-1}, x_3t^{-2}, \ldots$$

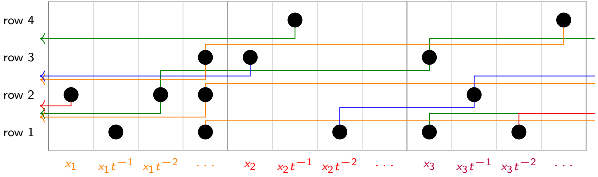
- To replace t by t^{-1}, we reverse the pairing process: pairing weakly to the right becomes strictly to the left

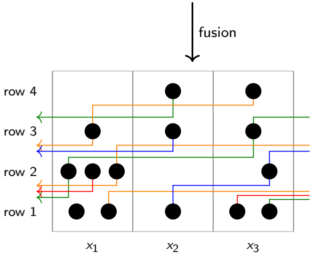


From multiline queues to $\widetilde{H}_\lambda$

- By compressing, we get bosonic multiline queues with no restriction on the number of particles at each site. This corresponds to fusion in the integrable systems setting

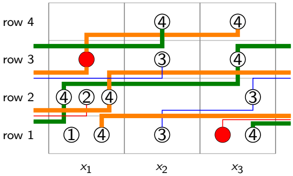
  (More generally, we call an object bosonic if it allows multiple particles per site. Otherwise, it is fermionic.)
- We end up with a formula for $\widetilde{H}_\lambda(X; q, t)$. Conj: Corteel–Haglund–M–Mason–Williams '20 Proof: Ayyer–M–Martin '21
- Garbali–Wheeler '20 used a similar idea to get a vertex model for $\widetilde{H}_\lambda(X; q, t)$





bosonic multiline queues as tableaux

Each **string** is mapped to a **column** in the tableau:



$\longrightarrow$ $\sigma =$

2	3			
3	1	2		
1	1	3	1	
3	1	2	3	1

$x^\sigma = x_1^6 x_2^3 x_3^5$

$\text{maj}(\sigma) = 5$

$\text{quinv}(\sigma) = 6$

multiline diagram of type (λ, n) $\rightarrow$ a tableau $\sigma : \lambda' \rightarrow [n]$

column content of the diagram $\rightarrow$ the monomial weight x^σ

wrapping pairings $\rightarrow$ maj(σ)

skipped particles $\rightarrow$ queue inversions (quinv)

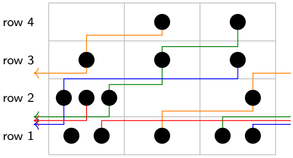
x / y ... z where $x < y < z$ (cyclically mod n)

Theorem (Ayyer–M–Martin '21)

$$\widetilde{H}_\lambda(x_1, \ldots, x_n; q, t) = \sum_{\sigma:\lambda' \rightarrow [n]} q^{\text{maj}(\sigma)} t^{\text{quinv}(\sigma)} x^\sigma$$

$t = 0$ specialization: modified Hall-Littlewood

- At $t = 0$, pairing lines are uniquely determined by the configuration of balls.
- Bosonic multiline queue of type (λ, n) is an integer matrix with row sums λ'
- We want to compute $\widetilde{H}_\lambda(X; q, 0) = \sum_{M \in bMLQ(\lambda)} x^M q^{\mathrm{maj}(M)}$



$$M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

$$\widetilde{cw}(M) = 1\ 1\ 2\ 3\ 4|1\ 3\ 4|1\ 1\ 2\ 2\ 2\ 3$$

$$\mathrm{charge}(M) = (4-2) + (4-1) + (3-1) + (2-1) = 8$$

$$\mathrm{wt}(M) = x_1^6 x_2^3 x_3^5 q^8$$

- Define $\widetilde{cw}(M)$ to be the **column reading word** of the $M = (a_{ij})$, recording a_{ij} i's when reading the entries from right to left and bottom to top.

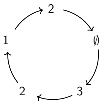
Theorem (M.–Valencia '23)

$$\widetilde{H}_\lambda(X; q, 0) = \sum_{M \in bMLQ(\lambda)} x^M q^{\mathrm{charge}(\widetilde{cw}(M))}.$$

TAZRP: a bosonic version of the ASEP

- fix a circular lattice on n sites, choose a set of particles $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k > 0)$. Fix $0 \leq t \leq 1$.

ASEP (fermionic process)



$n = 5$, $\lambda = (3, 2, 2, 1)$

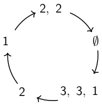
$\alpha = (2, 0, 3, 2, 1)$

- States are the compositions $\alpha \in S_n \cdot \lambda$ that are rearrangements of λ (on a circle: $\alpha_{n+1} = \alpha_1$)
- Transitions are swaps of adjacent particles:

$$\text{prob}(XABY \to XBAY) = \begin{cases} 1, & A > B \\ t, & A < B \end{cases}$$

- $t = 0$ is TASEP: particles only move to the right

TAZRP (bosonic process)



$n = 5$, $\lambda = (3, 3, 2, 2, 2, 1, 1)$

$\tau = (\ 2, 2 \mid \cdot \mid 3, 3, 1 \mid 2 \mid 1\ )$

- States are multiset compositions τ that are rearrangements of the parts of λ
- Transitions: a particle can jump from site j to site $j + 1 \mod n$ with rate

$$x_j^{-1} t^m,$$

where m is the number of particles at site j with higher priority.

- studied by Takeyama '15, related variants by Kuniba–Maruyama–Okado (2015+)
- $t = 0$ means only the strongest particle can hop from any site

Bosonic multiline queues project to the TAZRP

Theorem (Ayyer–M–Martin '22)

Fix λ, n. The (unnormalized) stationary probability of $\tau \in \mathrm{TAZRP}(\lambda, n)$ is

$$\widetilde{\Pr}(\tau) = \sum_{\substack{\sigma:\mathrm{dg}(\lambda)\to[n] \\ \pi(\sigma)=\tau}} x^{\sigma} t^{\mathrm{quinv}(\sigma)}.$$

where the sum is over bosonic multiline queues (or quinv tableaux) that correspond to state τ.

Proof: construction of a Markov chain on tableaux that lumps to the TAZRP.

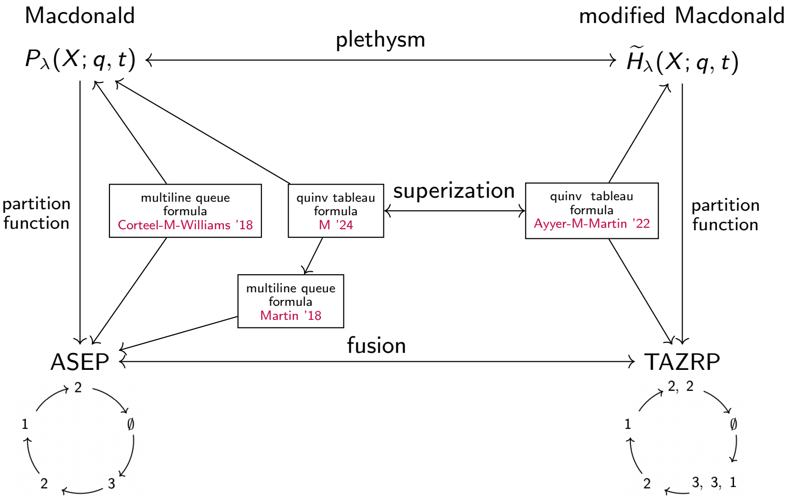
Corollary

The partition function of $\mathrm{TAZRP}(\lambda, n)$ *is*

$$\mathcal{Z}_{\lambda,n}(x_1, \ldots, x_n; t) = \widetilde{H}_{\lambda}(x_1, \ldots, x_n; 1, t)$$

- Can compute certain correlations and observables (current, density, local correlations) in terms of Macdonald/LLT polynomials

Formulas for Macdonald polynomials via ASEP and TAZRP



conclusion

Thank you Jim for your passion and enthusiasm, and for inspiring me to explore this beautiful topic!

Some highlights:

Studying Macdonald polynomials through combinatorics of the ASEP has led us to..

- natural interpretation of Macdonald coefficients (with Corteel and Williams)
- natural interpretations for some classical results (with Jerónimo Valencia)
- a quasisymmetric refinement of $P_\lambda(X; q, t)$ (with Corteel, Haglund, Mason, Williams)
- better/more compact formulas for $P_\lambda(X; q, t)$ and $\widetilde{H}_\lambda(X; q, t)$ (with C, H, M, W)
- a quasisymmetric refinement of the Kostka-Faulkes coefficient $K_{\lambda\mu}(q)$

Studying the ASEP through combinatorics of Macdonald polynomials has led us to..

- new connection between particle models and Macdonald polys (with Ayyer, Martin)

discussion

- What is the suitable quasisymmetric version of modified Macdonald polynomials $\widetilde{H}_\lambda(X; q, t)$? Nonsymmetric version?
- Can we use integrability of the TAZRP/bosonic multiline queue process to find operators acting on the TAZRP polynomials (nonsymmetric analogue of the modified Macdonald polynomials)?
- the ASEP with open boundaries is connected to Koornwinder polynomials (Macdonald type BC). Can we find a multiline process that captures the dynamics of the open ASEP?