Multinomial Dimer Model

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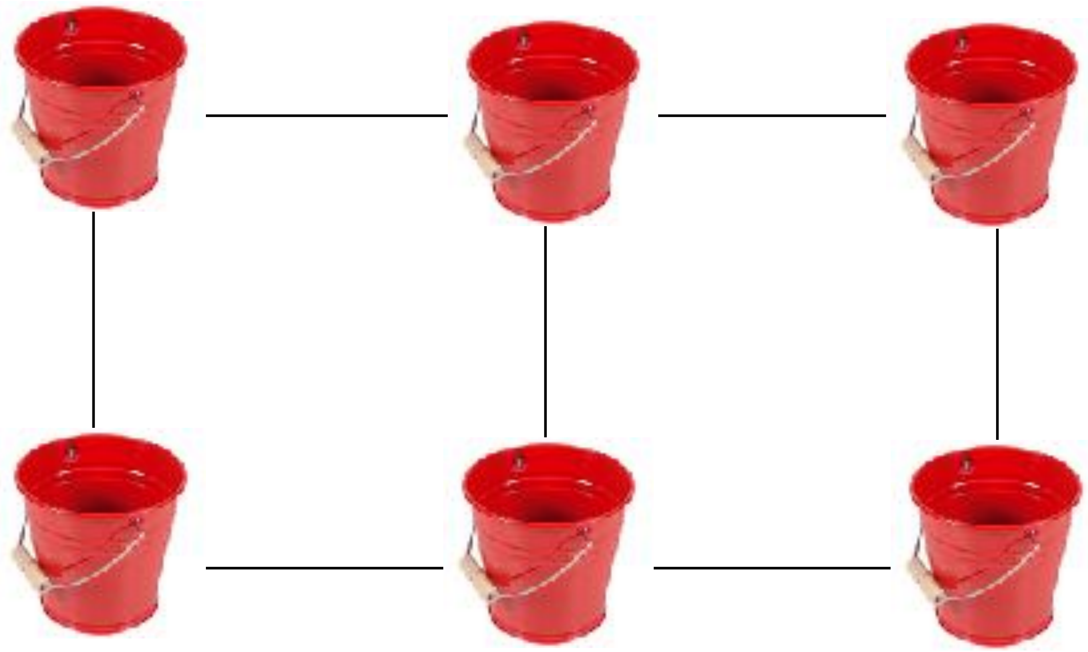
based on earlier work with Cosmin Pohoata (Emory)

The bucket game



Each bucket has capacity n

Select an edge at random; drop a ball into its two buckets. Repeat K times.

Condition on the event that all buckets filled after $K = \frac{1}{2}|V|n$ steps.

Q. How many times is each edge used?

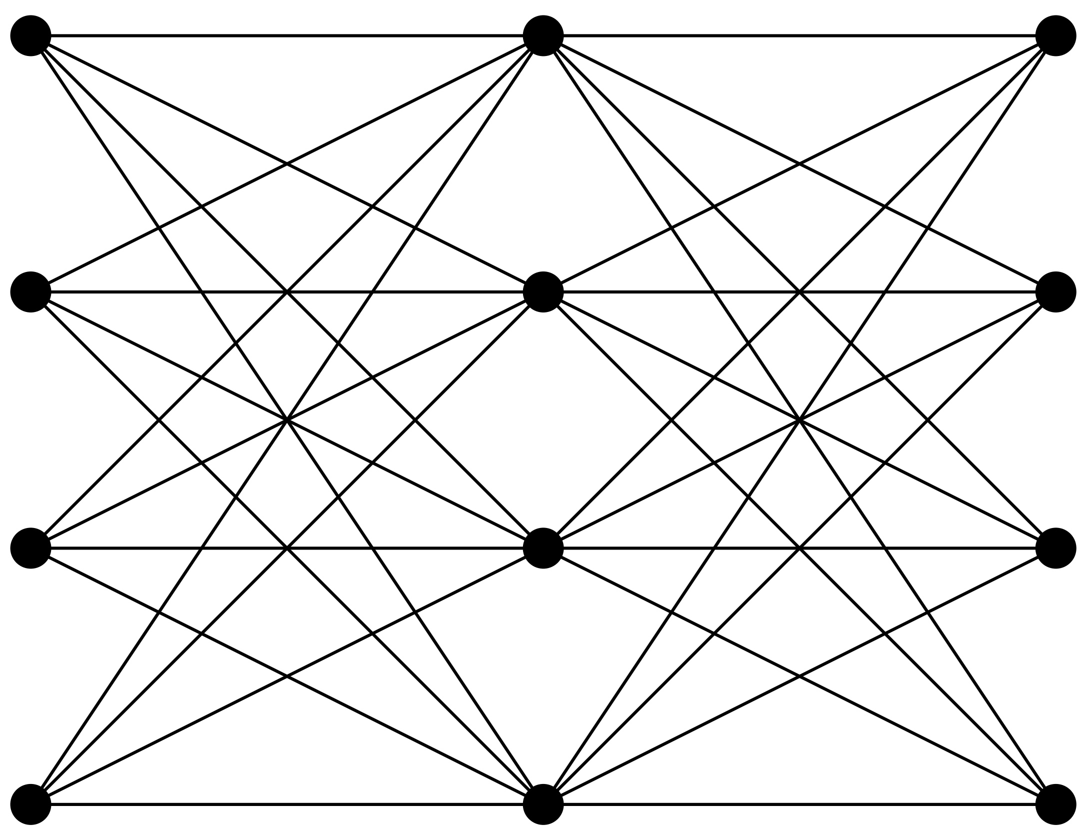
$G = (V, E)$ is a finite graph

Let G_n be the "blow-up graph" of G

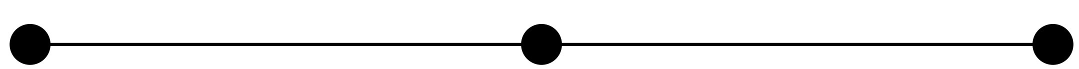
G_n has vertices $V \times \{1, 2, \ldots, n\}$

G_n has edges $(u, i) \sim (v, j)$ whenever $u \sim v$.

G_4



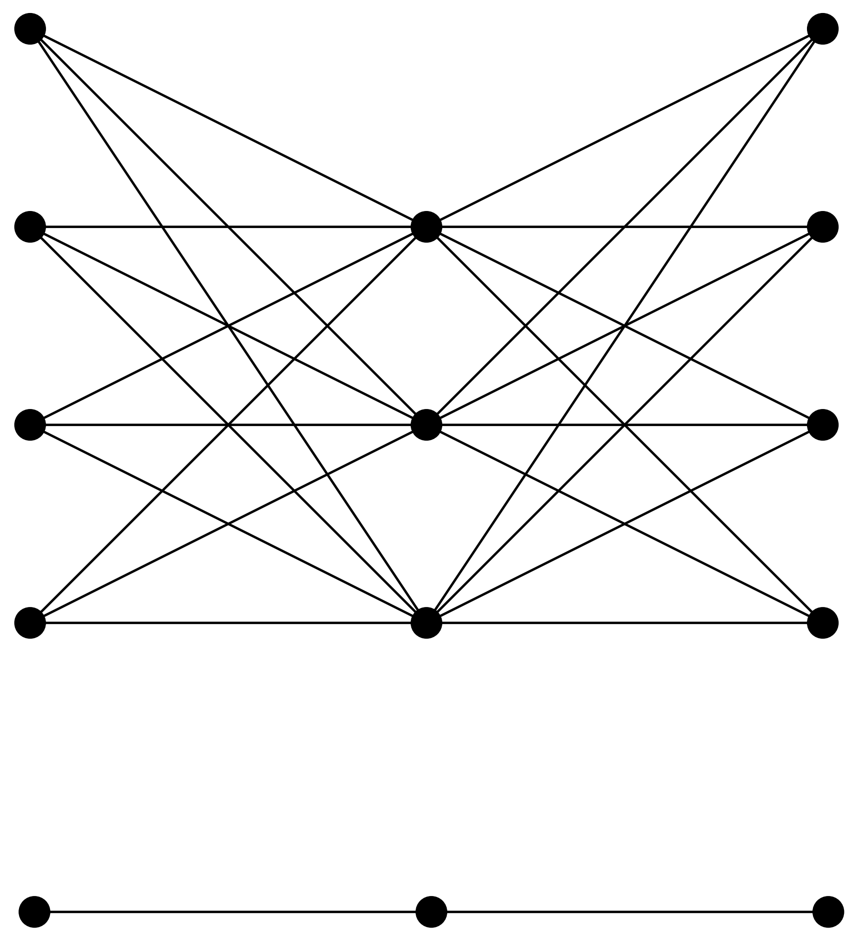
G



We can also let n vary from vertex to vertex: $\mathbf{n} = (n_1, \ldots, n_V)$.

$G_{4,3,4}$

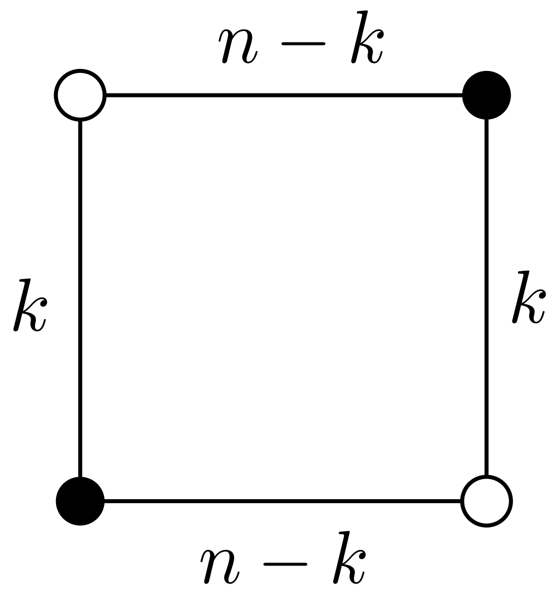
G



Let $Z(\mathbf{n})$ be the number of dimer covers of $G_{\mathbf{n}}$.

("$\mathbf{n}$-multinomial dimer covers of G.")

Example.



$$Z(n,n,n,n) = \sum_{k=0}^{n} \binom{n}{k}^4 k!^2 (n-k)!^2 = \sum_{k=0}^{n} \frac{n!^4}{k!^2 (n-k)!^2}$$

$$= n!^2 \sum_{k=0}^{n} \binom{n}{k}^2$$

$$= (2n)!$$

Let x_v a variable for each vertex v of G.

Let $P(\mathbf{x}) = \sum_{uv \in E} x_u x_v$ be the "edge polynomial".

Thm [K'-Pohoata 2021]:

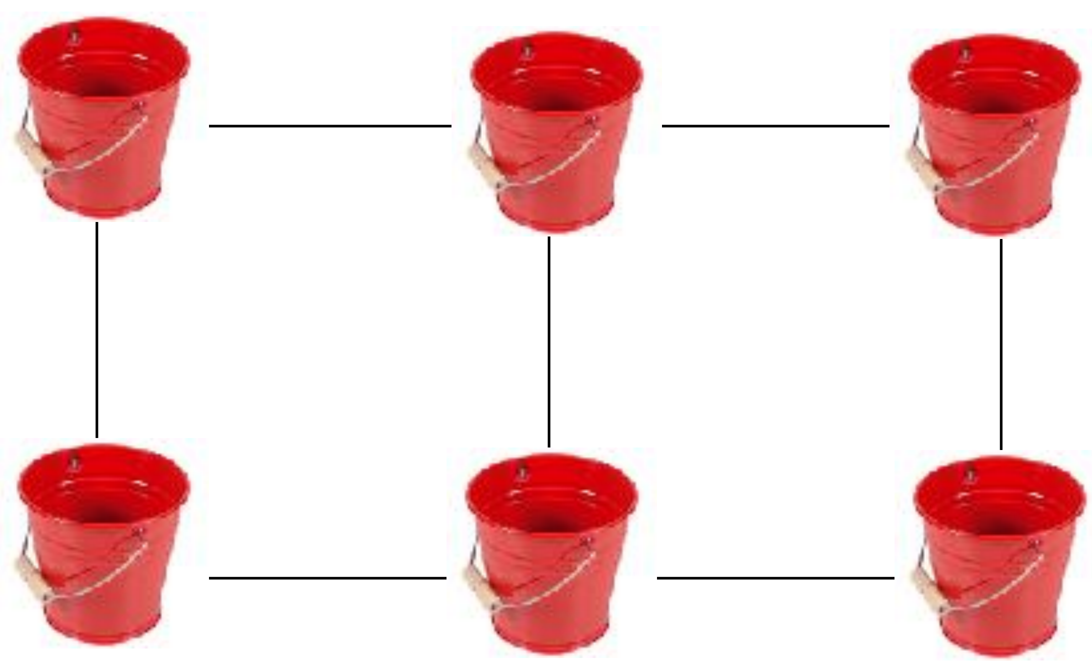
$$Z := \sum_{\mathbf{n} \geq 0} Z(\mathbf{n}) \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} = e^P.$$

$$\frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} := \prod_v \frac{x_v^{n_v}}{n_v!}$$

Note: If use K dimers: $\frac{P^K}{K!}$.

Probabilistic interpretation

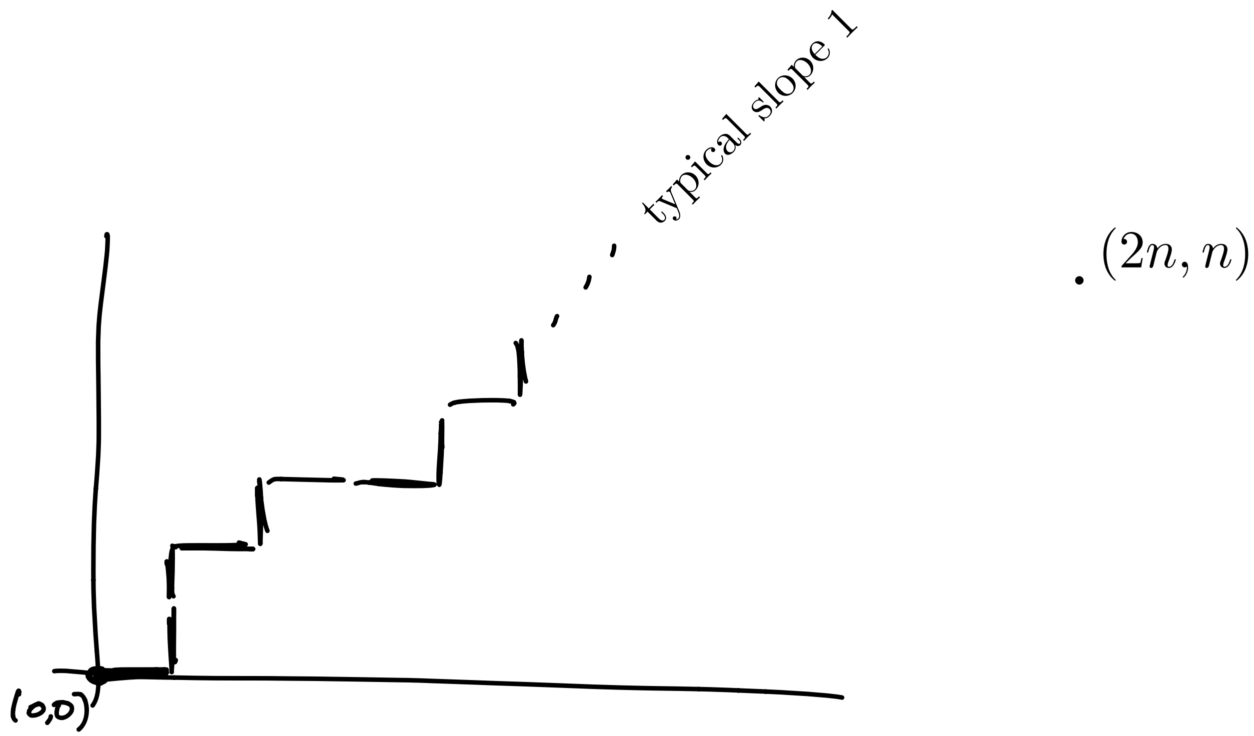
Think of P as a (scaled) probability generating function.



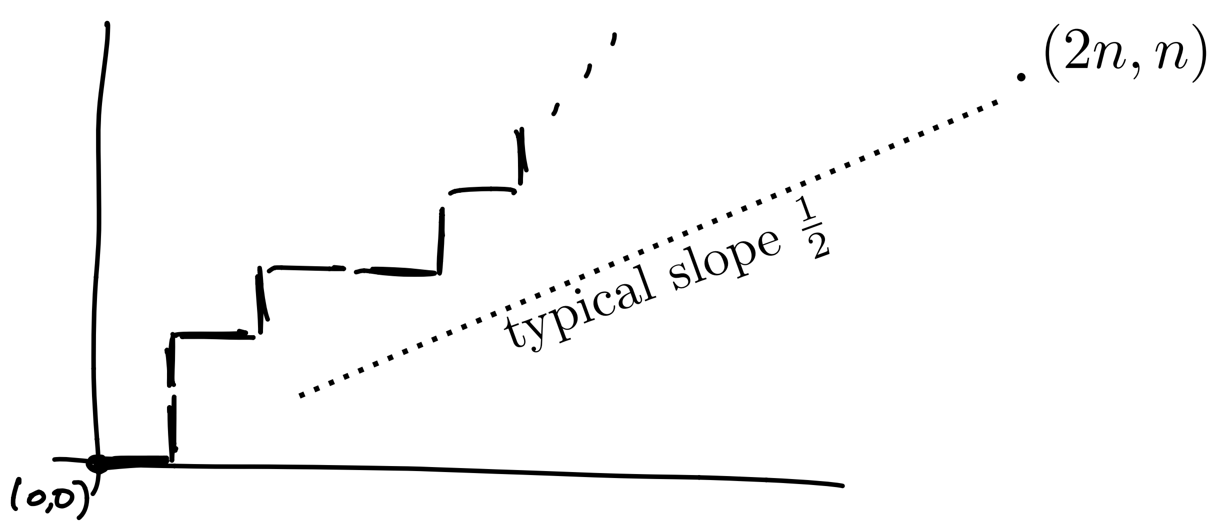
Select an edge at random; drop a ball into its two buckets. Repeat K times.
Condition on the event that all buckets filled after $K = 3n$ steps.

Problem: central buckets fill up faster...

Analogous problem: find coefficient of $x^{2n}y^n$ in $(x+y)^{3n}$.

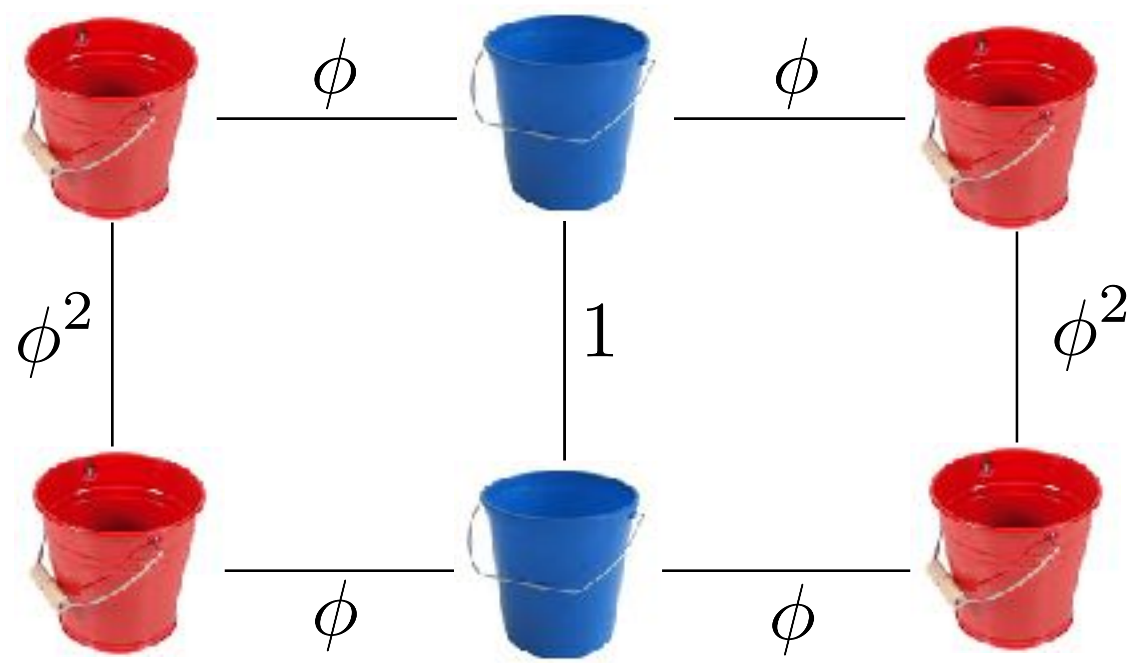


Bias by changing x to $\frac{2}{3}x$ and y to $\frac{1}{3}y$.



This bias does NOT change the resulting conditional distribution

Gauge change: change edge probabilities to $pr(uv) = f(u)f(v)$, so that all buckets fill *at the same rate*, but the conditional distribution is unchanged.



$= \phi$

$$\phi^2 + \phi = 2\phi + 1$$

$$\phi = \frac{\sqrt{5}+1}{2}$$

Asymptotics of $Z(\mathbf{n})$

Let K = number of dimers $= \frac{1}{2}\sum n_v$.

Suppose $\mathbf{n} \to \infty$ with $\frac{n_v}{K} \to \alpha_v$.

(So α_v is the fraction of dimers covering v.)

Thm[KP]: We have $Z(\mathbf{n}) = K! e^{cK+o(K)}$ where

$$c = \log P(\mathbf{x}) - \sum_v \alpha_v \log(x_v/\alpha_v)$$

and where the x_v are the (essentially) unique positive solution to

$$\frac{x_v P_{x_v}}{P} = \alpha_v.$$

we call $\{x_v\}$ the *critical gauge.*

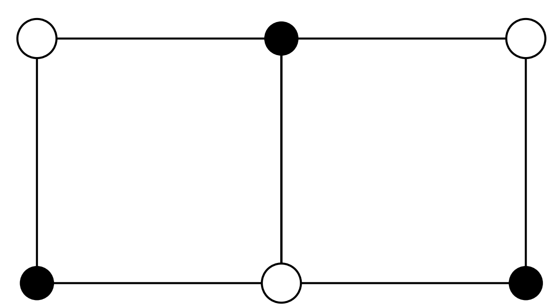
$$\forall v, \qquad \sum_{u\sim v} x_u x_v = \alpha_v P.$$

the critical gauge equation is homogeneous.

If $\mathbf{n} \equiv n$, we can take $\alpha_v P \equiv 1$, so that the critical gauge is one where the sum of edge weights around each vertex is 1.

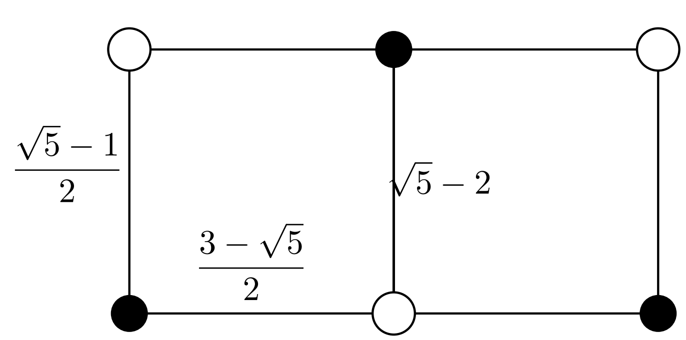
Then "dimer probabilities" (edge fractions) are $x_u x_v$.

Example.

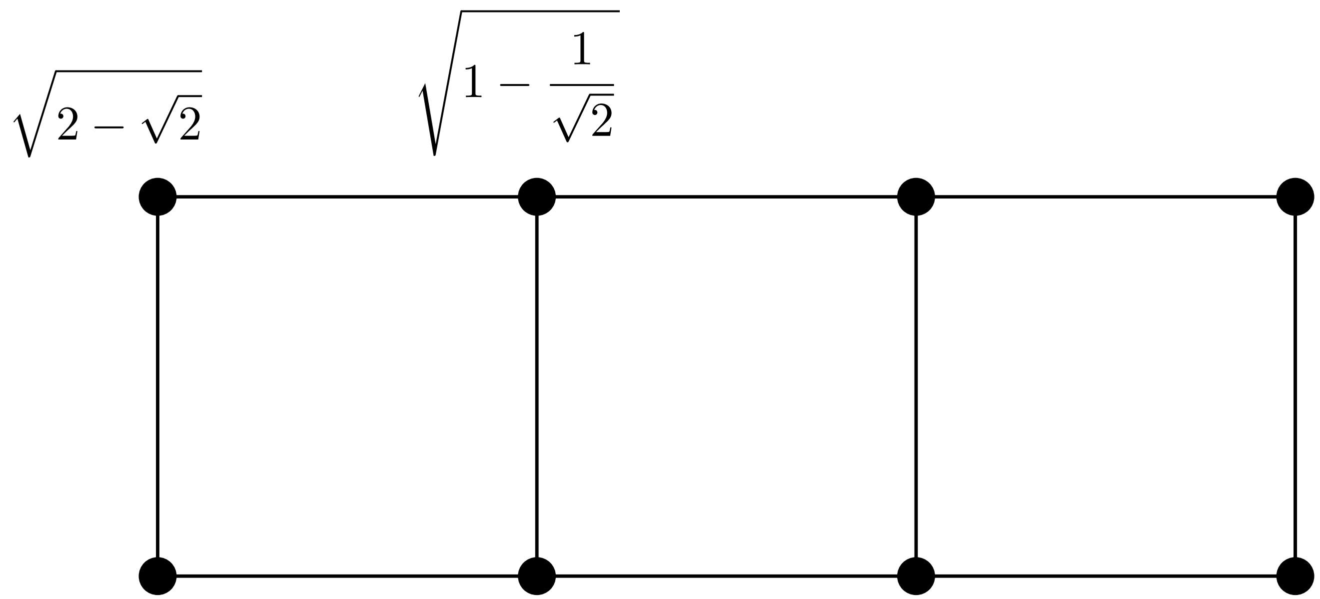


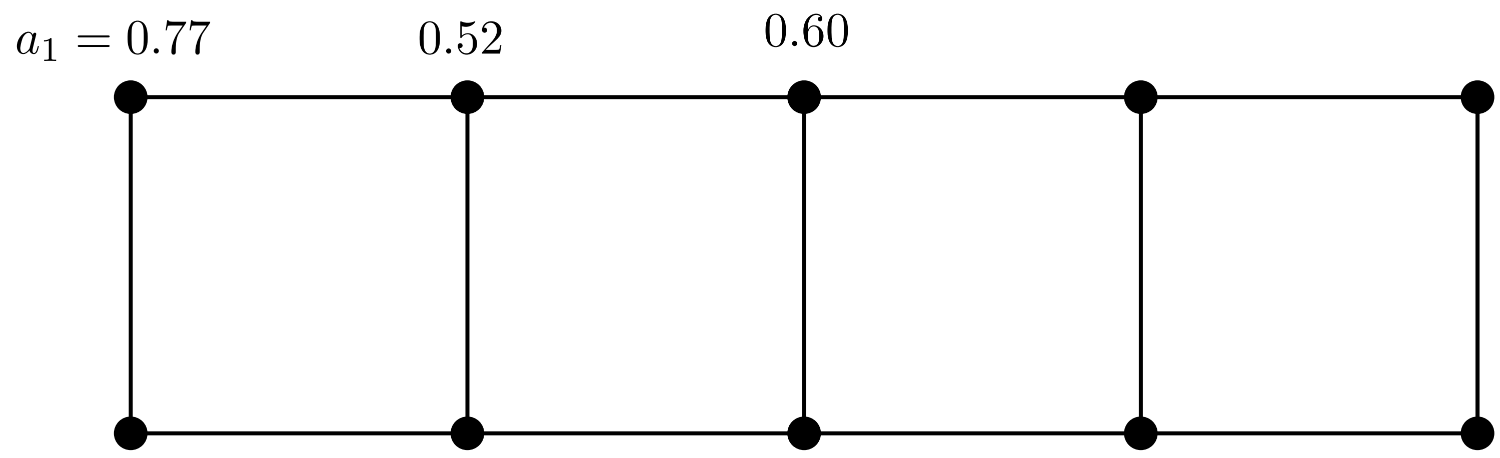
make the (W to B) adjacency matrix bistochastic!

$$\begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_4 & 0 & 0 \\ 0 & x_5 & 0 \\ 0 & 0 & x_6 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{5}-1}{2} & \frac{3-\sqrt{5}}{2} & 0 \\ \frac{3-\sqrt{5}}{2} & \sqrt{5}-2 & \frac{3-\sqrt{5}}{2} \\ 0 & \frac{3-\sqrt{5}}{2} & \frac{\sqrt{5}-1}{2} \end{pmatrix}$$



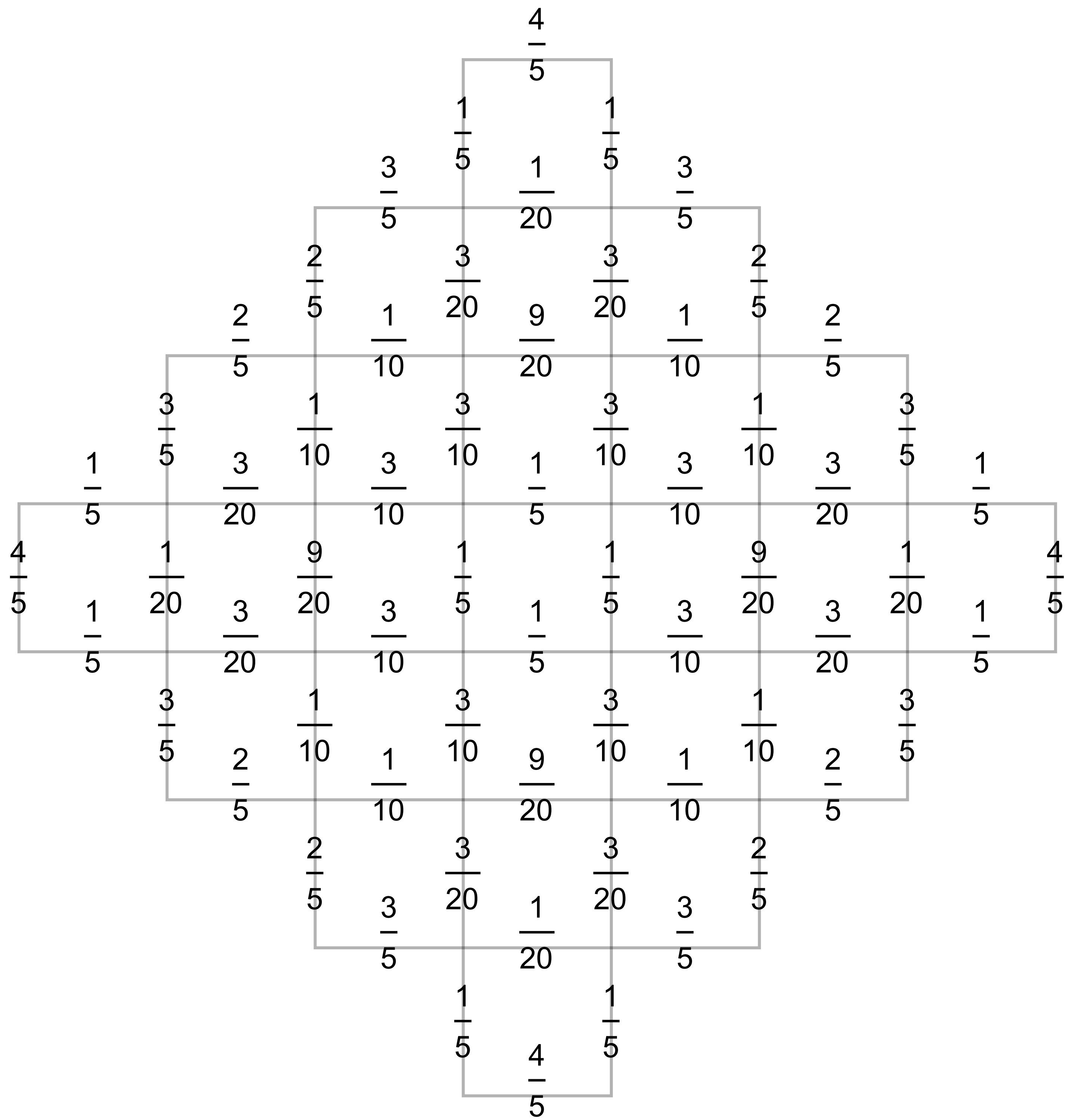
critical gauges





$$3a_1^6 - 4a_1^4 + 3a_1^2 - 1 = 0$$

critical gauge for Aztec diamond



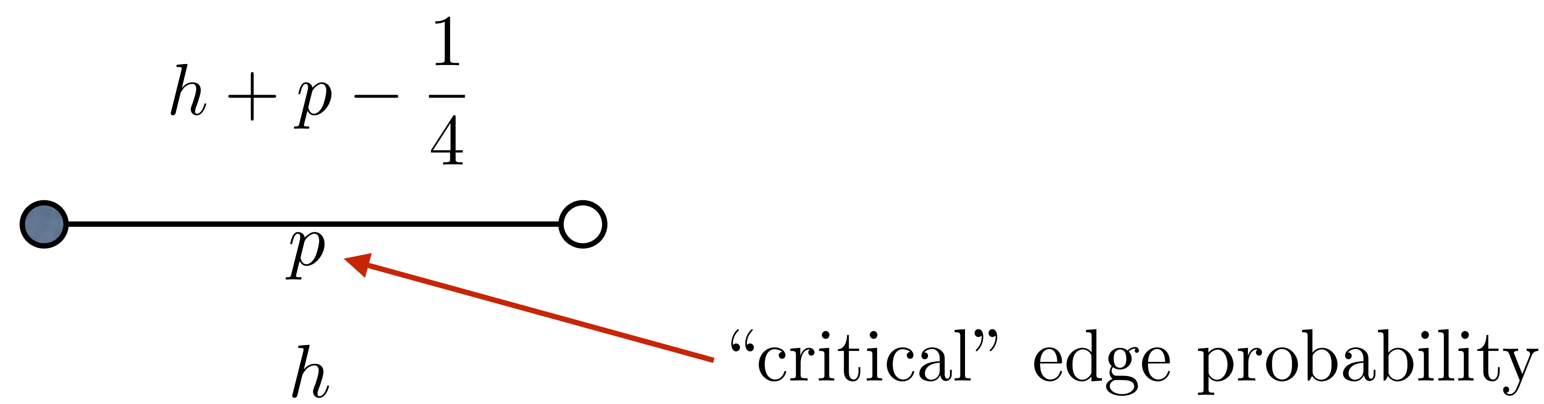
Why is the critical gauge for the AD nice?

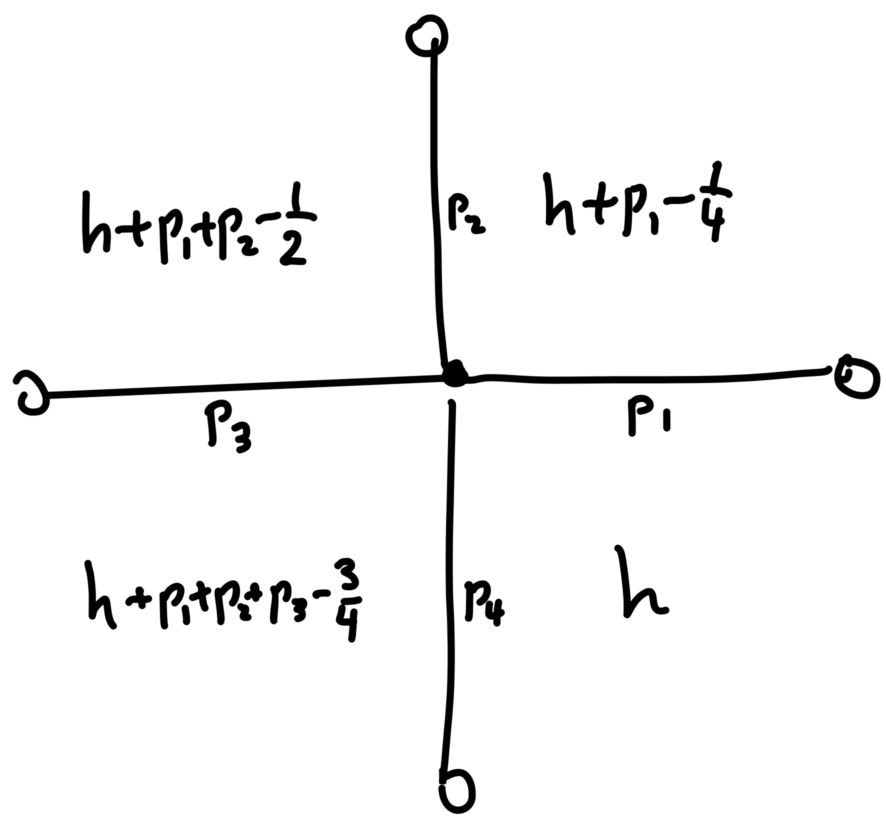
$$\frac{\binom{n-1}{j}}{\binom{n}{i+1}}\left(\frac{\binom{n-1}{i}}{\binom{n}{j}}+\frac{\binom{n-1}{i}}{\binom{n}{j+1}}+\frac{\binom{n-1}{i+1}}{\binom{n}{j}}+\frac{\binom{n-1}{i+1}}{\binom{n}{j+1}}\right)=\frac{n}{n+1}$$

$$\frac{\binom{n-1}{i}}{\binom{n}{j}}\left(\frac{\binom{n-1}{j-1}}{\binom{n}{i}}+\frac{\binom{n-1}{j-1}}{\binom{n}{i+1}}+\frac{\binom{n-1}{j}}{\binom{n}{i}}+\frac{\binom{n-1}{j}}{\binom{n}{i+1}}\right)=\frac{n+1}{n}$$

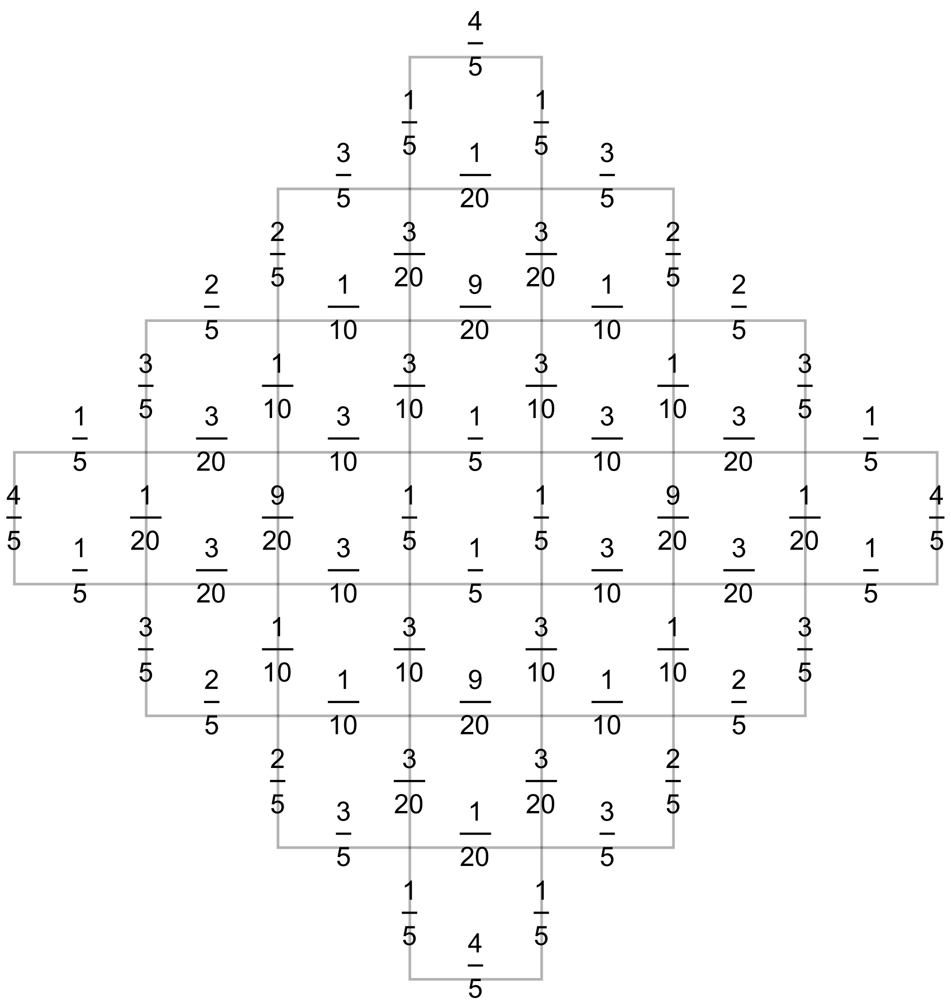
Open Question: "Explain" these identities.

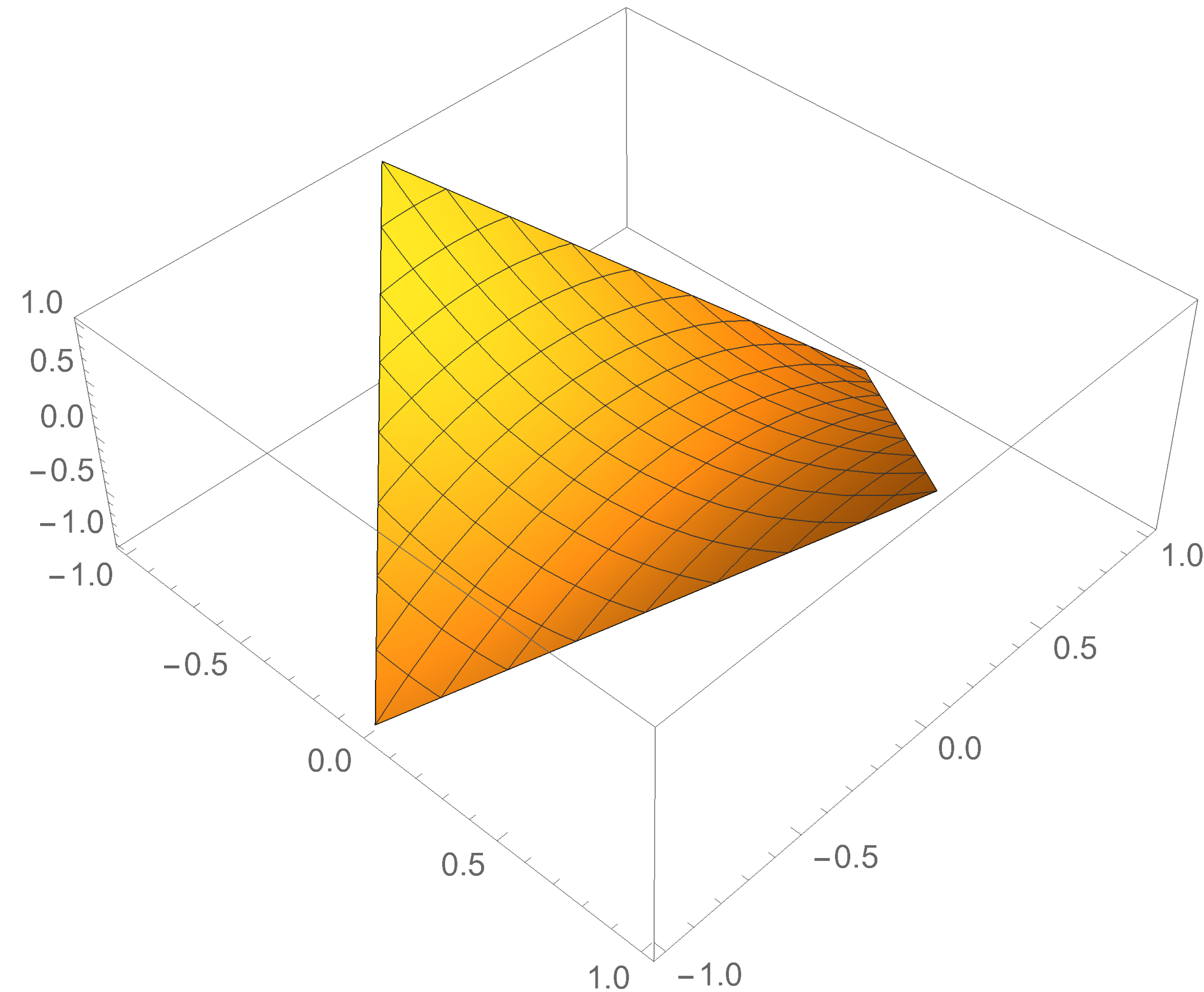
Defining the average height function for the square grid dimer model





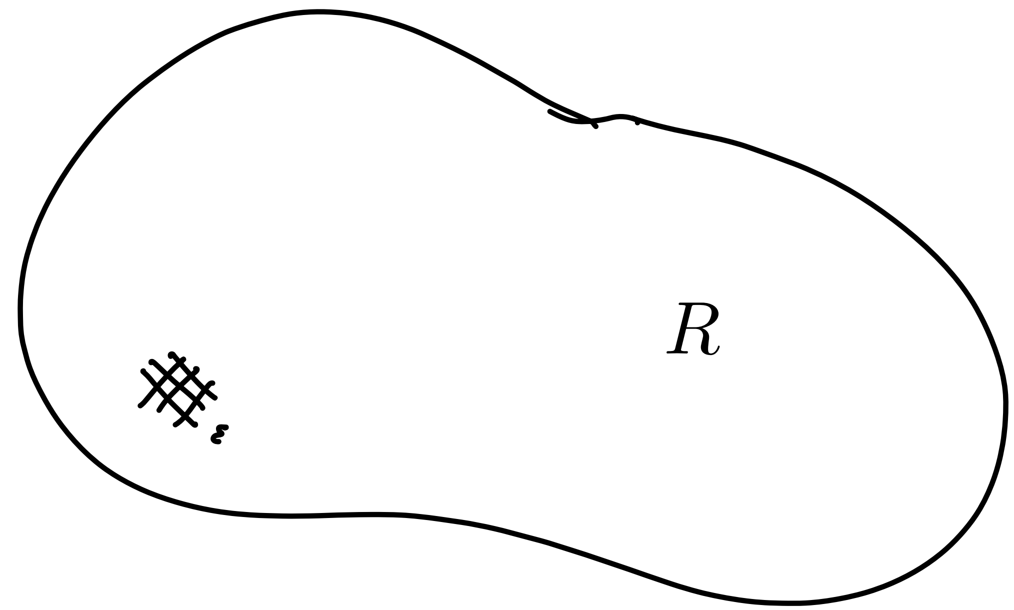
For the critical gauge as above, the tile fractions (edge probabilities) are $x_u x_v$.





The scaling limit height function for the aztec diamond is $h(x, y) = x^2 - y^2$.

Variational principle:



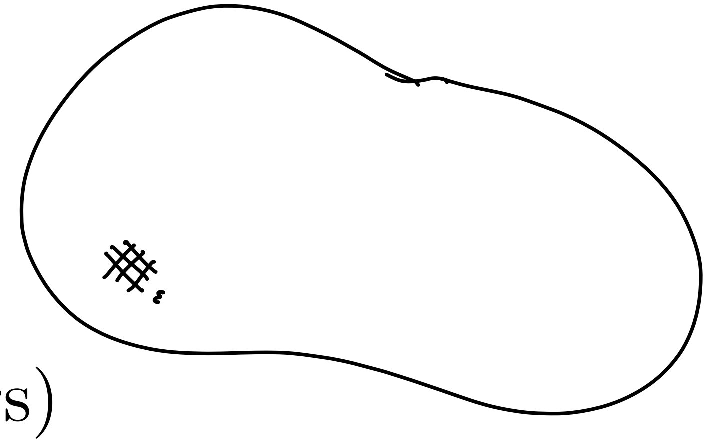
A region R in $\mathbb{R}^2$ and approximating graph $R_\epsilon \subset \mathbb{Z}^2$.

In limit $\epsilon \to 0$, what is the critical gauge?

What is the limiting height function?

(Note: boundary height is determined by choice of boundary conditions for R_ϵ)

Variational principle:



(Analog of [Cohn-K-Propp 2003] for multinomial dimers)

Thm [K-Wolfram]: For multinomial dimers on the scaling limit of (rotated) $\mathbb{Z}^2$, on a domain R with boundary height function $u : \partial R \to \mathbb{R}$, the limit height function h is the unique function with $h|_{\partial R} = u$ maximizing

$$\mathrm{Ent}(h) = \iint_R \sigma(\nabla h)\,dx\,dy$$

where

$$\sigma(s,t) = -\frac{1-s}{2}\log\frac{1-s}{2} - \frac{1+s}{2}\log\frac{1+s}{2} - \frac{1-t}{2}\log\frac{1-t}{2} - \frac{1+t}{2}\log\frac{1+t}{2}.$$

and $(s,t) \in [-1,1]^2$.

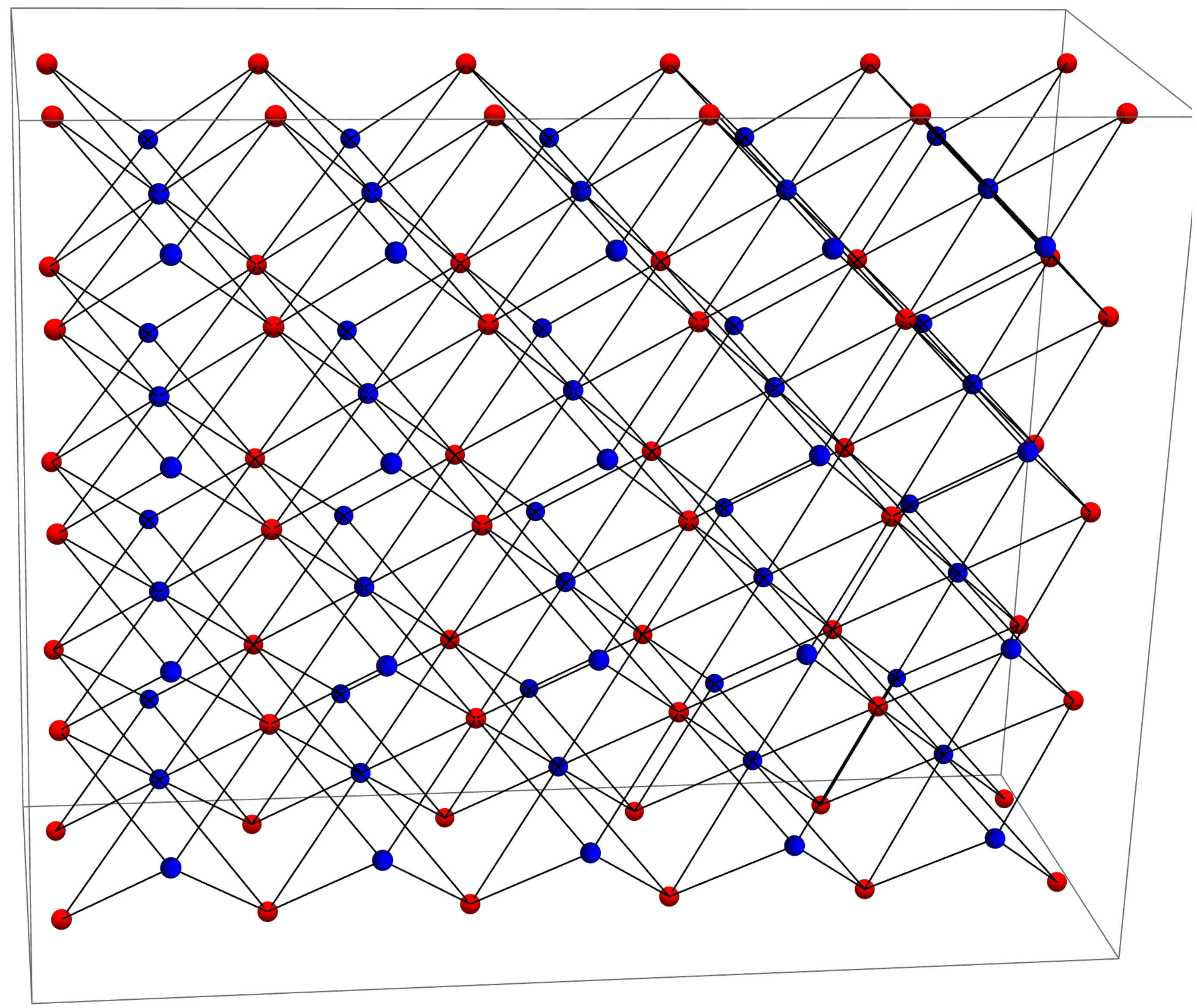
The Euler-Lagrange equation for the limiting height function h is

$$\frac{h_{xx}}{1-h_x^2} + \frac{h_{yy}}{1-h_y^2} = 0.$$

General solutions can be written in terms of ${}_2F_1$'s.

The Aztec brick

"3D Aztec diamond" on BCC lattice in $\mathbb{Z}^3$



Reds: $a \times b \times c$ box

Blues: $(a+1) \times (b-1) \times (c-1)$ box

balance condition $\quad abc = (a+1)(b-1)(c-1)$

The critical gauge is given by

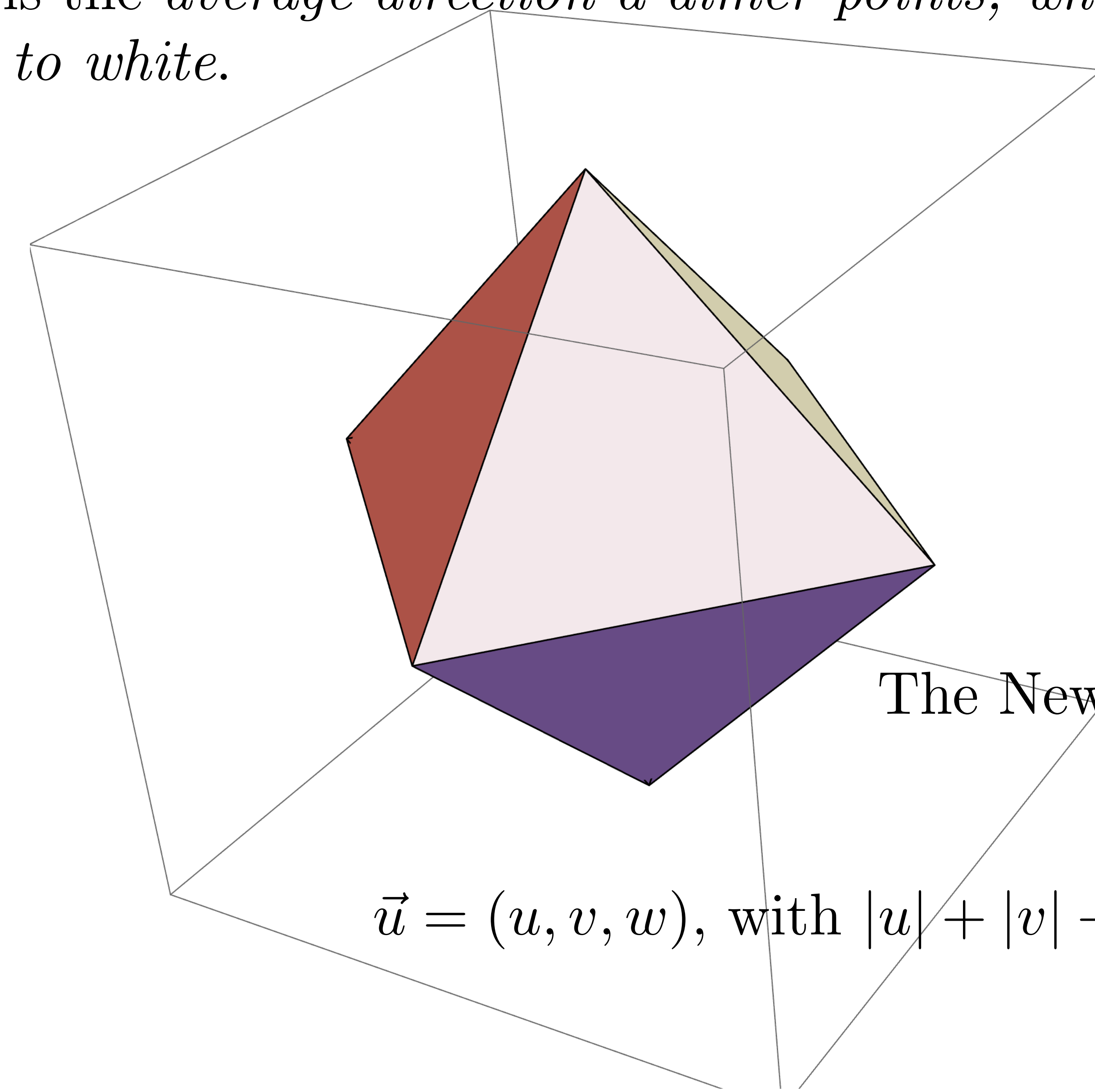
$$x(i, j, k) = \frac{\binom{a}{i}}{\binom{b}{j}\binom{c}{k}} \qquad \begin{array}{c} 0 \le i \le a \\ 0 \le j \le b \\ 0 \le k \le c \end{array}$$

at red vertices and

$$x(i', j', k') = \frac{\binom{b-1}{j}\binom{c-1}{k}}{\binom{a+1}{i+1}} \frac{bc}{(b+1)(c+1)} \qquad \begin{array}{c} -1 \le i \le a \\ 0 \le j \le b-1 \\ 0 \le k \le c-1 \end{array}$$

at blue vertices.

In $3d$, there is no "height function". We use the (divergence free) vector field $\vec{u}$ instead: $\vec{u}$ is the *average direction a dimer points, when all dimers are oriented from black to white.*



The Newton polytope of directions

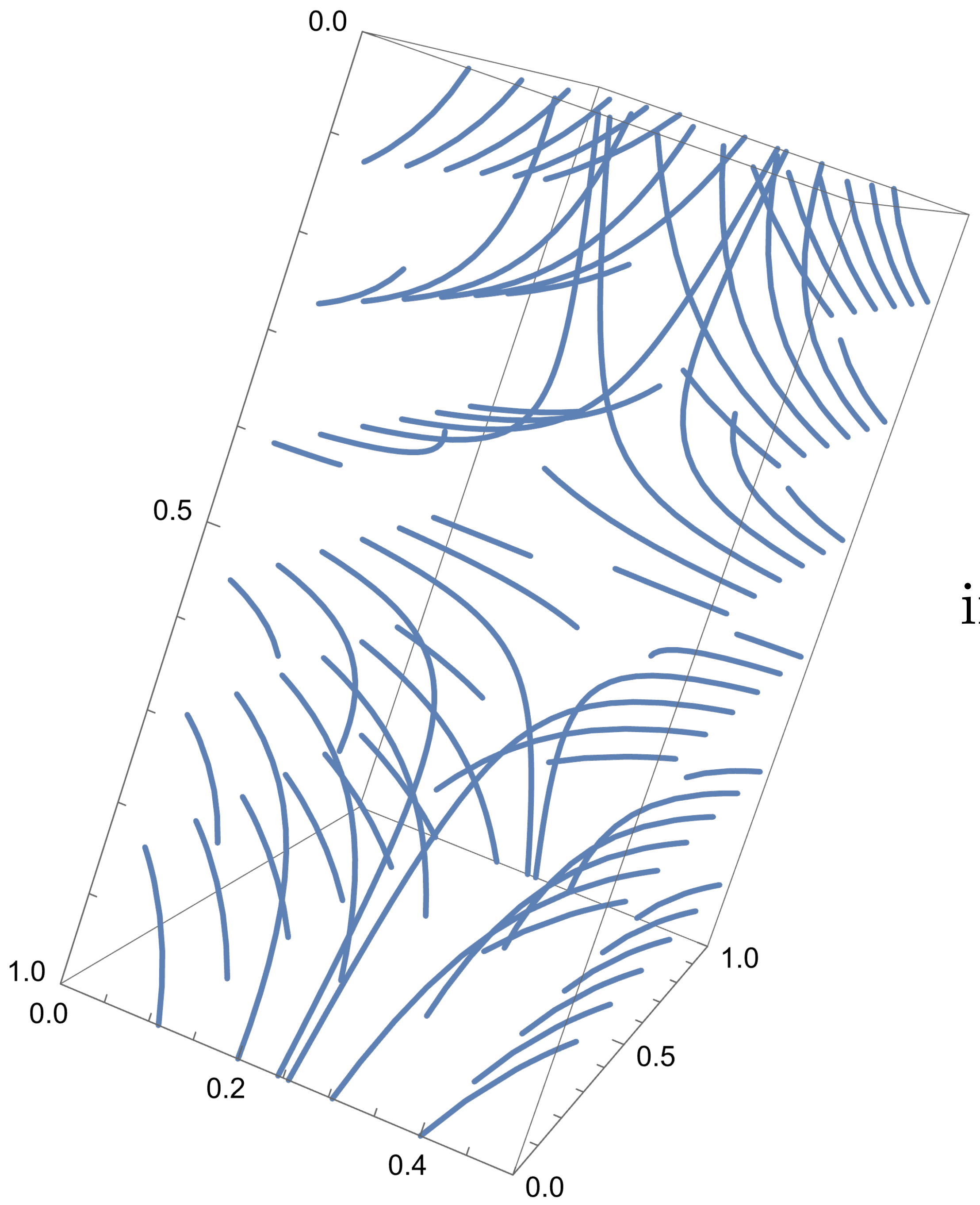
$$\vec{u} = (u, v, w), \text{ with } |u| + |v| + |w| \leq 1$$

Thm[K-Wolfram 24+] There is an analogous variational principle in $3d$.

(uses [**Chandgotia, Sheffield, Wolfram** arXiv:2304.08468].)

The limit vector field in $[0,\alpha]\times[0,\beta]\times[0,\gamma]$ is

$$\left(\frac{2x}{\alpha}-1, 1-\frac{2y}{\beta}, 1-\frac{2z}{\gamma}\right)$$



integral curves of the vector field

The surface tension $\sigma(u, v, w)$ is

$$\sigma(u, v, w) = S(\frac{1-u}{2}) + S(\frac{1-v}{2}) + S(\frac{1-w}{2})$$

$$S(p) = -p \log p - (1-p) \log(1-p)$$

The Euler-Lagrange equation for the divergence-free vector field $\vec{u} = (u, v, w)$ is

$$\begin{aligned} \frac{u_y}{1-u^2} &= \frac{v_x}{1-v^2} \\ \frac{v_z}{1-v^2} &= \frac{w_y}{1-w^2} \\ \frac{w_x}{1-w^2} &= \frac{u_z}{1-u^2}. \end{aligned}$$

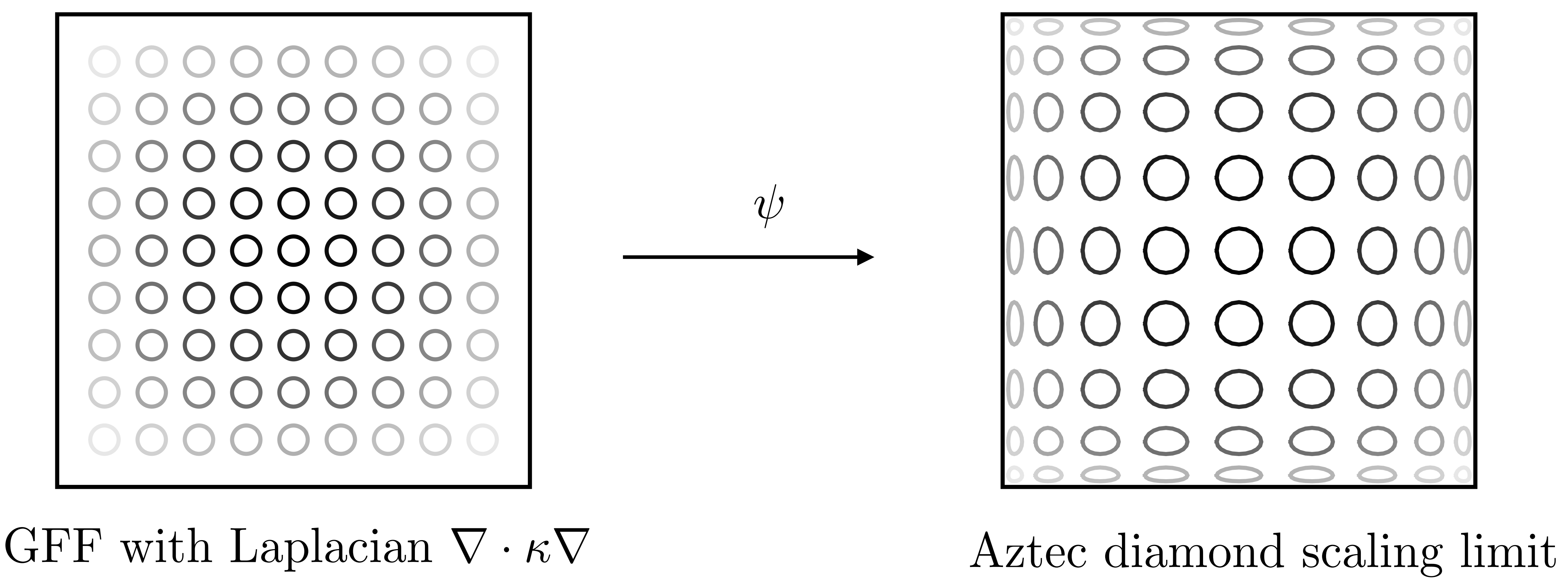
Fluctuations

Fluctuations for multinomial dimers (and multinomial tilings in general) are Gaussian.

Thm: (2D Aztec diamond) In the scaling limit, height fluctuations are given by the image of an *inhomogeneous Gaussian Free Field* on $[0,\pi]^2$ (with conductance κ) under a diffeomorphism $\Psi : [0,\pi]^2 \to R$:

$$\psi(u,v) = (\cos u, \cos v),$$

and $\kappa : [0,\pi]^2 \to \mathbb{R}$ is given by $\kappa(u,v) = \frac{1}{\sin u \sin v}$.



THANK YOU