Jim's List of Tiling Problems

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Enumeration of Matchings: Problems and Progress

JAMES PROPP

Dedicated to the memory of David Klarner (1940–1999)

ABSTRACT. This document is built around a list of thirty-two problems in enumeration of matchings, the first twenty of which were presented in a lecture at MSRI in the fall of 1996. I begin with a capsule history of the topic of enumeration of matchings. The twenty original problems, with commentary, comprise the bulk of the article. I give an account of the progress that has been made on these problems as of this writing, and include pointers to both the printed and on-line literature; roughly half of the original twenty problems were solved by participants in the MSRI Workshop on Combinatorics, their students, and others, between 1996 and 1999. The article concludes with a dozen new open problems.

1. Introduction

How many perfect matchings does a given graph G have? That is, in how many ways can one choose a subset of the edges of G so that each vertex of G belongs to one and only one chosen edge? (See Figure 1(a) for an example of a

The List


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- The list of 32 problems in enumeration of tilings/matchings.

- The first twenty were presented at MSRI in the fall of 1996.

The List

- The first twenty were presented at MSRI in the fall of 1996.
- Roughly half of the original twenty problems were solved between 1996 and 1999.

Comments on the List

Christian Krattenthaler on MathSciNet of the American Mathematical Society (AMS):

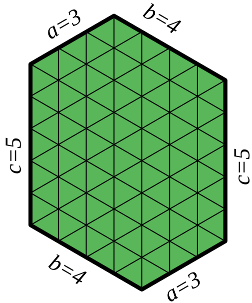
"*This list of problems was very influential; it called forth tremendous activity, resulting in the solution of several of these problems (but by no means all), in the development of interesting new techniques, and, very often, in results that move beyond the problems.*"

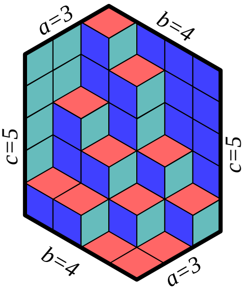
MacMahon's tiling formula

Theorem (MacMahon 1900s)

$$\mathrm{M}(Hex(a,b,c)) = \mathrm{PP}(a,b,c) = \frac{\mathrm{H}(a)\,\mathrm{H}(b)\,\mathrm{H}(c)\,\mathrm{H}(a+b+c)}{\mathrm{H}(a+b)\,\mathrm{H}(b+c)\,\mathrm{H}(c+a)},$$

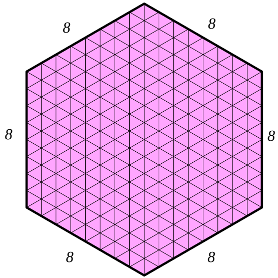
where the hyperfactorial $\mathrm{H}(n) := 0! \cdot 1! \cdot 2! \ldots (n-1)!$.

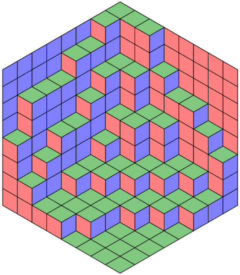




Denote by $\mathrm{M}(R)$ the number of lozenge tilings of the region R.

Example

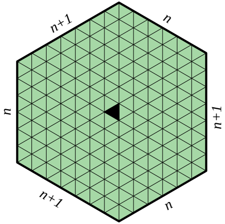




The number of tilings is

$$5,055,160,684,040,254,910,720 = 2^8 \cdot 5 \cdot 13^2 \cdot 17^7 \cdot 19^5 \cdot 23.$$

Problem 2: Enumerate the lozenge tilings of the region obtained from the n, $n+1$, n, $n+1$, n, $n+1$ hexagon by removing the central triangle.



n=1: 2

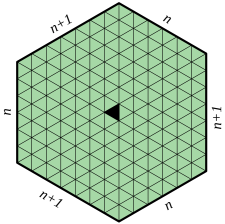
n=2: $2 \cdot 3^3$

n=3: $2^5 \cdot 3^3 \cdot 5$

n=4: $2^5 \cdot 5^7$

n=5: $2^2 \cdot 5^7 \cdot 7^5$

Problem 2: Enumerate the lozenge tilings of the region obtained from the n, $n+1$, n, $n+1$, n, $n+1$ hexagon by removing the central triangle.



The problem was solved independently by Ciucu (1998) and Helfgott–Gessel (1999).

Problem 2: Ciucu's Solution

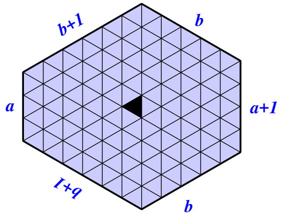


Figure: Ciucu's Region here

Ciucu used his Factorial Theorem to show that:

$$N(a, b, b) = \mathrm{SC}(a+1, b, b) \cdot \mathrm{SC}(a, b+1, b+1)$$

where $\mathrm{SC}(a, b, c)$ is the number of self-complementary plane partitions fitting in an $a \times b \times c$ box.

Self-Complementary Plane Partitions

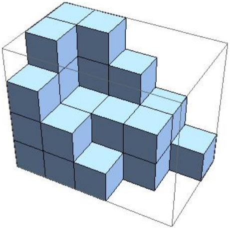


Figure: Self-Complementary Plane Partition (Source: Wikipedia).

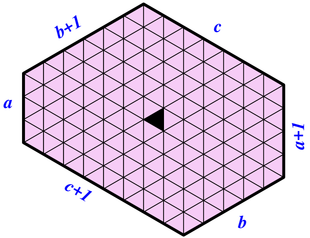
Stanley proves that

$$\mathrm{SC}(2a, 2b, 2c) = \mathrm{PP}(a, b, c)^2$$

$$\mathrm{SC}(2a+1, 2b, 2c) = \mathrm{PP}(a, b, c)\,\mathrm{PP}(a+1, b, c)$$

$$\mathrm{SC}(2a+1, 2b+1, 2c) = \mathrm{PP}(a+1, b, c)\,\mathrm{PP}(a, b+1, c)$$

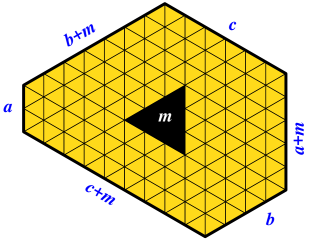
Problem 2: Okada–Krattenthaler's Generalization



Generalized by Okada–Krattenthaler (1998)

$$N(a,b,c) = \mathrm{PP}\left(\left\lceil\frac{a}{2}\right\rceil, \left\lceil\frac{b}{2}\right\rceil, \left\lceil\frac{c}{2}\right\rceil\right) \cdot \mathrm{PP}\left(\left\lceil\frac{a+1}{2}\right\rceil, \left\lfloor\frac{b}{2}\right\rfloor, \left\lceil\frac{c}{2}\right\rceil\right)$$
$$\times \mathrm{PP}\left(\left\lceil\frac{a}{2}\right\rceil, \left\lceil\frac{b+1}{2}\right\rceil, \left\lfloor\frac{c}{2}\right\rfloor\right) \cdot \mathrm{PP}\left(\left\lfloor\frac{a}{2}\right\rfloor, \left\lceil\frac{b}{2}\right\rceil, \left\lceil\frac{c+1}{2}\right\rceil\right).$$

Problem 2: Cored Hexagon



Generalized further by Ciucu–Eisenkölbl–Krattenthaler–Zare (2001) by removing an arbitrary triangle in the "**center***."

Problem 2: Cored Hexagon

Generalized by Ciucu–Eisenkölbl–Krattenthaler–Zare:

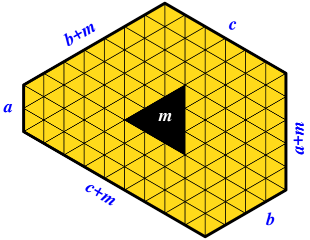
$$\mathrm{M}(C_{a,b,c}(m)) =$$
$$\frac{\mathrm{H}(a+m)\,\mathrm{H}(b+m)\,\mathrm{H}(c+m)\,\mathrm{H}(a+b+c+m)\,\mathrm{H}\left(m+\left\lceil\frac{a+b+c}{2}\right\rceil\right)\mathrm{H}\left(m+\left\lfloor\frac{a+b+c}{2}\right\rfloor\right)}{\mathrm{H}(a+b+m)\,\mathrm{H}(a+c+m)\,\mathrm{H}(b+c+m)\,\mathrm{H}(\frac{a+b}{2}+m)\,\mathrm{H}(\frac{b+c}{2}+m)\,\mathrm{H}(\frac{c+a}{2}+m)}$$
$$\times\frac{\mathrm{H}(\lceil\frac{a}{2}\rceil)\,\mathrm{H}(\lceil\frac{b}{2}\rceil)\,\mathrm{H}(\lceil\frac{c}{2}\rceil)\,\mathrm{H}(\lfloor\frac{a}{2}\rfloor)\,\mathrm{H}(\lfloor\frac{b}{2}\rfloor)\,\mathrm{H}(\lfloor\frac{c}{2}\rfloor)}{\mathrm{H}(\frac{m}{2}+\lceil\frac{a}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lceil\frac{b}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lceil\frac{c}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{a}{2}\rfloor)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{b}{2}\rfloor)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{c}{2}\rfloor)}$$
$$\times\frac{\mathrm{H}(\frac{m}{2})^2\,\mathrm{H}(\frac{a+b+m}{2})^2\,\mathrm{H}(\frac{b+c+m}{2})^2\,\mathrm{H}(\frac{c+a+m}{2})^2}{\mathrm{H}(\frac{m}{2}+\lceil\frac{a+b+c}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{a+b+c}{2}\rfloor)\,\mathrm{H}(\frac{a+b}{2})\,\mathrm{H}(\frac{b+c}{2})\,\mathrm{H}(\frac{c+a}{2})}.$$

when a, b, c have the same parity. (Other cases are similar.)

$$\mathrm{H}(n) := \begin{cases} \prod_{k=0}^{n-1}\Gamma(k+1) & n \text{ an integer,}\\ \prod_{k=0}^{n-\frac{1}{2}}\Gamma\left(k+\frac{1}{2}\right) & n \text{ a half-integer.}\end{cases}$$

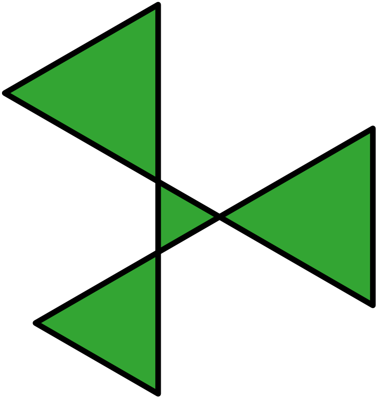
$\Gamma(k+1) = k!; \Gamma\left(k+\frac{1}{2}\right) = \sqrt{\pi}\cdot\frac{(2k-1)!!}{2^k}$

Problem 2: Can we generalize it further?

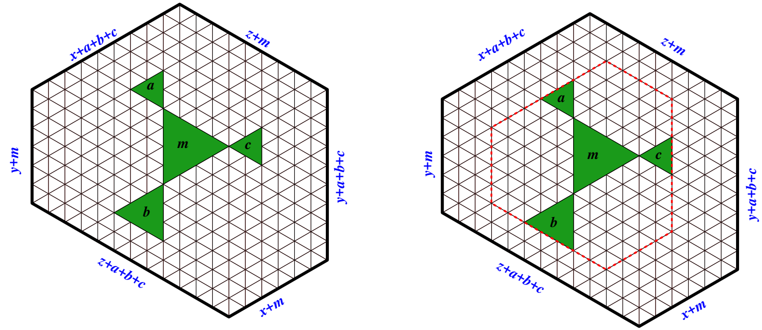


Ciucu–Krattenthaler (2013) expanded the "triangular holes" to a "*shamrock*" hole in the center.

What is a ‘shamrock’?



Problem 2: Shamrock Hole



The S-cored Hexagon: $SC_{x,y,z}(a,b,c,m)$

Problem 2: Ciucu–Krattenthaler's Shamrock Theorem

THEOREM 2.1. *Let x, y, z, a, b, c and m be nonnegative integers. If x, y and z have the same parity, we have*

$$
\begin{aligned}
\mathrm{M}(SC_{x,y,z}(a,b,c,m)) &= \frac{\mathrm{H}(m)^3\ \mathrm{H}(a)\ \mathrm{H}(b)\ \mathrm{H}(c)}{\mathrm{H}(m+a)\ \mathrm{H}(m+b)\ \mathrm{H}(m+c)} \\
&\times \frac{\mathrm{H}(\frac{x+y}{2}+m+a+b)\ \mathrm{H}(\frac{x+z}{2}+m+a+c)\ \mathrm{H}(\frac{y+z}{2}+m+b+c)}{\mathrm{H}(\frac{x+y}{2}+m+c)\ \mathrm{H}(\frac{x+z}{2}+m+b)\ \mathrm{H}(\frac{y+z}{2}+m+a)} \\
&\times \frac{\mathrm{H}(\frac{x+y}{2}+c)\ \mathrm{H}(\frac{x+z}{2}+b)\ \mathrm{H}(\frac{y+z}{2}+a)}{\mathrm{H}(\frac{x+y}{2}+a+b)\ \mathrm{H}(\frac{x+z}{2}+a+c)\ \mathrm{H}(\frac{y+z}{2}+b+c)} \\
&\times \frac{\mathrm{H}(x+m+a+b+c)\,\mathrm{H}(y+m+a+b+c)}{\mathrm{H}(x+y+m+a+b+c)\,\mathrm{H}(x+z+m+a+b+c)} \\
&\times \frac{\mathrm{H}(z+m+a+b+c)\,\mathrm{H}(x+y+z+m+a+b+c)}{\mathrm{H}(y+z+m+a+b+c)} \\
&\times \frac{\mathrm{H}(\lceil\frac{x+y+z}{2}\rceil+m+a+b+c)\,\mathrm{H}(\lfloor\frac{x+y+z}{2}\rfloor+m+a+b+c)}{\mathrm{H}(\frac{x+y}{2}+m+a+b+c)\,\mathrm{H}(\frac{x+z}{2}+m+a+b+c)\,\mathrm{H}(\frac{y+z}{2}+m+a+b+c)} \\
&\times \frac{\mathrm{H}(\lceil\frac{x}{2}\rceil\,\mathrm{H}(\lfloor\frac{x}{2}\rfloor)\,\mathrm{H}(\lceil\frac{y}{2}\rceil)}{\mathrm{H}(\lceil\frac{x}{2}\rceil+\frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor\frac{x}{2}\rfloor+\frac{m+a+b+c}{2})\,\mathrm{H}(\lceil\frac{y}{2}\rceil+\frac{m+a+b+c}{2})} \\
&\times \frac{\mathrm{H}(\lfloor\frac{y}{2}\rfloor)\,\mathrm{H}(\lceil\frac{z}{2}\rceil)\,\mathrm{H}(\lfloor\frac{z}{2}\rfloor)}{\mathrm{H}(\lfloor\frac{y}{2}\rfloor+\frac{m+a+b+c}{2})\,\mathrm{H}(\lceil\frac{z}{2}\rceil+\frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor\frac{z}{2}\rfloor+\frac{m+a+b+c}{2})} \\
&\times \frac{\mathrm{H}(\frac{m+a+b+c}{2})^2\ \mathrm{H}(\frac{x+y}{2}+\frac{m+a+b+c}{2})^2\ \mathrm{H}(\frac{x+z}{2}+\frac{m+a+b+c}{2})^2\ \mathrm{H}(\frac{y+z}{2}+\frac{m+a+b+c}{2})^2}{\mathrm{H}(\lceil\frac{x+y+z}{2}\rceil+\frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor\frac{x+y+z}{2}\rfloor+\frac{m+a+b+c}{2})\,\mathrm{H}(\frac{x+y}{2})\,\mathrm{H}(\frac{x+z}{2})\,\mathrm{H}(\frac{y+z}{2})}.
\end{aligned}
\tag{2.1}
$$

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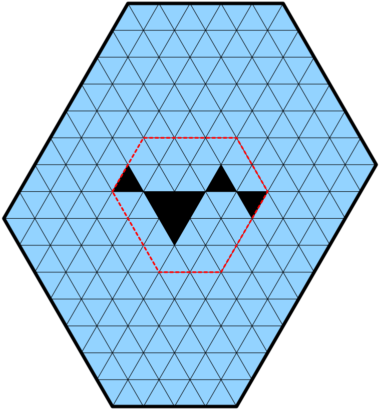
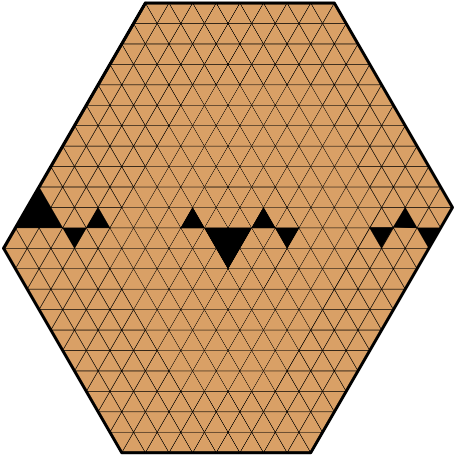
(screenshot of Ciucu–Krattenthaler's paper)

Problem 2: Removing an array of triangles



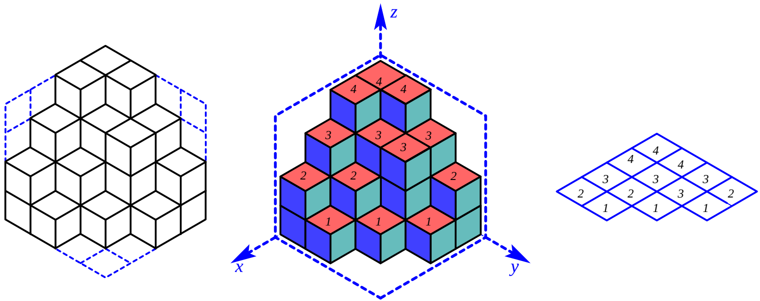
(Ciucu 2017)

Problem 2: Removing three array of triangles

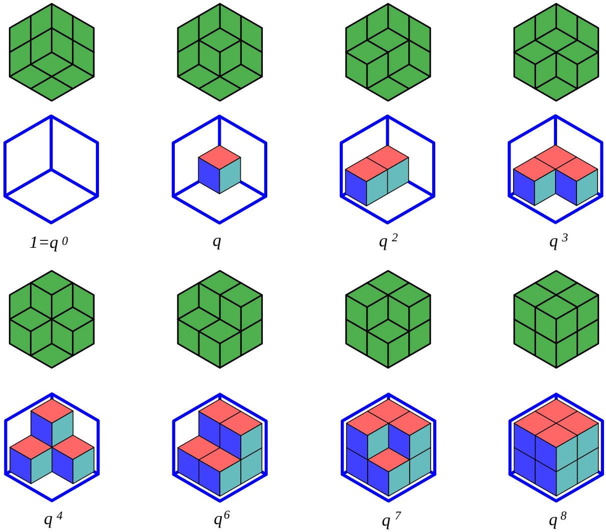


(L. 2020)

Lozenge Tilings and Plane Partitions



Weighting Lozenge Tilings



MacMahon's q-formula

Theorem (MacMahon 1900s)

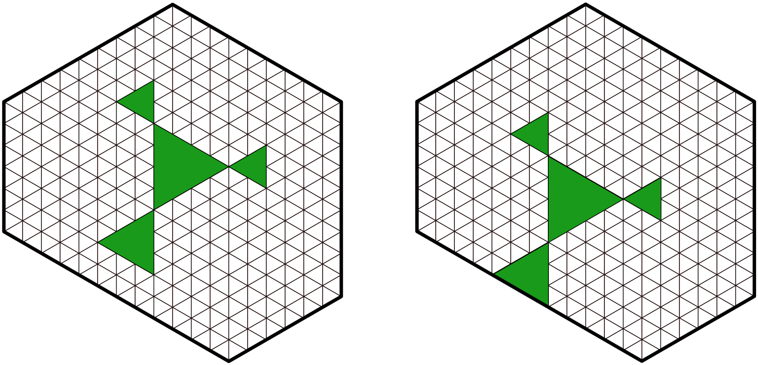
$$\sum_{\pi} q^{|\pi|} = \frac{\mathrm{H}_q(a)\,\mathrm{H}_q(b)\,\mathrm{H}_q(c)\,\mathrm{H}_q(a+b+c)}{\mathrm{H}_q(a+b)\,\mathrm{H}_q(b+c)\,\mathrm{H}_q(c+a)},$$

where the sum is over all stacks of unit cubes fitting in an $a \times b \times c$-box, and $|\pi|$ denote the volume of the stack π.

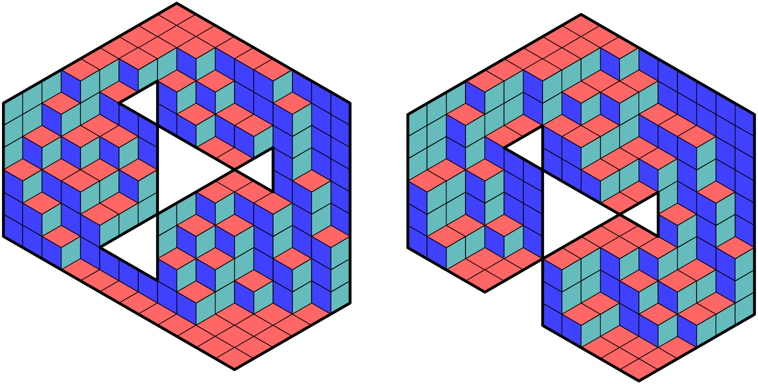
Definition:

- q-integer $[n]_q := 1 + q + q^2 + \ldots + q^{n-1}$
- q-factorial $[n]_q! := [1]_q \cdot [2]_q \cdot [3]_q \ldots [n]_q$,
- q-hyperfactorial $\mathrm{H}_q(n) := [0]_q! \cdot [1]_q! \cdot [2]! \ldots [n-1]_q!$.

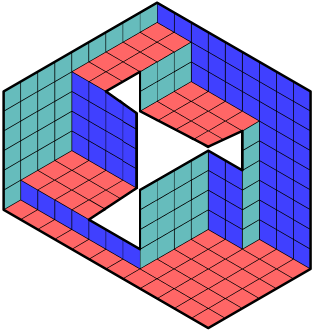
Shamrock Hole v.s. Shamrock Dent

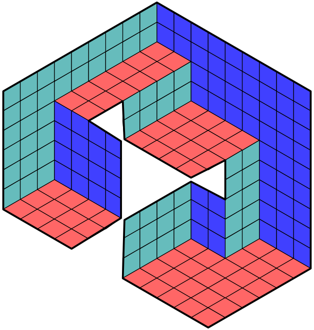


Shamrock Hole v.s. Shamrock Dent

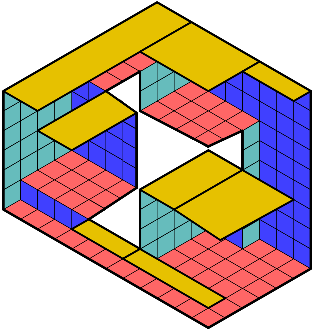


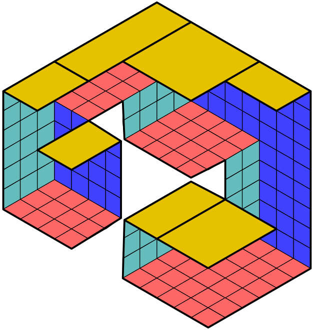
Shamrock Hole v.s. Shamrock Dent



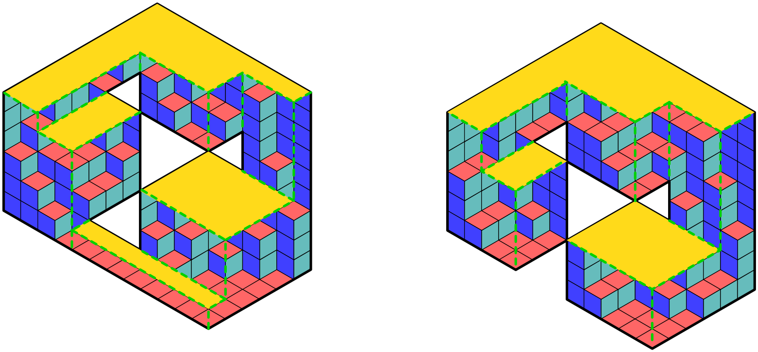


Shamrock Hole v.s. Shamrock Dent

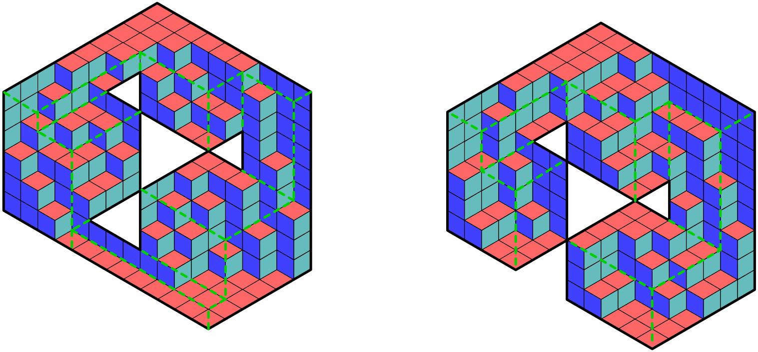




Shamrock Hole v.s. Shamrock Dent



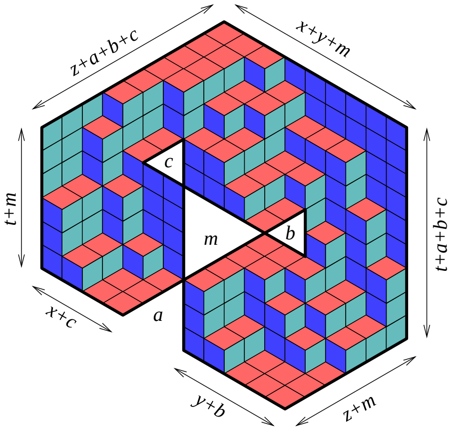
Shamrock Hole v.s. Shamrock Dent



We would like to find the (volume) generating functions of the stacks

$$\sum_{\pi} q^{|\pi|}$$

q-enumeration for shamrock dent case



$$Q\begin{pmatrix} x & y & z & t \\ m & a & b & c \end{pmatrix}$$

q-enumeration for shamrock dent case

Theorem (L. 2015)

For nonnegative integers x, y, z, t, m, a, b, c

$$\sum_{\pi} q^{|\pi|} = \frac{\mathrm{H}_q(\Delta+x+y+z+t)}{\mathrm{H}_q(\Delta+x+y+t)\,\mathrm{H}_q(\Delta+x+y+z)}$$

$$\times \frac{\mathrm{H}_q(\Delta+x+t)\,\mathrm{H}_q(\Delta+x+y)\,\mathrm{H}_q(\Delta+y+z)\,\mathrm{H}_q(\Delta)}{\mathrm{H}_q(\Delta+z+t)\,\mathrm{H}_q(\Delta+x)\,\mathrm{H}_q(\Delta+y)}$$

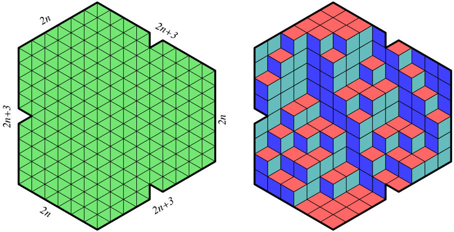
$$\times \frac{\mathrm{H}_q(m+b+c+z+t)\,\mathrm{H}_q(m+a+c+x)\,\mathrm{H}_q(m+a+b+y)}{\mathrm{H}_q(m+b+y+z)\,\mathrm{H}_q(m+c+x+t)}$$

$$\times \frac{\mathrm{H}_q(c+x+t)\,\mathrm{H}_q(b+y+z)}{\mathrm{H}_q(a+c+x)\,\mathrm{H}_q(a+b+y)\,\mathrm{H}_q(b+c+z+t)}$$

$$\times \frac{\mathrm{H}_q(m)^3\,\mathrm{H}_q(a)^2\,\mathrm{H}_q(b)\,\mathrm{H}_q(c)\,\mathrm{H}_q(x)\,\mathrm{H}_q(y)\,\mathrm{H}_q(z)\,\mathrm{H}_q(t)}{\mathrm{H}_q(m+a)^2\,\mathrm{H}_q(m+b)\,\mathrm{H}_q(m+c)\,\mathrm{H}_q(x+t)\,\mathrm{H}_q(y+z)},$$

where $\Delta = m+a+b+c$.

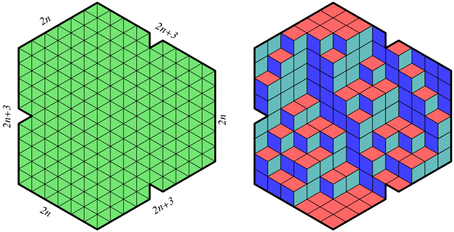
Problem 3: Find the number of lozenge tilings in a hexagon of side-lengths $2n+3, 2n, 2n+3, 2n, 2n+3, 2n$, where the central unit triangles are removed from the long sides.



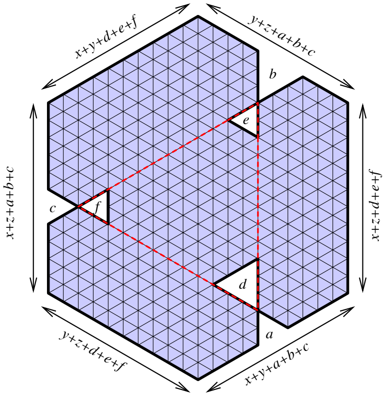
n=1: $2^7 \cdot 7^2$
n=2: $2^2 \cdot 7^4 \cdot 11^4 \cdot 13^2$
n=3: $2^{10} \cdot 3^3 \cdot 5^8 \cdot 13^2 \cdot 17^4 \cdot 19^2$
n=4: $2^2 \cdot 5^2 \cdot 7^2 \cdot 11^3 \cdot 13^4 \cdot 17^4 \cdot 19^8 \cdot 23^4$

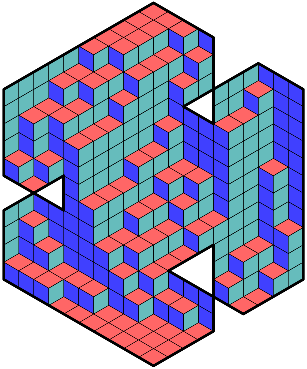
Eisenkölbl solved the problem (1999):

$$\left(\prod_{i=1}^{2n}(n+i+1)\right)^6 \frac{\left(\prod_{k=0}^{2n} k!\right)^3 \prod_{k=0}^{6n+2} k!}{\left(\prod_{k=0}^{4n+2} k!\right)^3}(n+1)^3(3n+1)(3n+2)^2$$



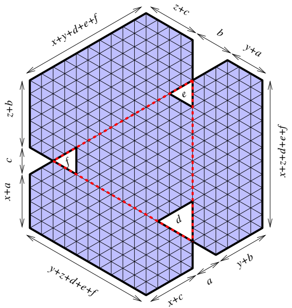
Generalizing Propp's Problem 3





$$F := F\begin{pmatrix} x & y & z \\ a & b & c \\ d & e & f \end{pmatrix}$$

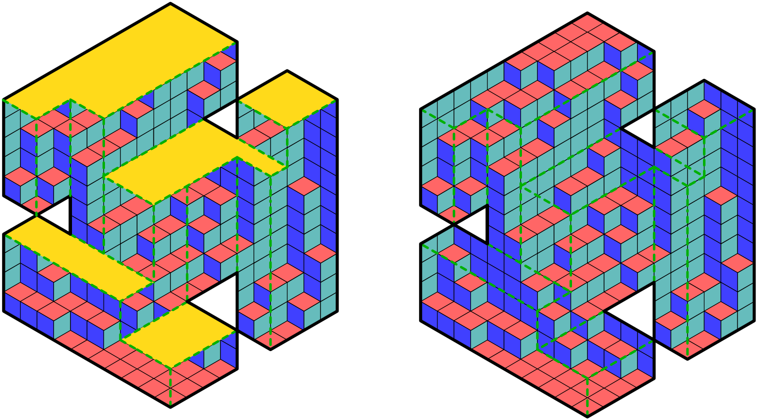
Example



$$\mathrm{M}\left(F\begin{pmatrix}2&1&2\\2&3&2\\3&2&2\end{pmatrix}\right)=391,703,752,601,434,880,582,976,000,000,000$$

$$=2^{15}\cdot 3^3\cdot 5^9\cdot 7^9\cdot 11^3\cdot 13^3\cdot 17^4\cdot 23$$

Generalizing Propp's Problem



$$M_q\left(F\begin{pmatrix} x & y & z \\ a & b & c \\ d & e & f \end{pmatrix}\right) = \sum_{\pi} q^{|\pi|} = ?$$

q-enumeration of a hexagon with three dents

Theorem (L. 2016)

For nonnegative integers $x, y, z, a, b, c, d, e, f$

$$\mathrm{M}_q\left(F\begin{pmatrix} x & y & z \\ a & b & c \\ d & e & f \end{pmatrix}\right) =$$

$$\frac{\mathrm{H}_q(x)\,\mathrm{H}_q(y)\,\mathrm{H}_q(z)\,\mathrm{H}_q(a)^2\,\mathrm{H}_q(b)^2\,\mathrm{H}_q(c)^2\,\mathrm{H}_q(d)\,\mathrm{H}_q(e)\,\mathrm{H}_q(f)\,\mathrm{H}_q(d+e+f+x+y+z)^4}{\mathrm{H}_q(a+d)\,\mathrm{H}_q(b+e)\,\mathrm{H}_q(c+f)\,\mathrm{H}_q(d+e+x+y+z)\,\mathrm{H}_q(e+f+x+y+z)\,\mathrm{H}_q(f+d+x+y+z)}$$

$$\times\frac{\mathrm{H}_q(A+2x+2y+2z)\,\mathrm{H}_q(A+x+y+z)^2}{\mathrm{H}_q(A+2x+y+z)\,\mathrm{H}_q(A+x+2y+z)\,\mathrm{H}_q(A+x+y+2z)}$$

$$\times\frac{\mathrm{H}_q(a+b+d+e+x+y+z)\,\mathrm{H}_q(a+c+d+f+x+y+z)\,\mathrm{H}_q(b+c+e+f+x+y+z)}{\mathrm{H}_q(a+d+e+f+x+y+z)^2\,\mathrm{H}_q(b+d+e+f+x+y+z)^2\,\mathrm{H}_q(c+d+e+f+x+y+z)^2}$$

$$\times\frac{\mathrm{H}_q(a+d+x+y)\,\mathrm{H}_q(b+e+y+z)\,\mathrm{H}_q(c+f+z+x)}{\mathrm{H}_q(a+b+y)\,\mathrm{H}_q(b+c+z)\,\mathrm{H}_q(c+a+x)}$$

$$\times\frac{\mathrm{H}_q(A-a+x+y+2z)\,\mathrm{H}_q(A-b+2x+y+z)\,\mathrm{H}_q(A-c+x+2y+z)}{\mathrm{H}_q(b+c+e+f+x+y+2z)\,\mathrm{H}_q(c+a+d+f+2x+y+z)\,\mathrm{H}_q(a+b+d+e+x+2y+z)}.$$

where $A = a+b+c+d+e+f$.

Problem 16: Find a formula for the number of "diform" tilings in the (a, b, c)-"quasihexagon" in the dissection of the plane that arises from slicing the dissection into squares along every third upward-sloping diagonal.

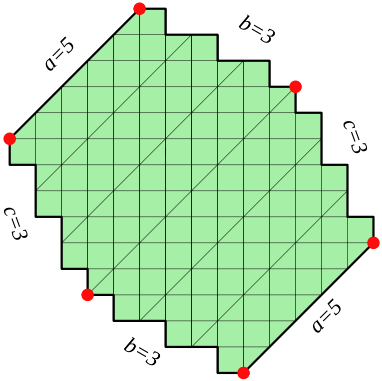
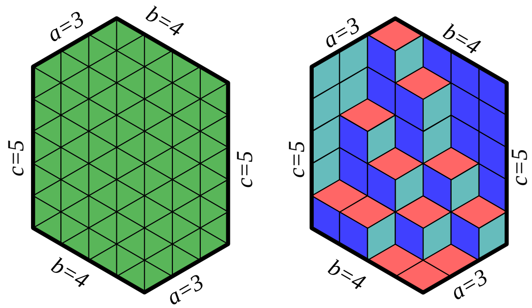


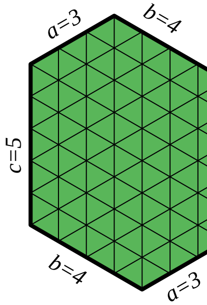
Figure: A "quasihexagon".

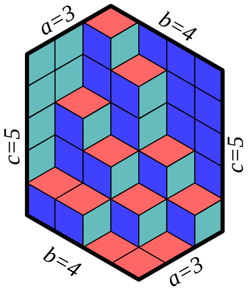
- A lattice divides the plane into small pieces, called "fundamental regions".



Definition of Tilings

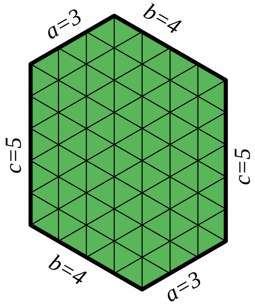
- A lattice divides the plane into small pieces, called "fundamental regions".
- A **tile** is the union of any two adjacent pieces.

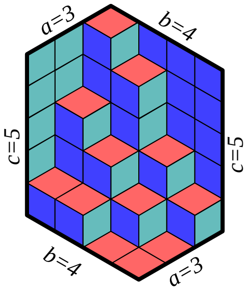




Definition of Tilings

- A lattice divides the plane into small pieces, called "fundamental regions".
- A **tile** is the union of any two adjacent pieces.
- A **tiling** of a region is a covering of the region by **tiles** so that there are no gaps or overlaps.





Definition of Tilings

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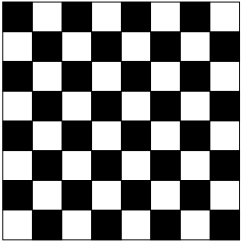
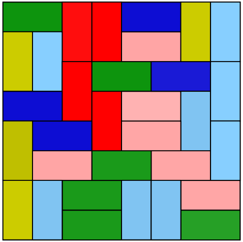
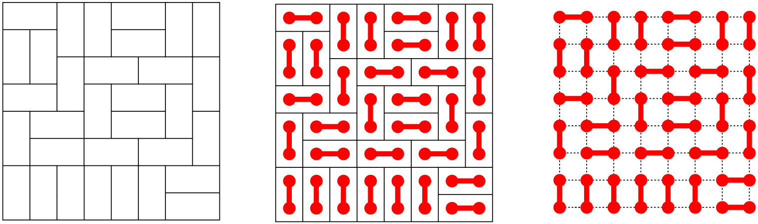


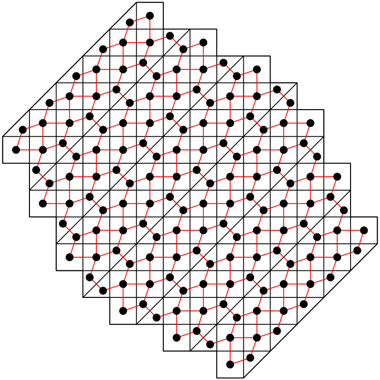


Tilings and perfect matching



The number of tilings of a region is equal to the number of perfect matchings of its dual graph.

Dual graph of the QuasiHexagon



Dual graph of the QuasiHexagon

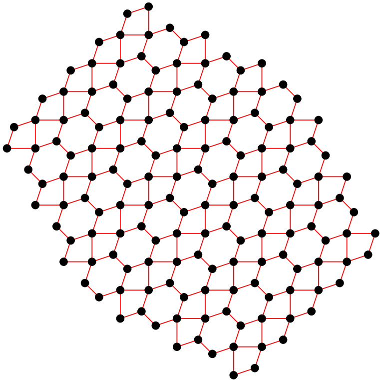


Figure: The dual graph of the quasihexagon

Transforming the dual graph

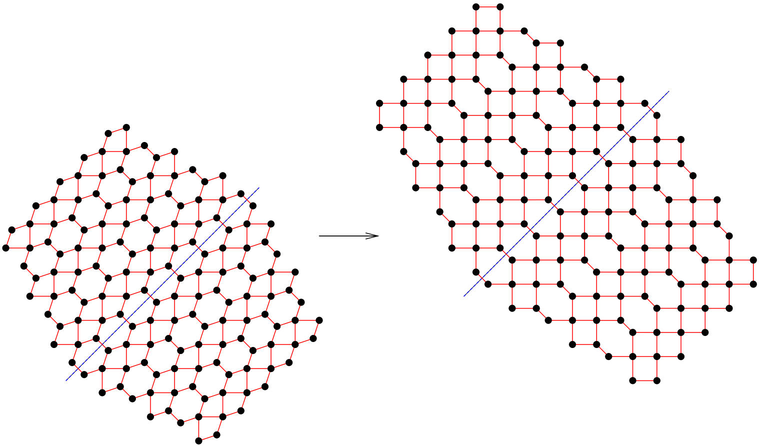


Figure: Transforming the dual graph

Transforming the dual graph

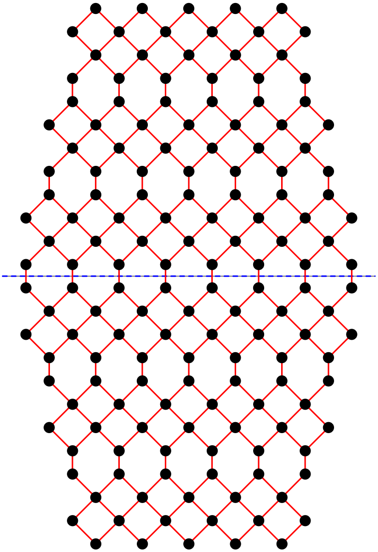
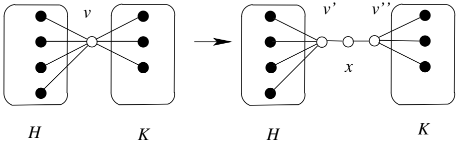
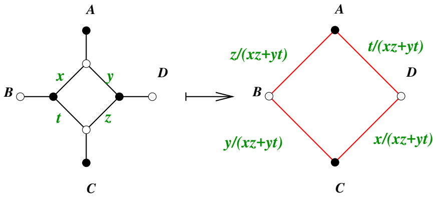


Figure: Transforming the dual graph

Vertex–Splitting Lemma



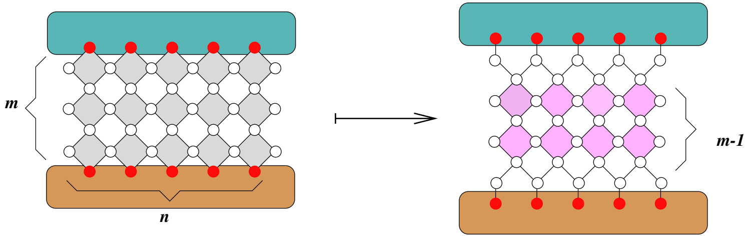
$$\mathrm{M}(G) = \mathrm{M}(G')$$



$$\mathrm{M}(G) = (xz + yt) \cdot \mathrm{M}(G')$$

- $\mathrm{M}(G) = \sum_{\mu} wt(\mu)$, where $wt(\mu)$ is the weight of perfect matching μ of G.
- $wt(\mu)$ is the product of weights of all edges in μ.

Sandwich Lemma



$$\mathrm{M}(G) = 2^m\,\mathrm{M}(G')$$

Transforming the dual graph

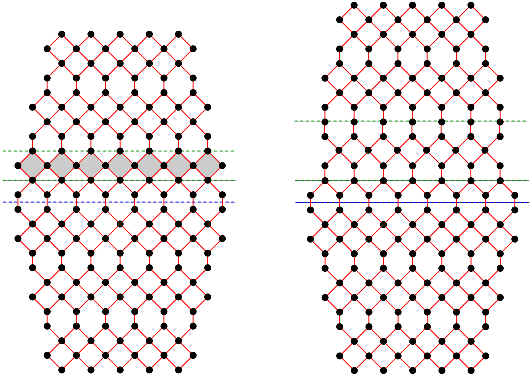


Figure: Transforming the dual graph

Transforming the dual graph

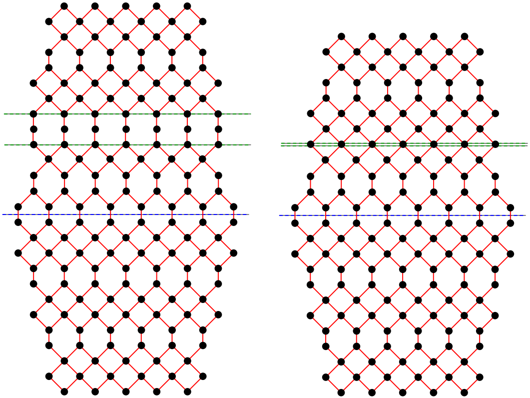


Figure: Transforming the dual graph

Transforming the dual graph

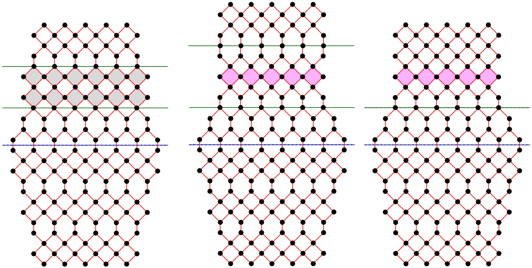


Figure: Transforming the dual graph

Transforming the dual graph

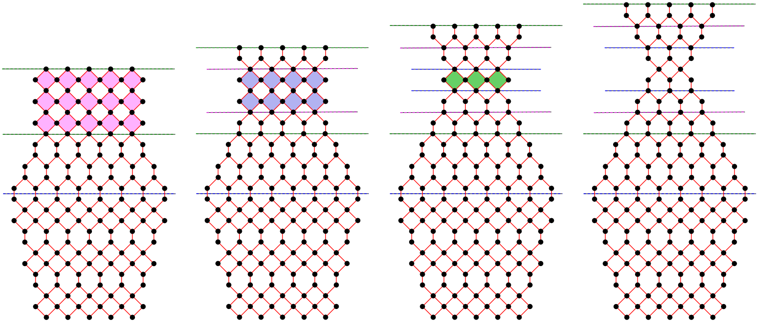


Figure: Transforming the dual graph

Transforming the dual graph

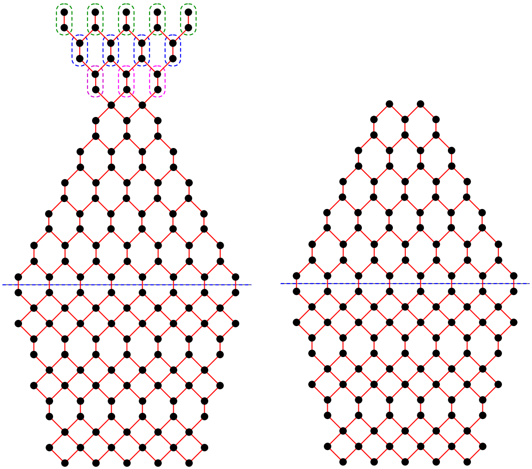


Figure: Transforming the dual graph

Transforming the dual graph

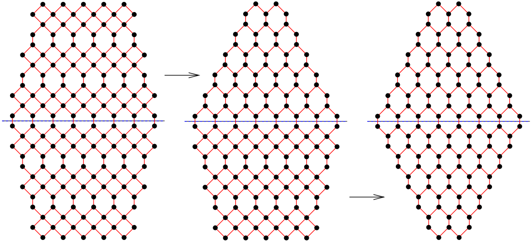
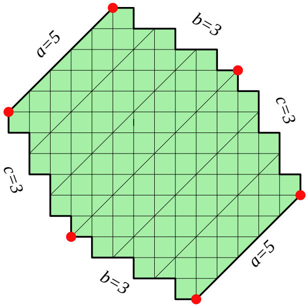


Figure: Transforming the dual graph

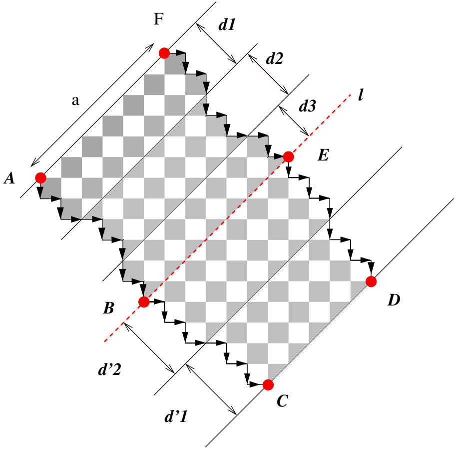
Problem 16



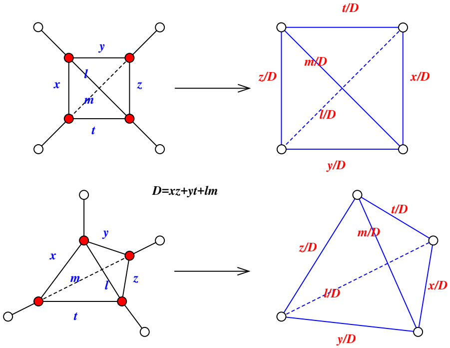
- There is no simple product formula for the general case.
- (L. 2014) However, in the case $b = c$:

$$\mathrm{M}(Q(a,b,b)) = 2^p \cdot \mathrm{M}(Hex(a',b',b')).$$

Problem 16: More general region



Genralized Urban Renewal



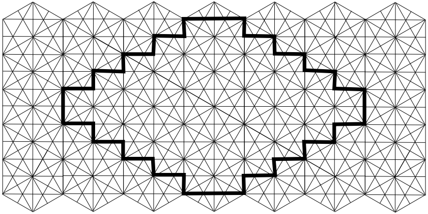
$$\mathrm{M}(G) = (xz + yt + lm)\,\mathrm{M}(G')$$

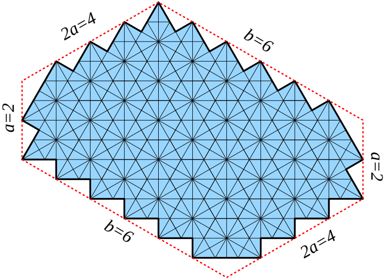
Question: *Is there any application of the generalized urban renewal?*

Propp's Problem 25

Theorem (Ciucu 2001)

The number of tilings of an Aztec Dungeon is always a power of 13 or twice a power of 13.





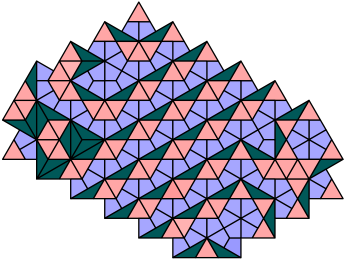


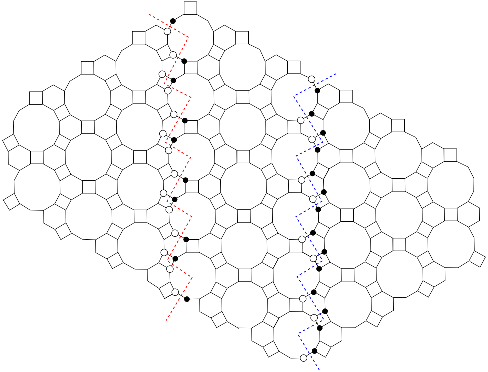
Figure: A "hexagonal dungeon".

Problem 25: (Blum's Conjecture) *Show that the hexagonal dungeon with sides $a, 2a, b, a, 2a, b$ has exactly*

$$13^{2a^2}14^{\lfloor \frac{a^2}{2} \rfloor}$$

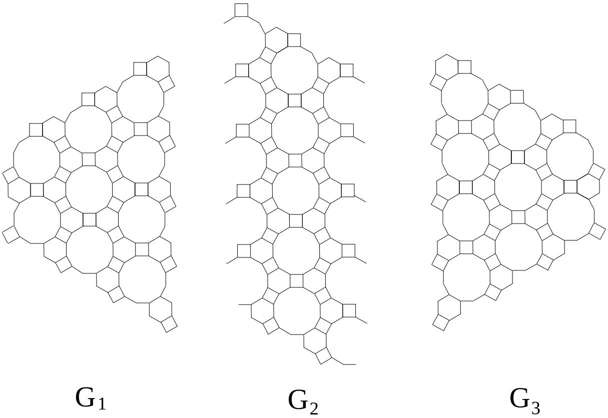
diform tilings, for all $b \geq 2a$.

Blum's Conjecture: Rephrasing



Blum's Conjecture: Rephrasing

$$\mathrm{M}(G) = \mathrm{M}(G_1)\,\mathrm{M}(G_2)\,\mathrm{M}(G_3) = \mathrm{M}(G_1)^2 = 13^{2a^2}14^{\lfloor \frac{a^2}{2} \rfloor}.$$



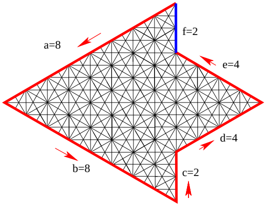
Blum's Conjecture: Rephrasing

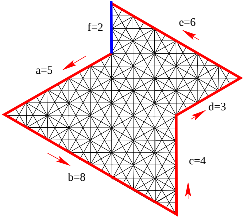
Conjecture (Blum's conjecture (an equivalent form))

$$\mathrm{M}(G_1) = 13^{a^2} 14^{\lfloor \frac{a^2}{4} \rfloor}$$

Defining the Contours

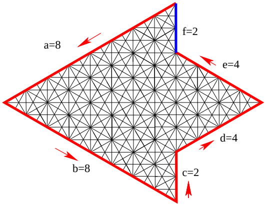
- *a* units southwest, *b* units southeast, *c* units north, *d* units northeast, and *e* units northwest.

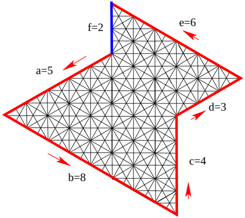




Defining the Contours

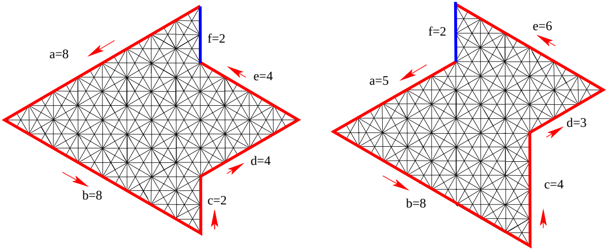
- a units southwest, b units southeast, c units north, d units northeast, and e units northwest.
- Choose e so that the ending point is on the same vertical lattice line as the starting point.



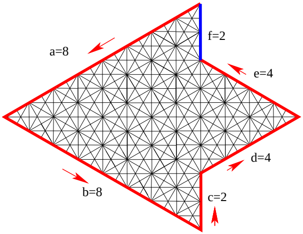


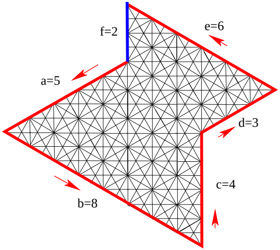
Defining the Contours

- The choice of e implies that $a + e = b + d$, or $e = b + d - a$.



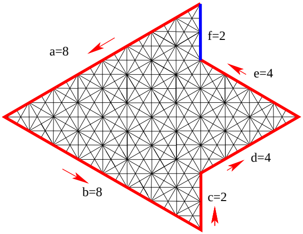
- The choice of e implies that $a + e = b + d$, or $e = b + d - a$.
- The closure of the contour requires $f = |a - c - d|$.

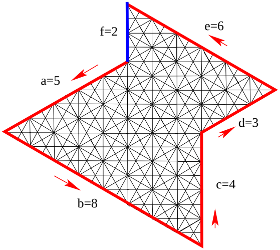




Defining the Contours

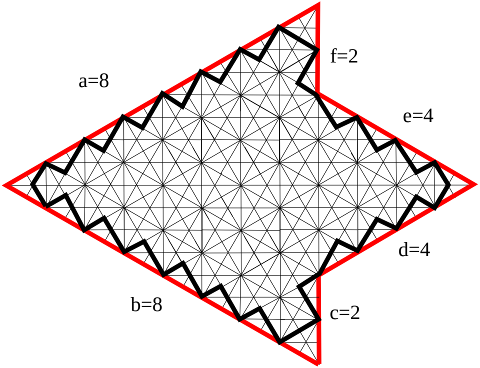
- The choice of e implies that $a + e = b + d$, or $e = b + d - a$.
- The closure of the contour requires $f = |a - c - d|$.
- Moreover, $d = 2b - a - 2c$.



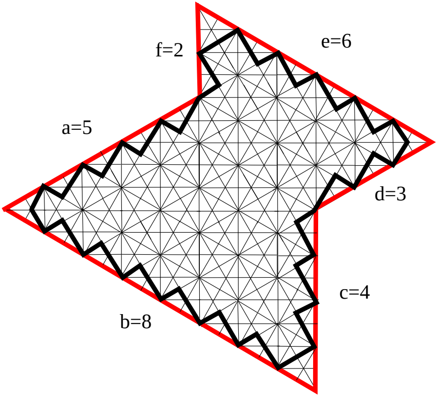


Definition of region $D_{a,b,c}$ when $a > c + d$.

Region $D_{8,8,2}$

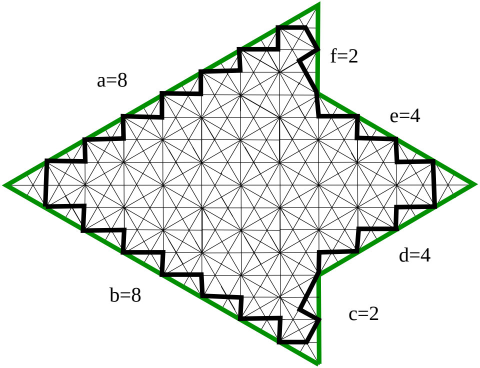


Region $D_{5,8,4}$

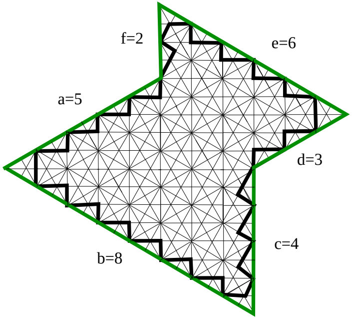


Definition of region $E_{a,b,c}$ when $a > c + d$.

Region $E_{8,8,2}$



Region $E_{5,8,4}$



Main result

Theorem (Ciucu–L. 2014)

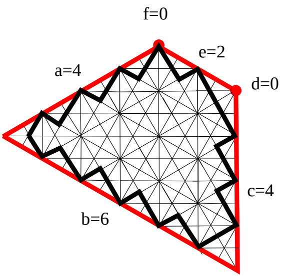
Assume that a, b, *and* c *are three nonnegative integers satisfying* $b \geq 2$, $(d =)2b - a - 2c \geq 0$ *and* $(e =)3b - 2a - 2c \geq 0$. *Then*

$$\mathrm{M}(D_{a,b,c}) = h(a,b,c)13^{g(a,b,c)}14^{f(a,b,c)}$$

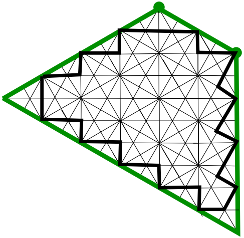
$$\mathrm{M}(E_{a,b,c}) = p(a,b,c)13^{g(a,b,c)}14^{f(a,b,c)}$$

where $h(a,b,c)$ *and* $p(a,b,c)$ *take values* $1, 2, 3$, *or* 5 *depending on* $3b + a - c \pmod 6$.

Implying Blum's conjecture.



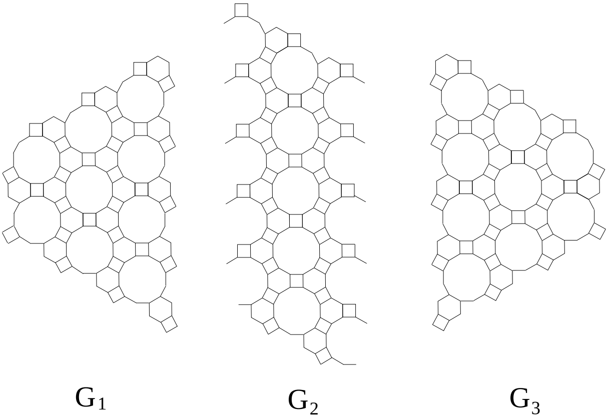
(a)



(b)

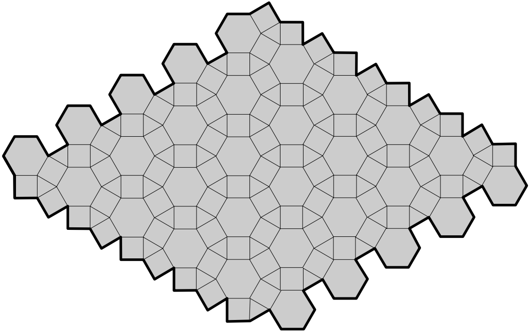
Implying Blum's conjecture

$$\mathrm{M}(G) = \mathrm{M}(G_1)\,\mathrm{M}(G_2)\,\mathrm{M}(G_3) = \mathrm{M}(G_1)^2 = 13^{2a^2}14^{\lfloor \frac{a^2}{2} \rfloor}.$$



Problem 15: Prove that the number of diform tilings of the dragon of order n is $2^{n(n+1)}$.

Solved by Ben Wieland



Problem 15: Generalizing the Aztec Dragon

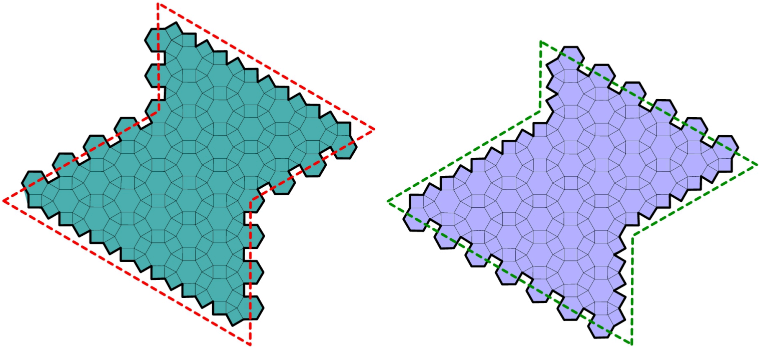


Figure: Generalized Aztec dragons

(L. 2016)

$$\mathrm{M}(D(a,b,c)) = 2^p \cdot 3^q$$

Connection to Cluster Algebra

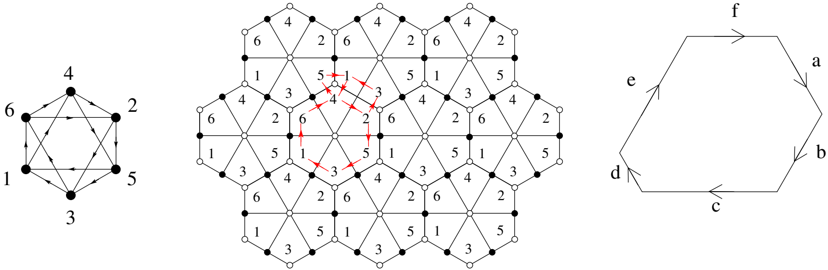


Figure: The dP_3 quiver, its associated brane tiling, and the hexagonal contour $(+,+,+,+,+,+)$.

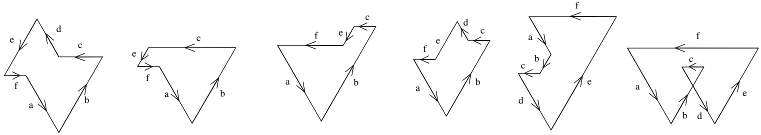
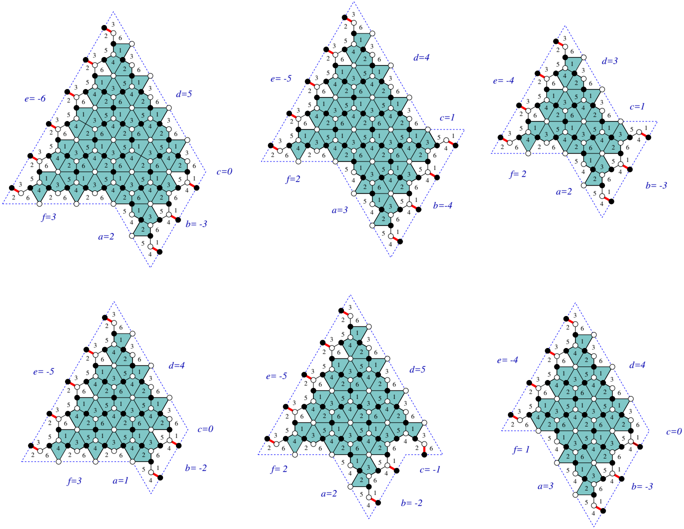
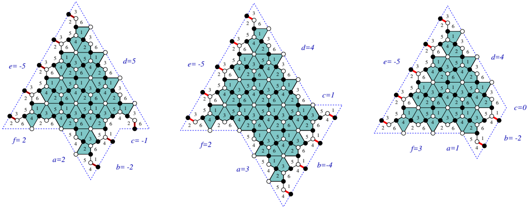


Figure: Different shapes of the hexagonal contour.

Connection to Cluster Algebra



Connection to Cluster Algebra



- Face are weighted by $x_1, x_2, \ldots, x_6$
- Edge adjacent to faces i and j is weighted by $\frac{1}{x_i x_j}$
- *Covering monomial* $F(G)$ is the product of weights of all shaded faces and their neighbor faces.
- The weight $W(G) := F(G) \sum_{\mu} wt(\mu)$, where the sum is taken over all perfect matchings μ of G, and $wt(\mu)$ is the product of weights of edges in μ.

Connection to Cluster Algebra

- Face i is weighted by x_i
- Edge adjacent to faces i and j is weighted by $\frac{1}{x_i x_j}$
- *Covering monomial* $F(G)$ is the product of weights of all shaded faces and their neighbor faces.
- The weight $W(G) := F(G) \sum_\mu wt(\mu)$, where the sum is taken over all perfect matchings μ of G, and $wt(\mu)$ is the product of weights of edges in μ.

Theorem (Musiker and L. 2017)

Let G be any six-sided graph in the above family. The weight $W(G)$ is equal to the cluster variable obtained by a sequence of toric mutations on the dP_3 quiver. Moreover, the latter cluster variable can be written as a closed-form product formula in the variables $x_1, x_2, \ldots, x_6$.

Connection to Cluster Algebra

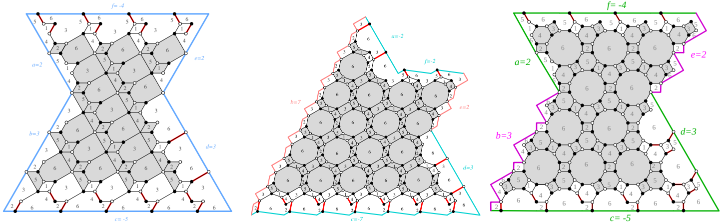
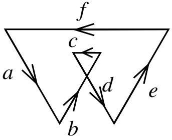
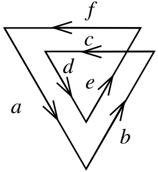


Figure: Several examples of the 6-sided graphs for Models 2, 3, and 4.

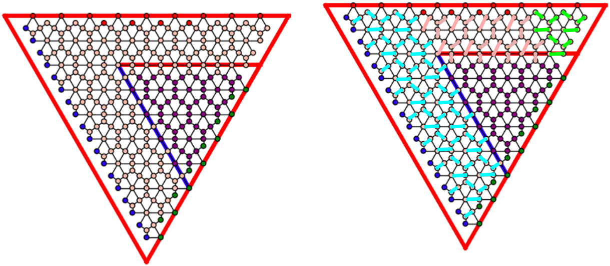
How about the intersecting contours?

(+,−,+,−,+,−)





Mixed-dimer Model

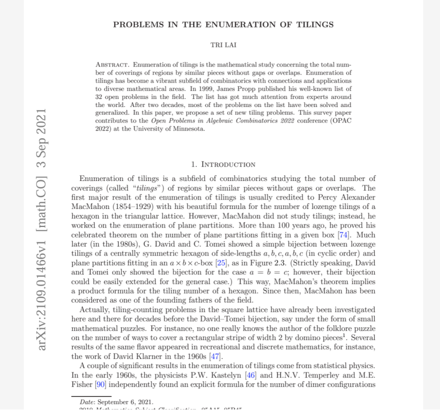


Conjecture (Musiker and L.)

In the case of self-intersecting contours, the weighted sum of the mixed-dimer configurations is equal to the cluster variable obtained by a sequence of toric mutations on the dP_3 quiver in the previous theorem.

Musiker–Jenne–L. work in progress

New List of Tiling Problems



PROBLEMS IN THE ENUMERATION OF TILINGS

TRI LAI

ABSTRACT. Enumeration of tilings is the mathematical study concerning the total number of coverings of regions by similar pieces without gaps or overlaps. Enumeration of tilings has become a vibrant subfield of combinatorics with connections and applications to diverse mathematical areas. In 1999, James Propp published his well-known list of 32 open problems in the field. The list has got much attention from experts around the world. After two decades, most of the problems on the list have been solved and generalized. In this paper, we propose a set of new tiling problems. This survey paper contributes to the *Open Problems in Algebraic Combinatorics 2022* conference (OPAC 2022) at the University of Minnesota.

1. INTRODUCTION

Enumeration of tilings is a subfield of combinatorics studying the total number of coverings (called "*tilings*") of regions by similar pieces without gaps or overlaps. The first major result of the enumeration of tilings is usually credited to Percy Alexander MacMahon (1854–1929) with his beautiful formula for the number of lozenge tilings of a hexagon in the triangular lattice. However, MacMahon did not study tilings; instead, he worked on the enumeration of plane partitions. More than 100 years ago, he proved his celebrated theorem on the number of plane partitions fitting in a given box [74]. Much later (in the 1980s), G. David and C. Tomei showed a simple bijection between lozenge tilings of a centrally symmetric hexagon of side-lengths a, b, c, a, b, c (in cyclic order) and plane partitions fitting in an $a \times b \times c$-box [25], as in Figure 2.3. (Strictly speaking, David and Tomei only showed the bijection for the case $a = b = c$; however, their bijection could be easily extended for the general case.) This way, MacMahon's theorem implies a product formula for the tiling number of a hexagon. Since then, MacMahon has been considered as one of the founding fathers of the field.

Actually, tiling-counting problems in the square lattice have already been investigated here and there for decades before the David–Tomei bijection, say under the form of small mathematical puzzles. For instance, no one really knows the author of the folklore puzzle on the number of ways to cover a rectangular stripe of width 2 by domino pieces[1]. Several results of the same flavor appeared in recreational and discrete mathematics, for instance, the work of David Klarner in the 1960s [47].

A couple of significant results in the enumeration of tilings come from statistical physics. In the early 1960s, the physicists P.W. Kastelyn [46] and H.N.V. Temperley and M.E. Fisher [90] independently found an explicit formula for the number of dimer configurations

Date: September 6, 2021.

New List of Tiling Problems

- Contains 33 new problems
- To appear in OPAC 2022 Proceedings of Symposia in Pure Mathematics (AMS)
- Available in arXiv.org: `https://arxiv.org/abs/2109.01466`

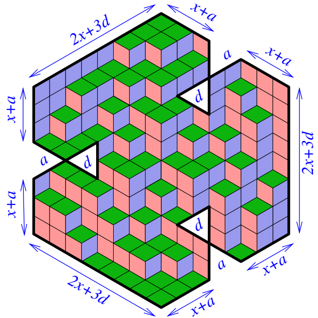
New Problem

- Macdonald conjectured that

$$\sum_{\pi} q^{|\pi|} = \prod_{i=1}^{n} \frac{1-q^{3i-1}}{1-q^{3i-2}} \prod_{1\leq i<j\leq n} \frac{1-q^{3(2i+j-1)}}{1-q^{3(2i+j-2)}} \prod_{1\leq i<j,k\leq n} \frac{1-q^{3(i+j+k-1)}}{1-q^{3(i+j+k-2)}},$$

  where the sum is over all cyclically symmetric plane partitions π that are contained in a $(n \times n \times n)$-box.
- The unweighted version ($q = 1$) was proved by Andrews in 1980.
- The conjecture was proved by Mills, Robbins, and Rumsey in 1982.

New Problem



New Problem 21: *Find the generating function of the cyclically symmetric tiling of the hexagon with three removed bowties.*

Happy Birthday Jim!

Email: tlai3@unl.edu
Website: https://sites.google.com/view/trilai/
Jim's list: https://faculty.uml.edu/jpropp/matchings.pdf
(with an update https://faculty.uml.edu/jpropp/eom.pdf)
My new list: https://arxiv.org/abs/2109.01466
Jim's list of problems in trimer coverings https://arxiv.org/abs/2206.06472