PERFECT MATCHINGS OF NEARLY SYMMETRIC GRAPHS: SOLUTIONS OF SOME OPEN PROBLEMS OF PROPP, LAI, AND SOME RELATED RESULTS

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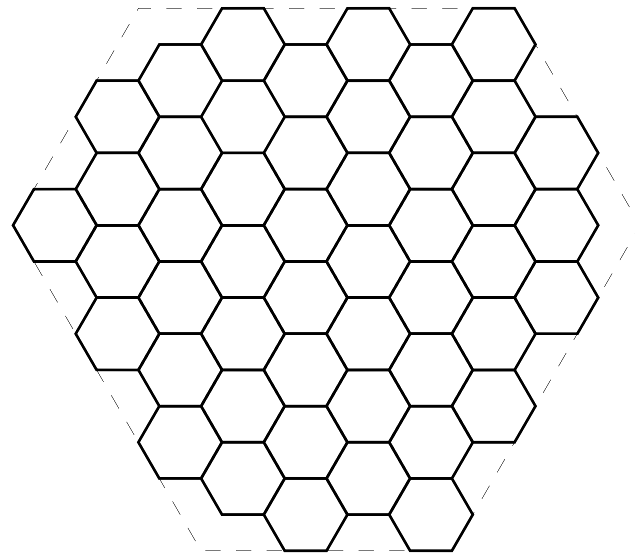
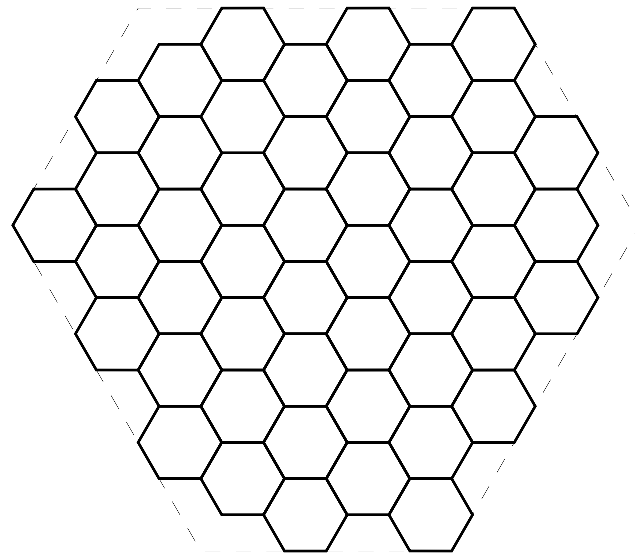


The (a, b)-benzel for $a = 7$, $b = 8$



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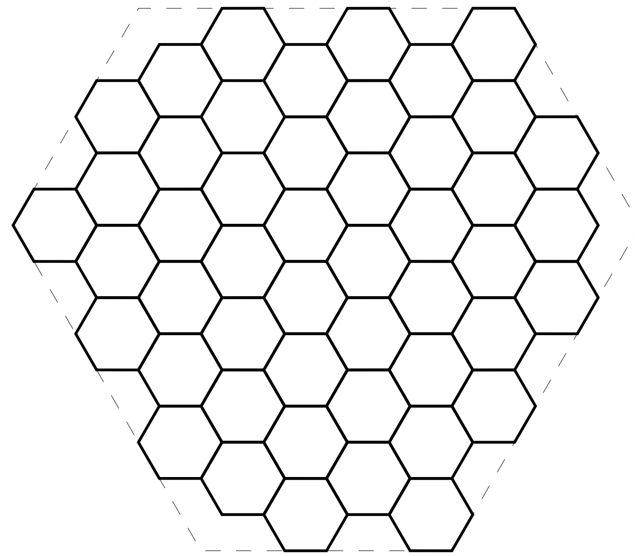

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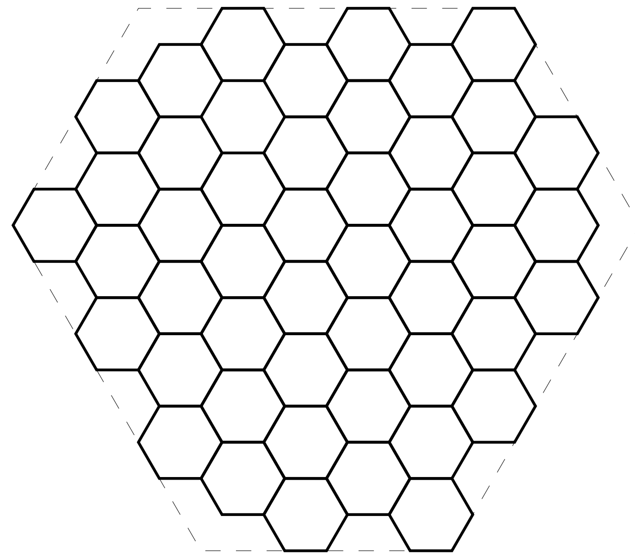

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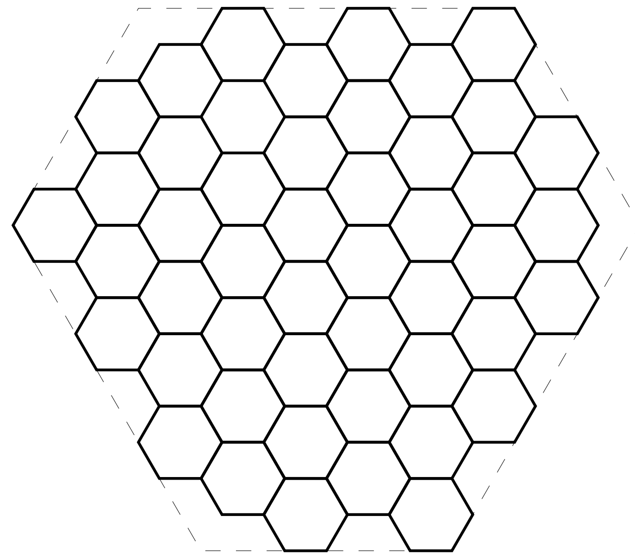

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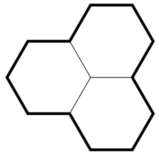
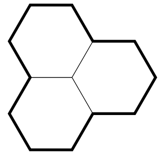
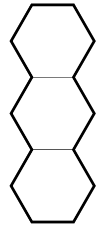
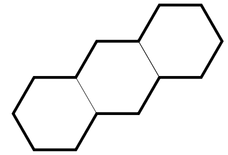
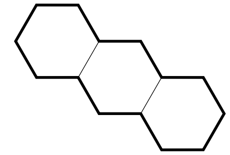
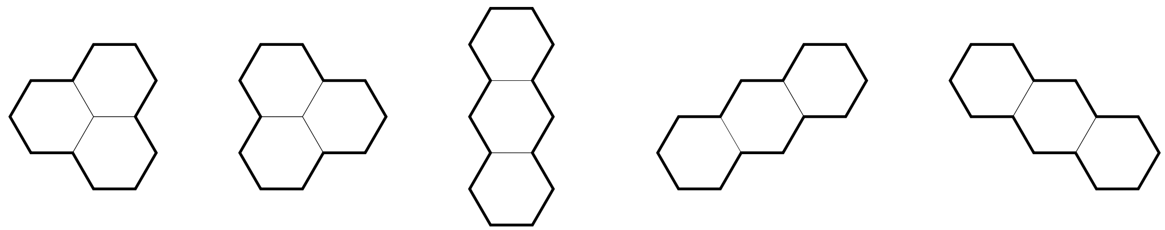




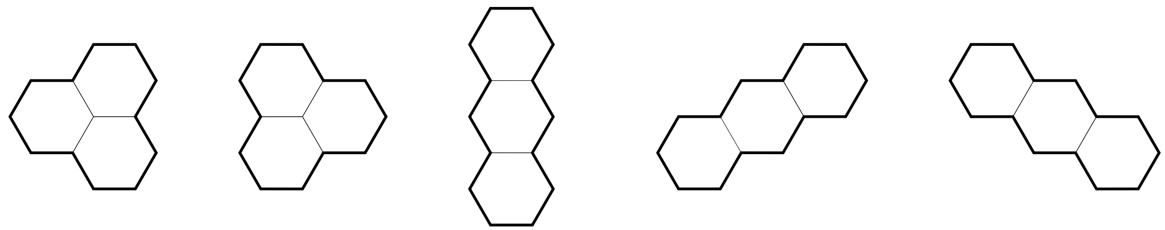








Left stone, right stone, vertical bone, rising bone, falling bone



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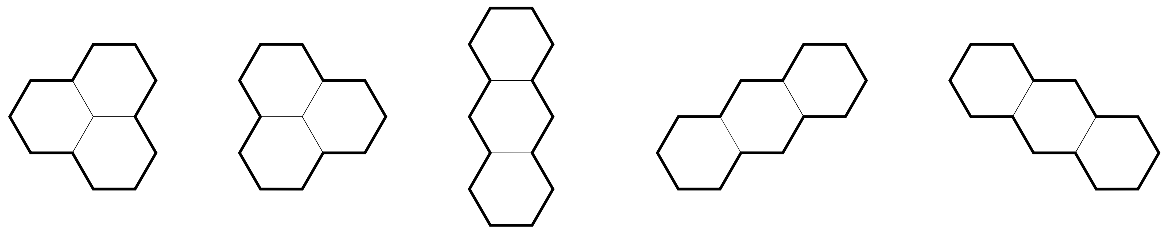
- *Stones and bones cover of a benzel:* A covering of the benzel by these shapes, with no gaps or overlaps



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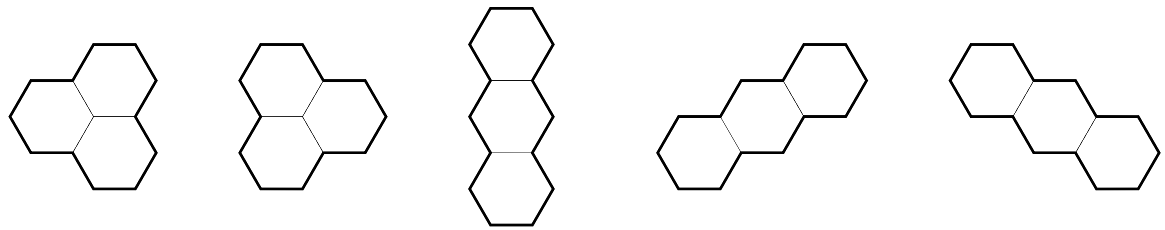
- Propp published in 2021 a list of 20 open problems on such covers of benzels.



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- Propp published in 2021 a list of 20 open problems on such covers of benzels.

- Problems 8–11 were still open; we show how to solve them in this talk

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Problem 8. If $a_n = \#$ vertical-bone-free covers of $(3n, 3n)$-benzel, then

$$\frac{a_n a_{n+2}}{a_{n+1}^2} = \frac{256(2n+3)^2(4n+1)(4n+3)^2(4n+5)}{27(3n+1)(3n+2)^2(3n+4)^2(3n+5)}, \quad n \geq 1.$$

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$$\frac{c_n}{c_{n-1}} = 2^{2n-3} \frac{(4n-2)!\,(4n-4)!\,(n-1)!!\,(n-3)!!}{(3n-1)!!\,(3n-2)!\,(3n-3)!\,(3n-5)!!}, \quad n \geq 2.$$

Problem 11. If $d_n = \#$ vertical-bone-free covers of $(3n+1, 3n+2)$-benzel, then

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Theorem (Byun, C. and Lee, 2024).

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Note: It would require some work to obtain these formulas directly from the recurrences.

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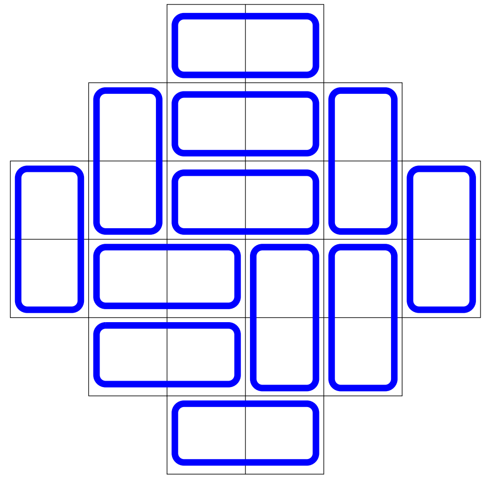
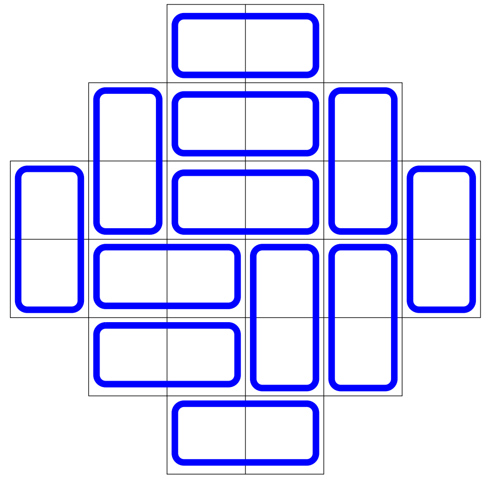
- Combine resulting expressions to obtain stated formulas



A domino tiling of the Aztec diamond AD_3



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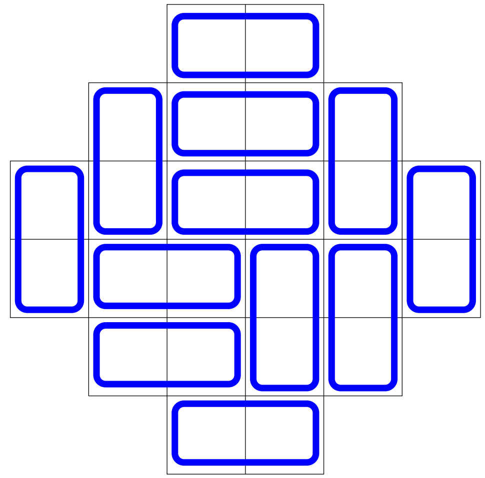
There are 64



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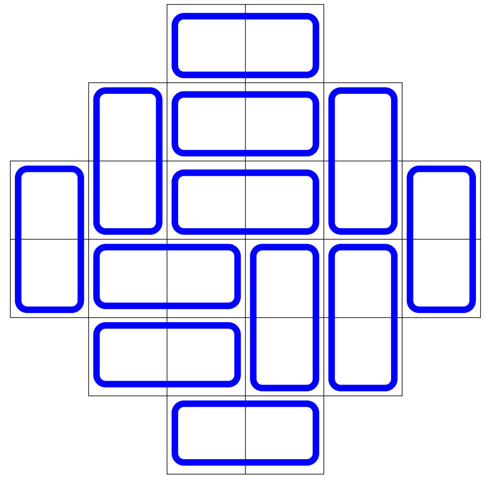
Write $\mathrm{M}(AD_3) = 64$



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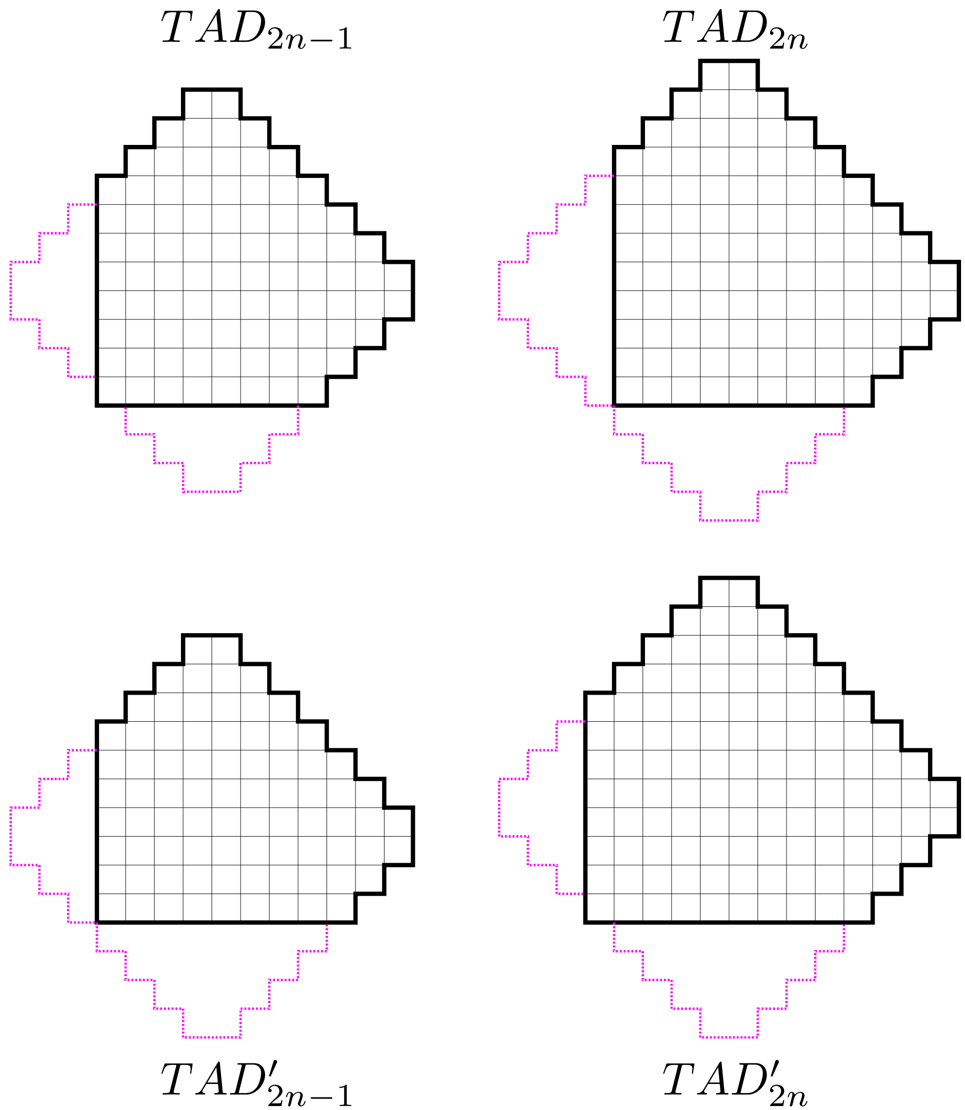
Theorem (Elkies, Kuperberg, Larsen and Propp, 1992).

$$\mathrm{M}(AD_n) = 2^{\binom{n+1}{2}}.$$

Truncated Aztec diamonds



Theorem. (Defant, Foster, Li, Propp and Young, 2024).

$$
\begin{aligned}
a_n &= \mathrm{M}(TAD_{2n-1}) \\
b_n &= \mathrm{M}(TAD_{2n}) \\
c_n &= \mathrm{M}(TAD'_{2n-1}) \\
d_n &= \mathrm{M}(TAD'_{2n})
\end{aligned}
$$

Thus, to prove the stated formulas for Propp's four problems it suffices to prove:

Proposition (Byun, C. and Lee, 2024).

$$\mathrm{M}(TAD_{2n-1}) = 2^{n^2} \cdot \frac{(2n-1)!!}{n!} \cdot \prod_{i=1}^{2n-1} \frac{(2i)!}{(n+i)!},$$

$$\mathrm{M}(TAD_{2n}) = 2^{n(n+1)} \cdot \prod_{i=0}^{n-1} \frac{(4i+2)!(4i+3)!}{(n+2i+1)!(n+2i+2)!},$$

$$\mathrm{M}(TAD'_{2n-1}) = 2^{n^2-1} \cdot \prod_{i=0}^{n-1} \frac{(4i+2)!}{(n+2i+1)!} \cdot \prod_{i=0}^{n-2} \frac{(4i+3)!}{(n+2i+1)!},$$

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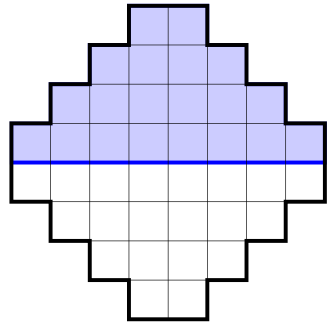
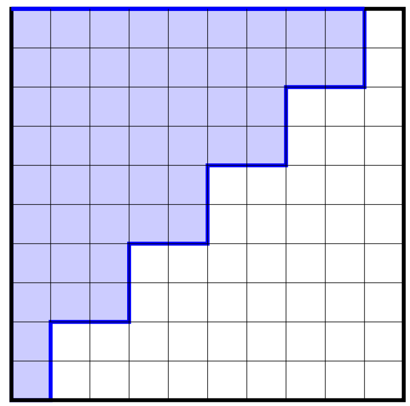
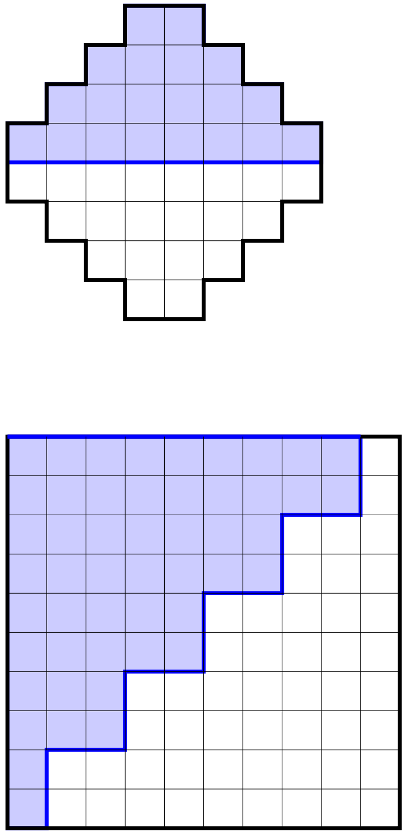
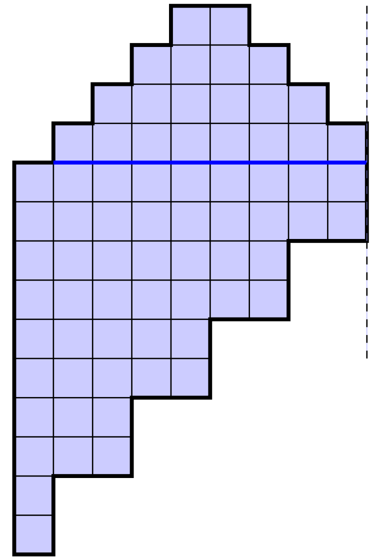


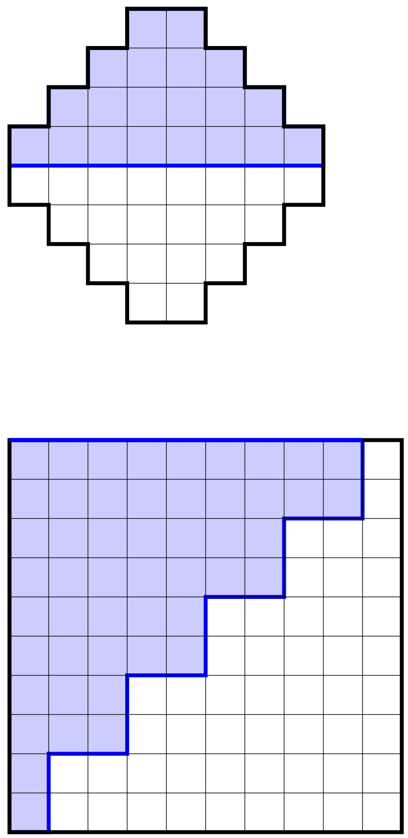




$\frac{1}{2}\ S_{2n} + \frac{1}{2}\ AD_{n-1}$



$\mathcal{T}_n$



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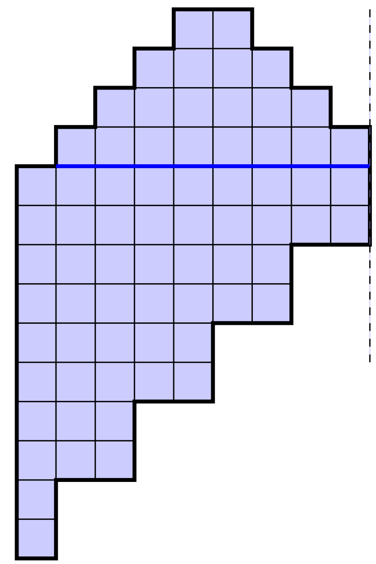
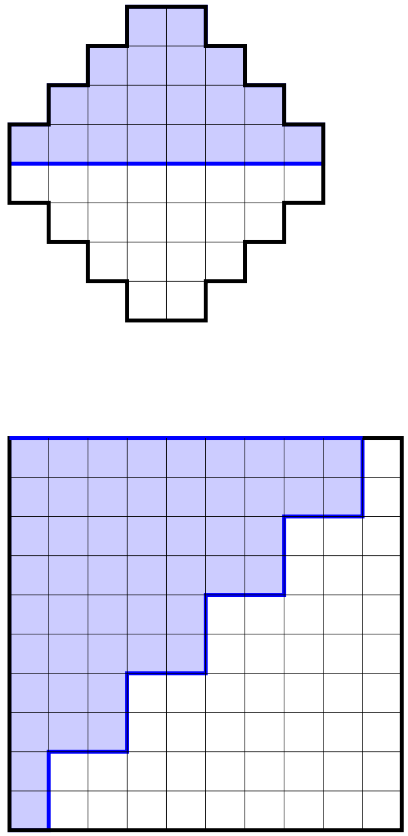


$\mathcal{T}_n$

$\mathcal{T}_n$: Aztec triangle of order n (Di Francesco, 2020).



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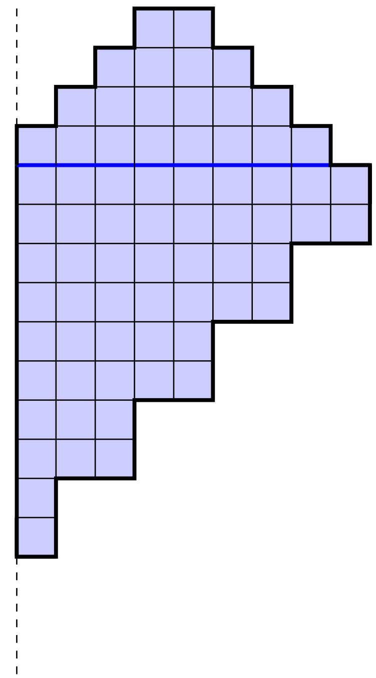
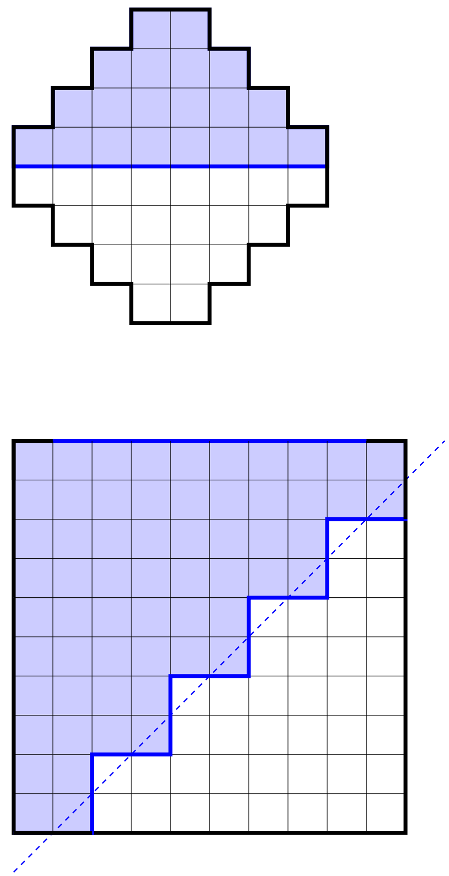
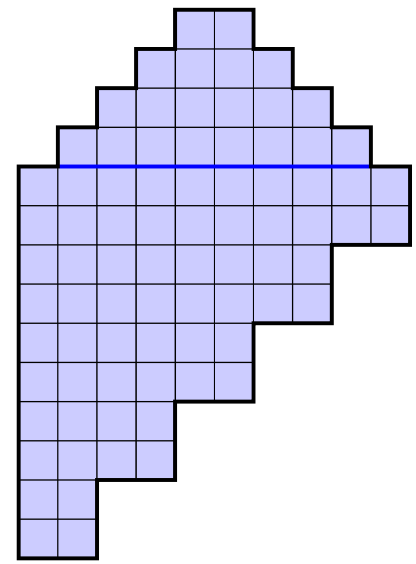


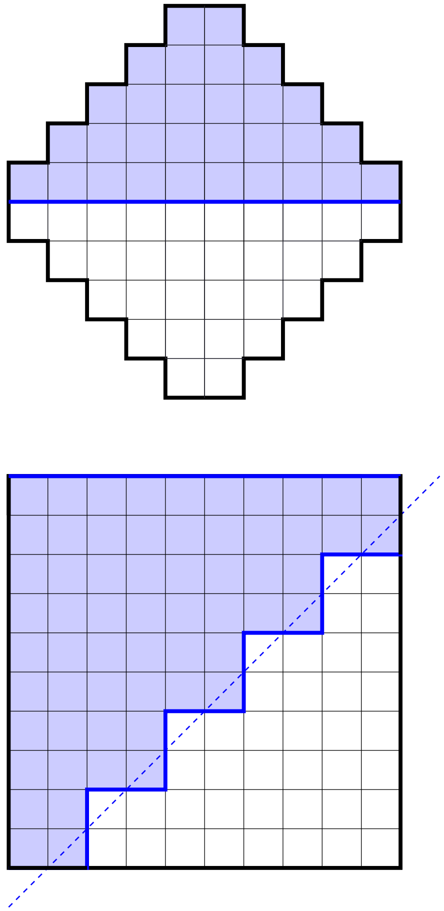
$\mathcal{T}_n'$



$\frac{1}{2}(1+\epsilon)\ S_{2n} + \frac{1}{2}\ AD_{n-1}$



$\mathcal{T}''_n$



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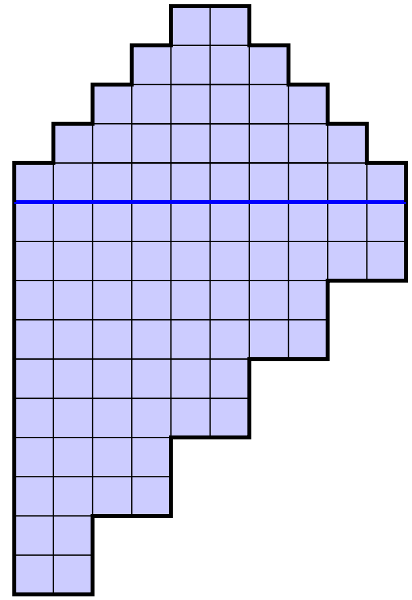

$\mathcal{T}'''_{n+1}$

Lemma. (special case of result of Corteel, Huang and Krattenthaler, 2023)

$$M(\mathcal{T}_n) = M(\mathcal{T}'_n) = 2^{n(n-1)/2} \cdot \prod_{i=0}^{n-1} \frac{(4i+2)!}{(n+2i+1)!}$$

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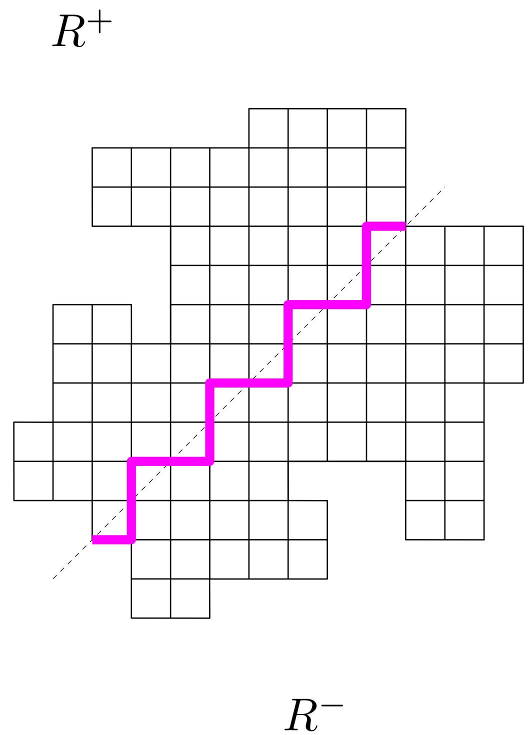
- Top formulas appear in CHK (direct bijection: Byun and C., 2024)
- Takes some work to extract bottom two from general result of CHK

Reducing truncated Aztec diamonds to Aztec triangles

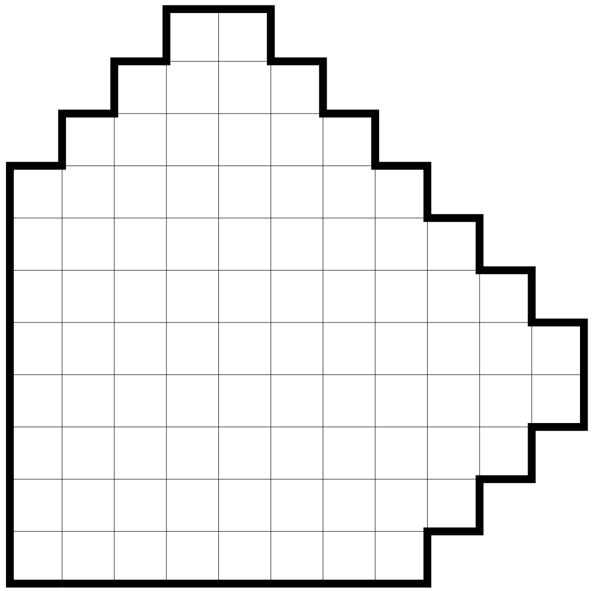
Factorization theorem (C., 1997) implies



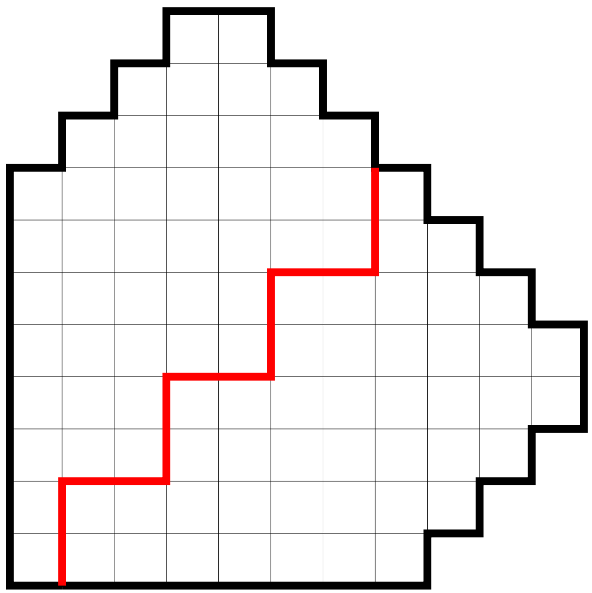
$$\mathrm{M}(R) = 2^{n/2}\,\mathrm{M}(R^+)\,\mathrm{M}(R^-),$$

where n= # (unit squares on symmetry axis).

Proof of formula for M(TAD_{2n-1})



Proof of formula for $\mathrm{M}(TAD_{2n-1})$



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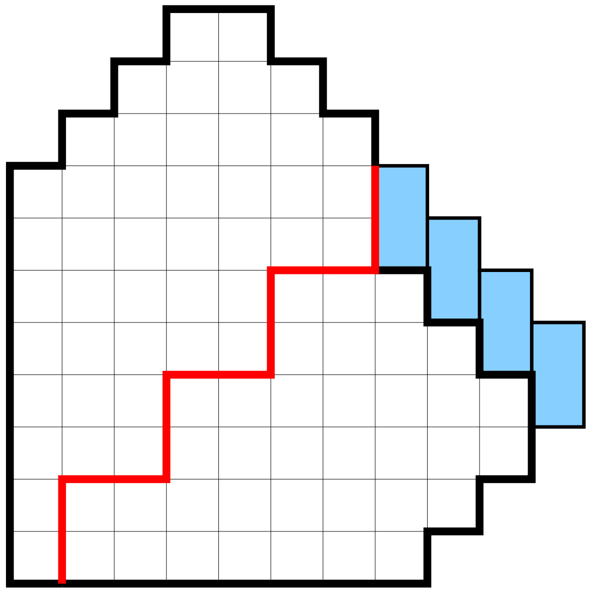

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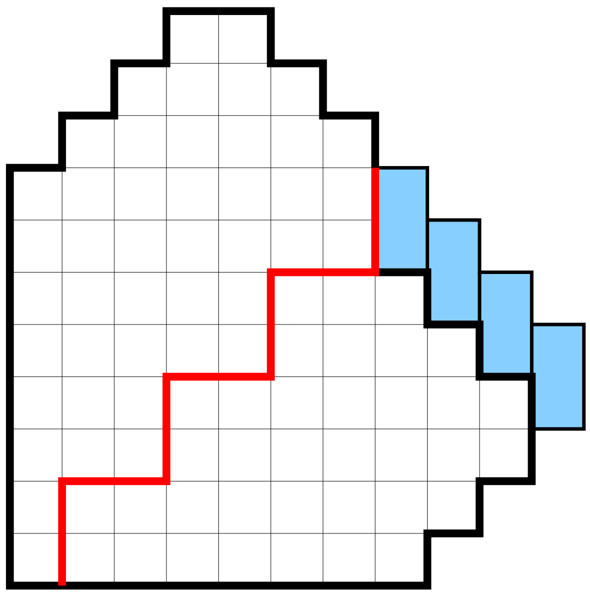
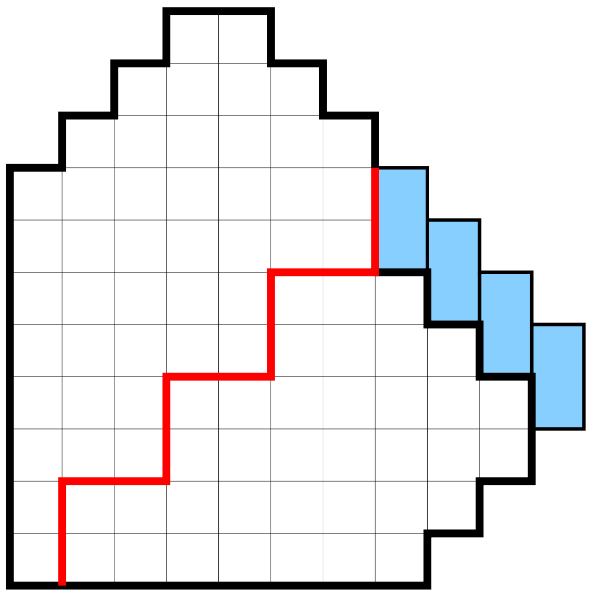


$\mathcal{T}_n$

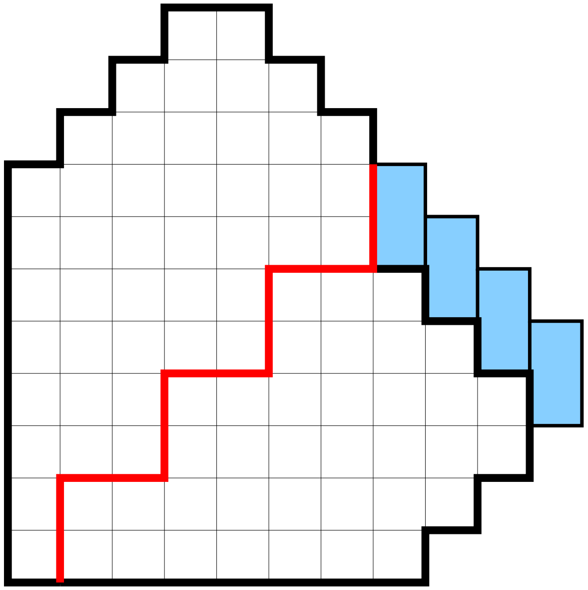
Proof of formula for M(TAD_{2n-1})

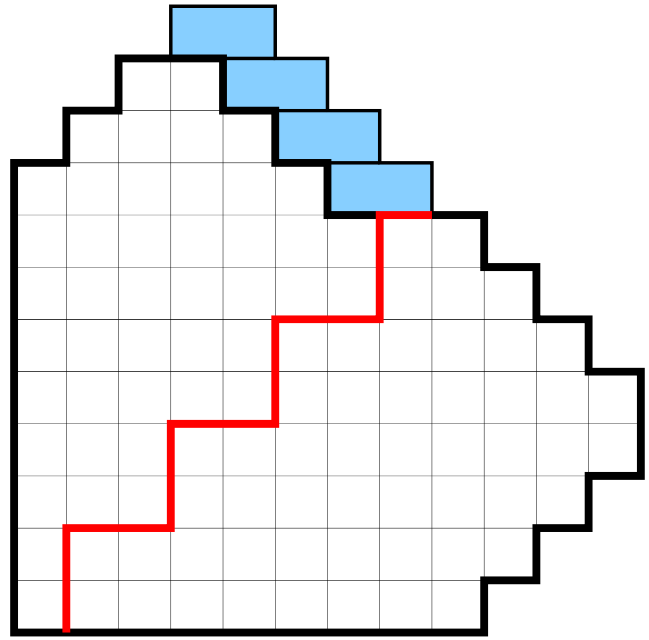


$\mathcal{T}_n$

$\mathcal{T}_n'''$

Proof of formula for M(TAD_{2n-1}) and M(TAD_{2n})





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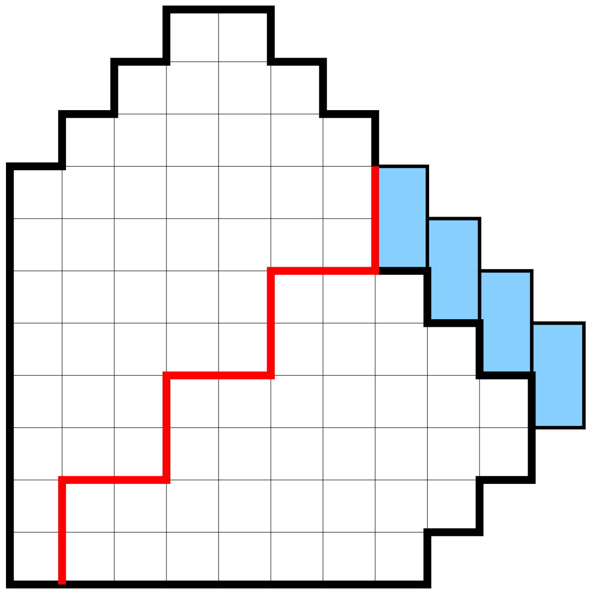
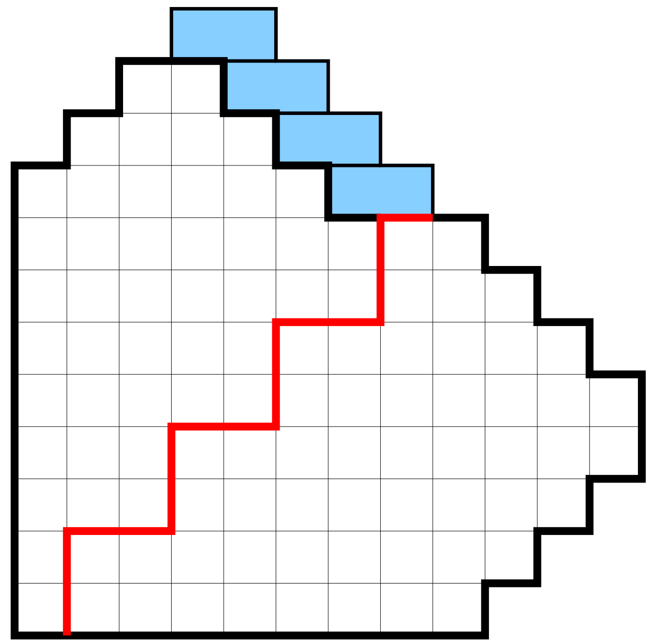
Proof of formula for M(TAD_{2n-1}) and M(TAD_{2n})



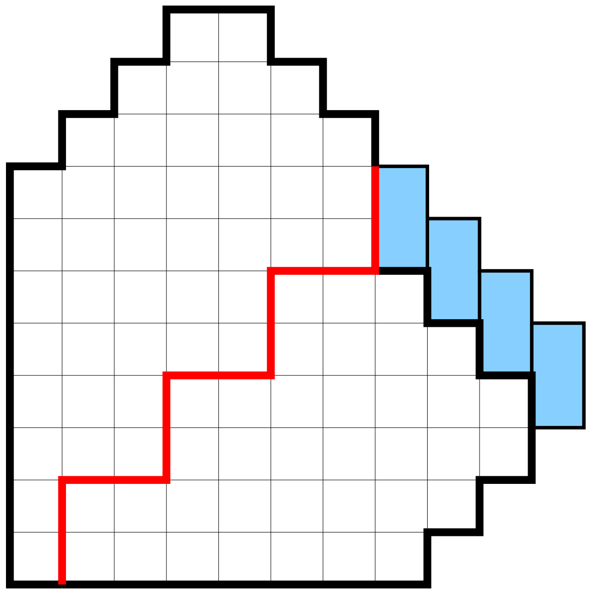


$\mathcal{T}_n$

$\mathcal{T}_n'$

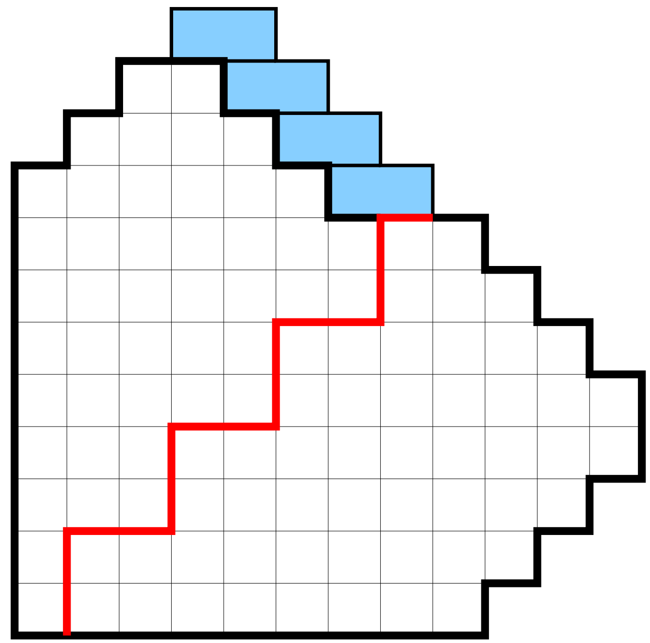
$\mathcal{T}_n'''$

Proof of formula for M(TAD_{2n-1}) and M(TAD_{2n})



$\mathcal{T}_n$

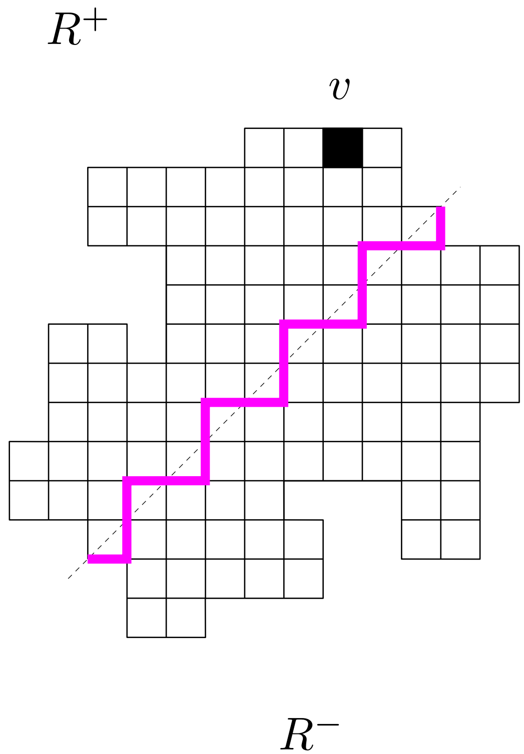
$\mathcal{T}_n'''$



$\mathcal{T}_n'$

$\mathcal{T}_n''$

A variant of the factorization theorem: symmetric region with unit dent



If v same color as diagonal:

$$\mathrm{M}(R \setminus v) = 2^{n/2}\, \mathrm{M}(R^+ \setminus v)\, \mathrm{M}(R^-),$$

- n= # (unit squares on symmetry axis) -1
- zig-zag cut starts below diagonal

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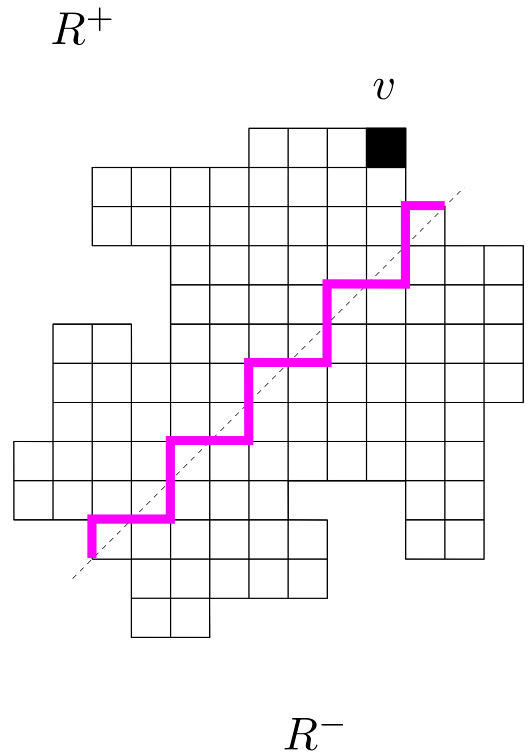
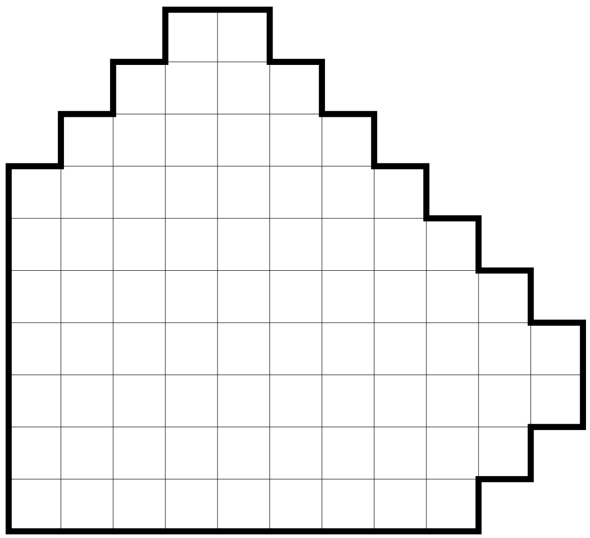


If v opposite color of diagonal:

$$\mathrm{M}(R \setminus v) = 2^{n/2}\,\mathrm{M}(R^+ \setminus v)\,\mathrm{M}(R^-),$$

- n= # (unit squares on symmetry axis) -1
- zig-zag cut starts above diagonal

Proof of formula for $\mathrm{M}(TAD'_{2n-1})$



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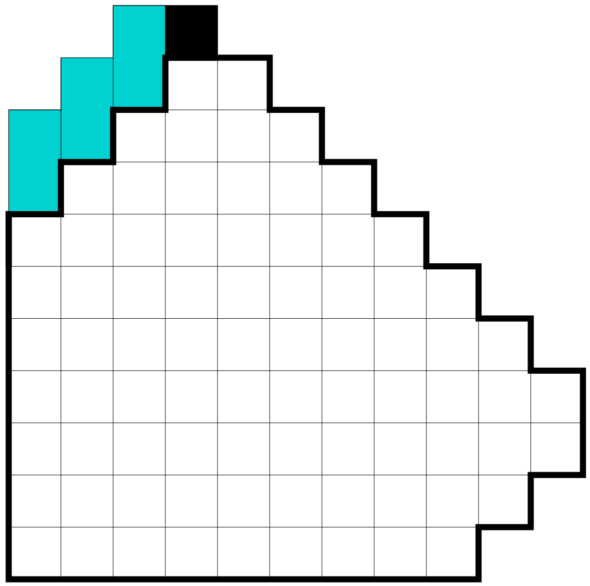

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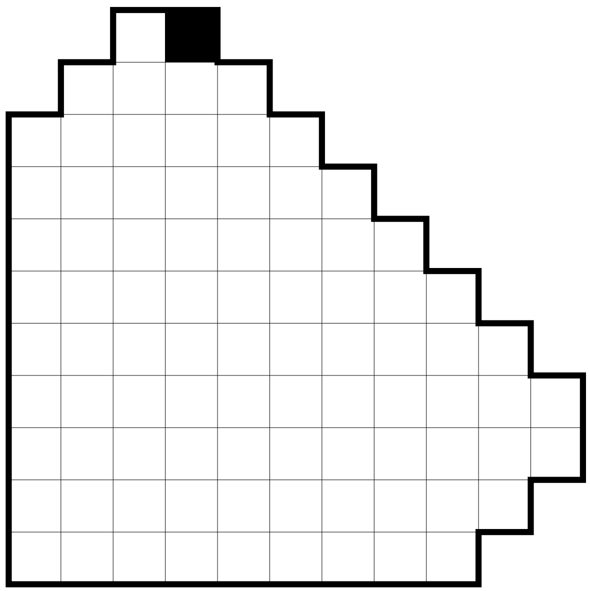

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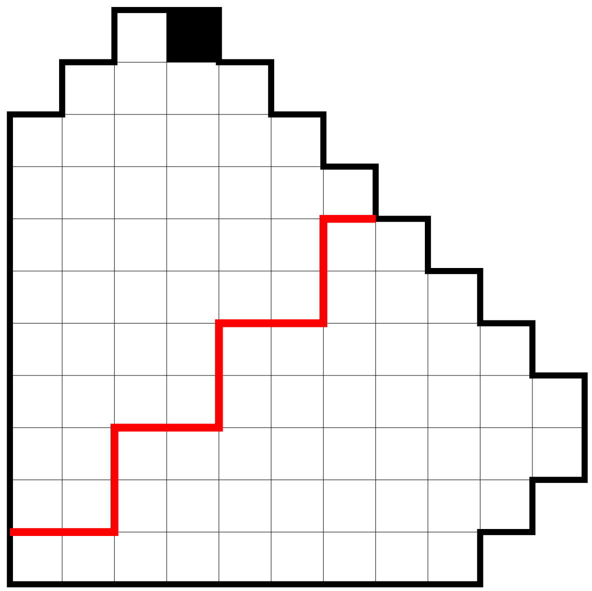

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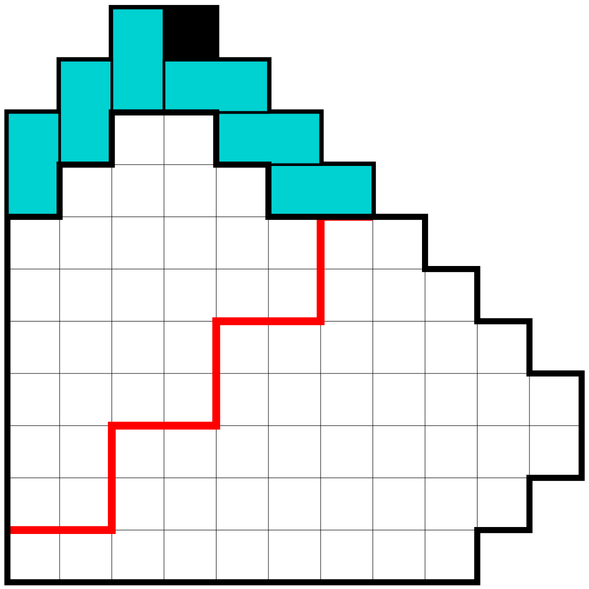

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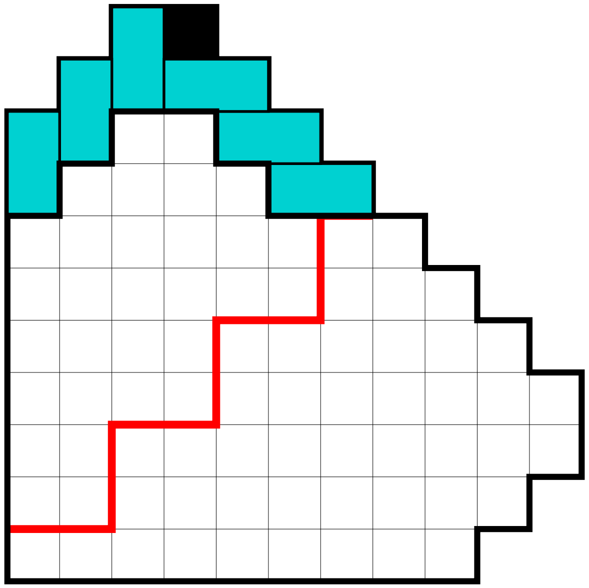

$\mathcal{T}''_{n-1}$

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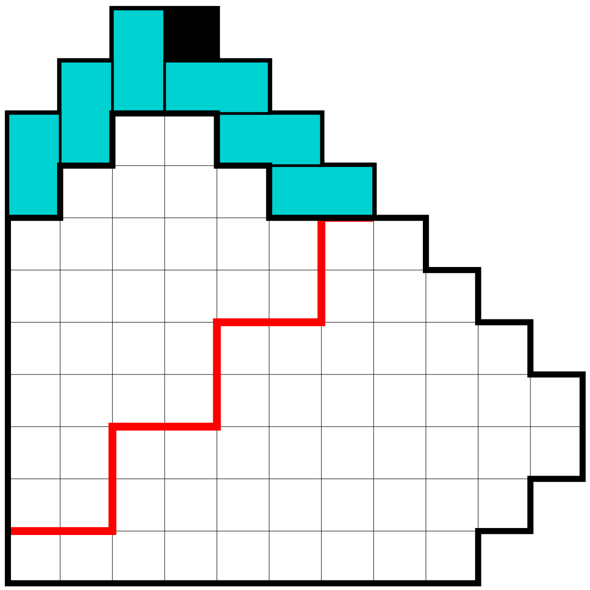
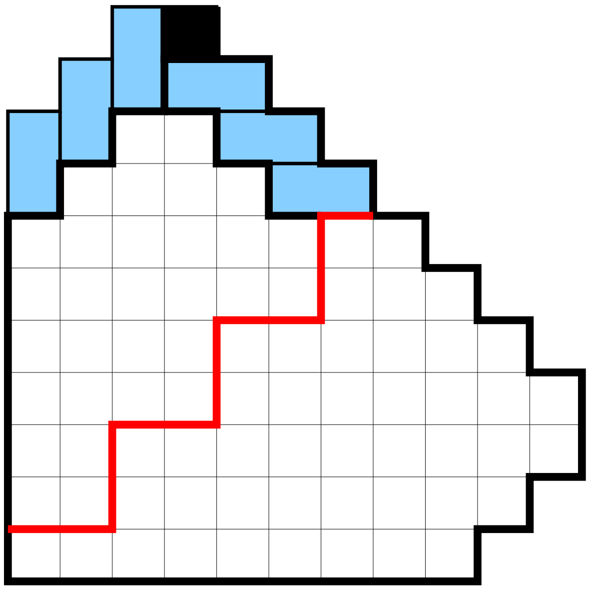


$\mathcal{T}''_{n-1}$

$\mathcal{T}'_{n}$

Proof of formula for $\mathrm{M}(TAD'_{2n-1})$ and $\mathrm{M}(TAD'_{2n})$



$\mathcal{T}''_{n-1}$

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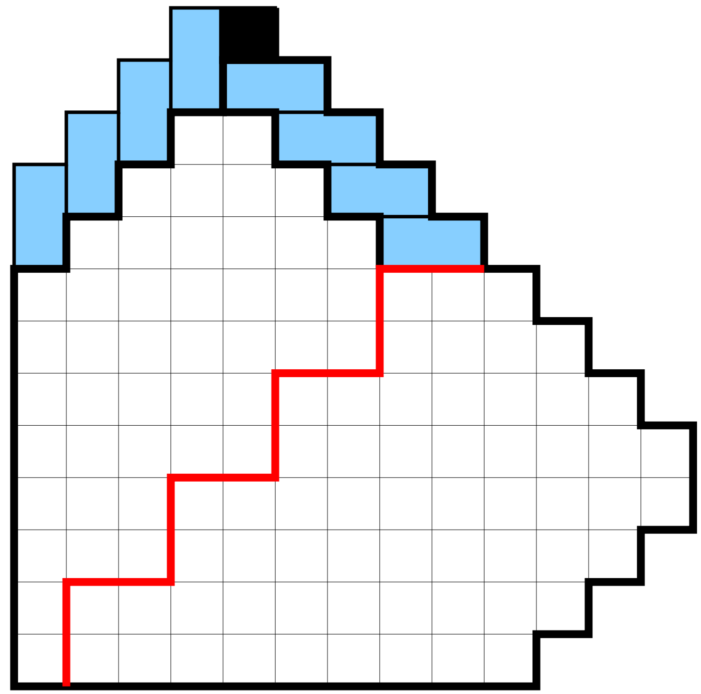

$\mathcal{T}_{n}$

$\mathcal{T}'''_{n+1}$

A classical result on the triangular lattice:

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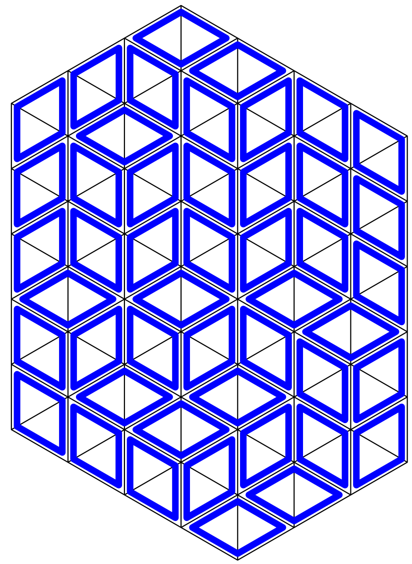

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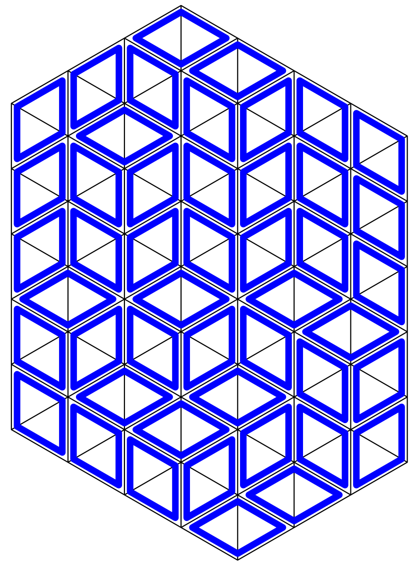

A lozenge tiling of the hexagon $H_{3,4,5}$.

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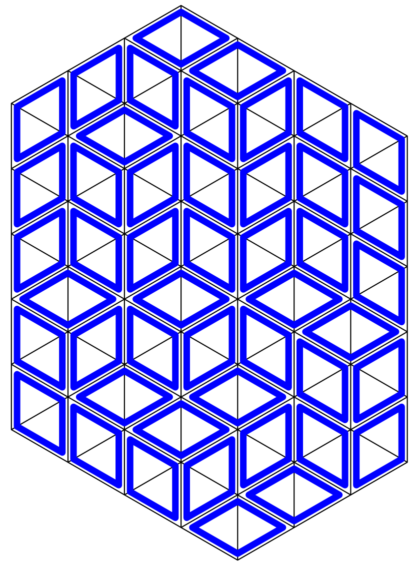

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MacMahon's Theorem (1900):

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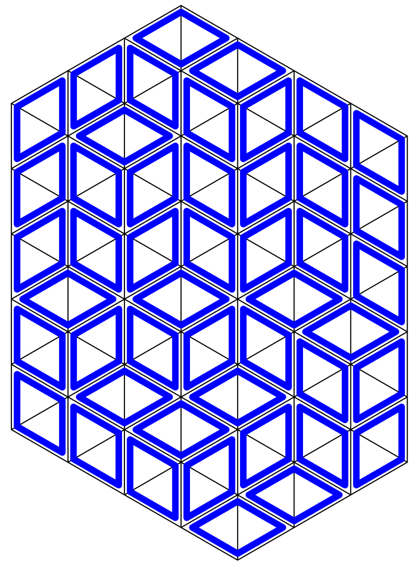


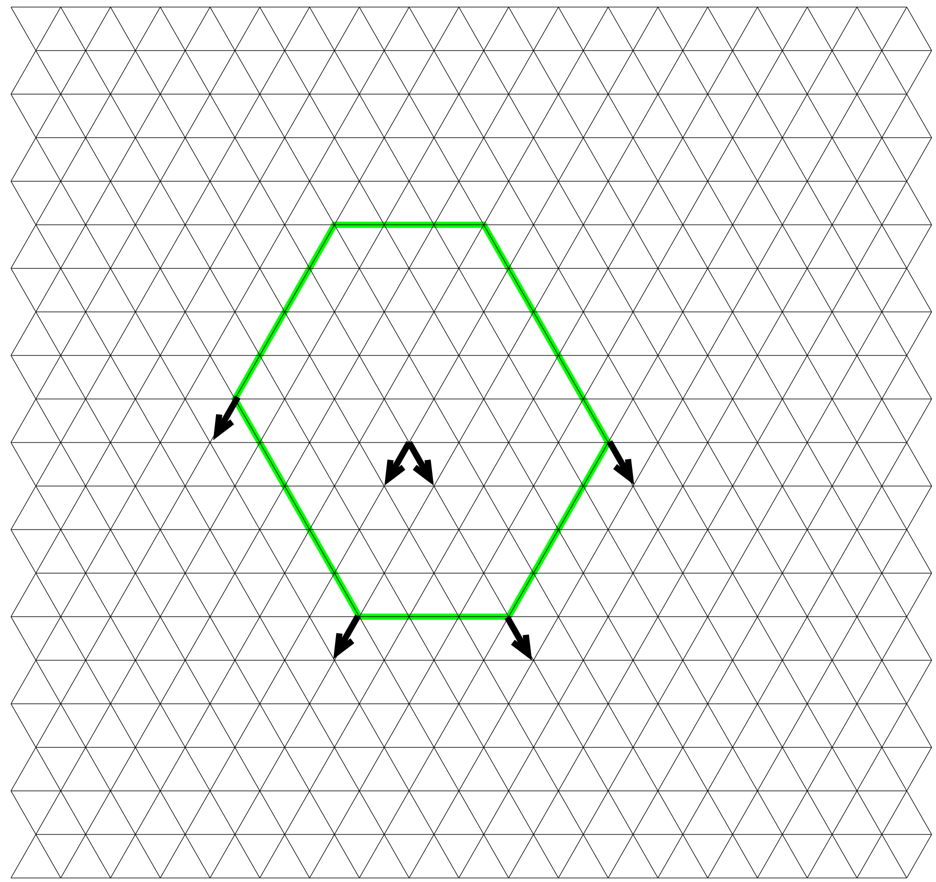
A lozenge tiling of the hexagon $H_{3,4,5}$.

MacMahon's Theorem (1900):

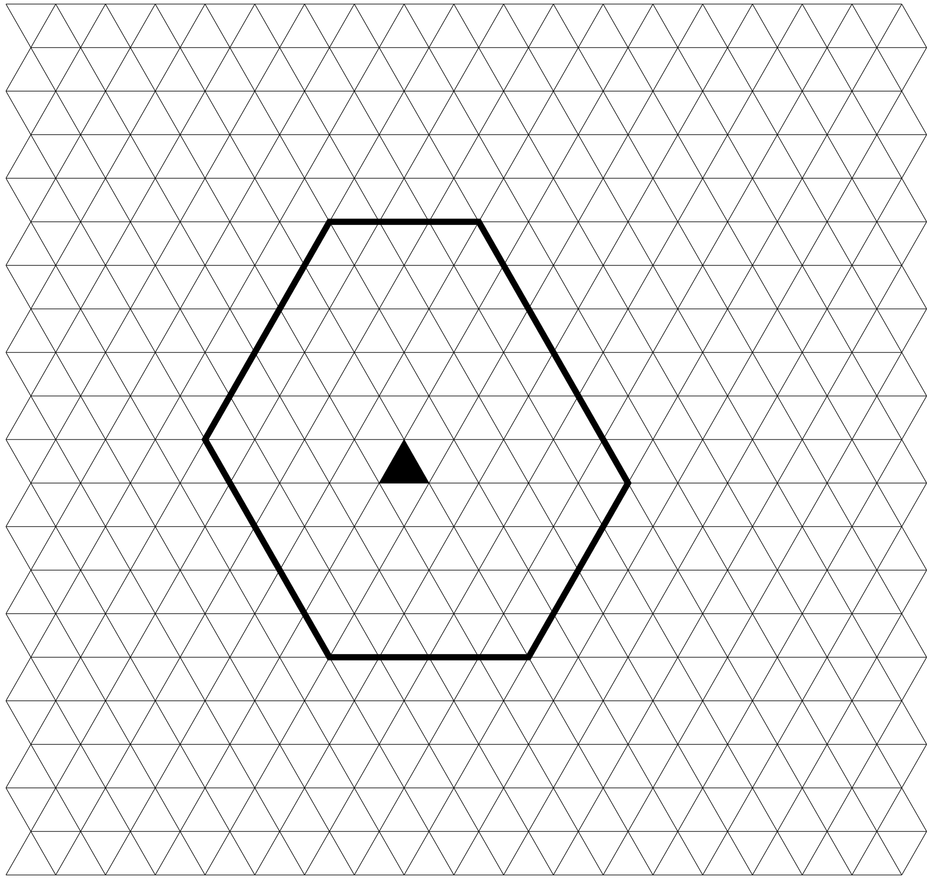
$$\mathrm{M}(H_{a,b,c}) = \frac{\mathrm{H}(a)\,\mathrm{H}(b)\,\mathrm{H}(c)\,\mathrm{H}(a+b+c)}{\mathrm{H}(a+b)\,\mathrm{H}(a+c)\,\mathrm{H}(b+c)},$$

where

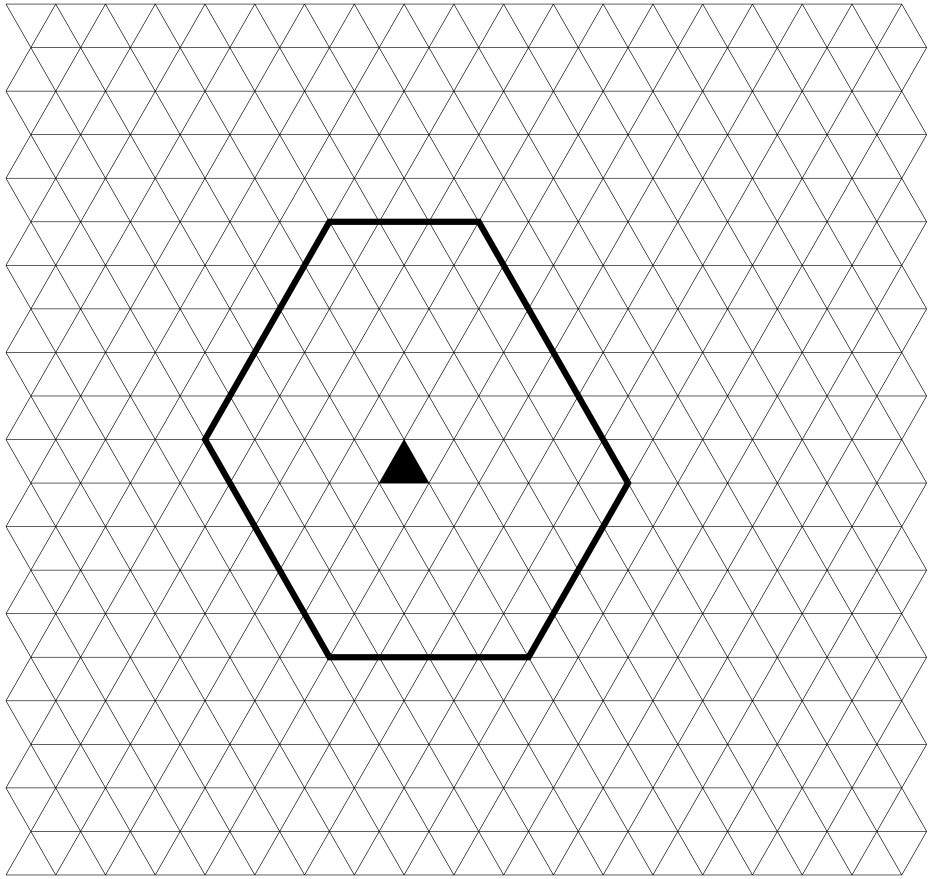
$$\mathrm{H}(n) = 0!\,1!\,\cdots\,(n-1)!$$



Creating a unit hole



Q: How many tilings?



Q: How many tilings?
A: 1,000,000

There are dozens of families of regions on various lattices in the plane whose number of tilings are given by “simple product formulas”.

Two open problems of Lai

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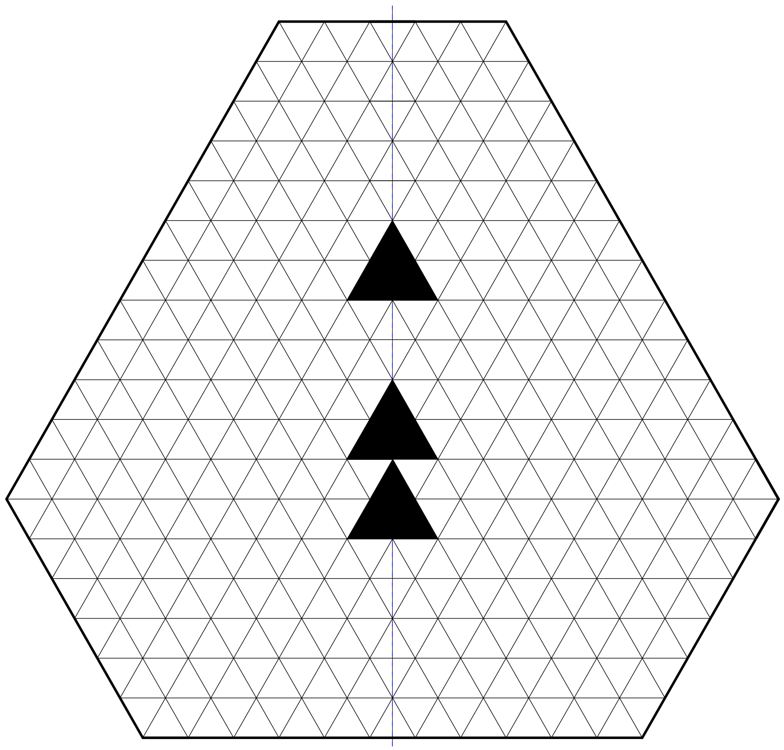
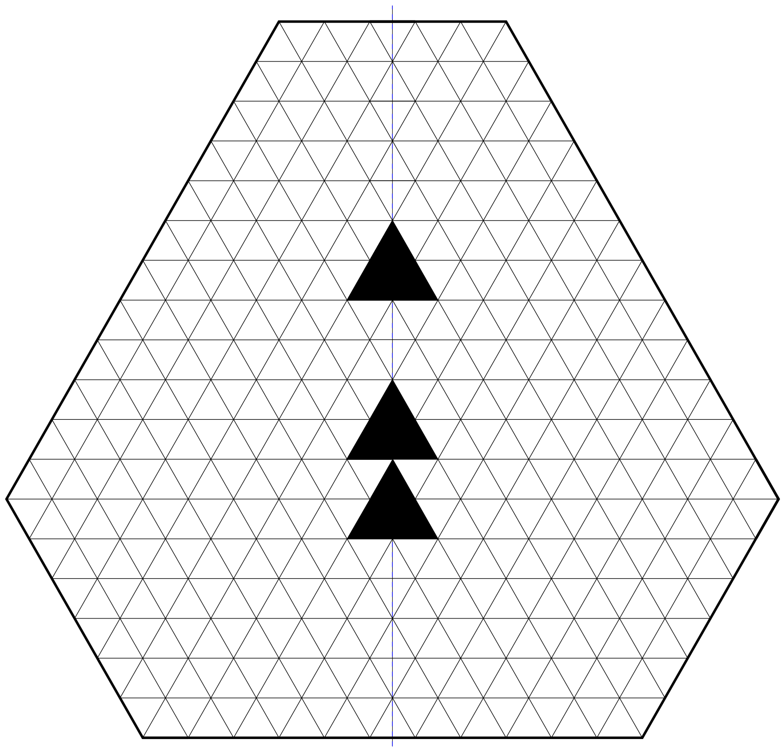


Two open problems of Lai



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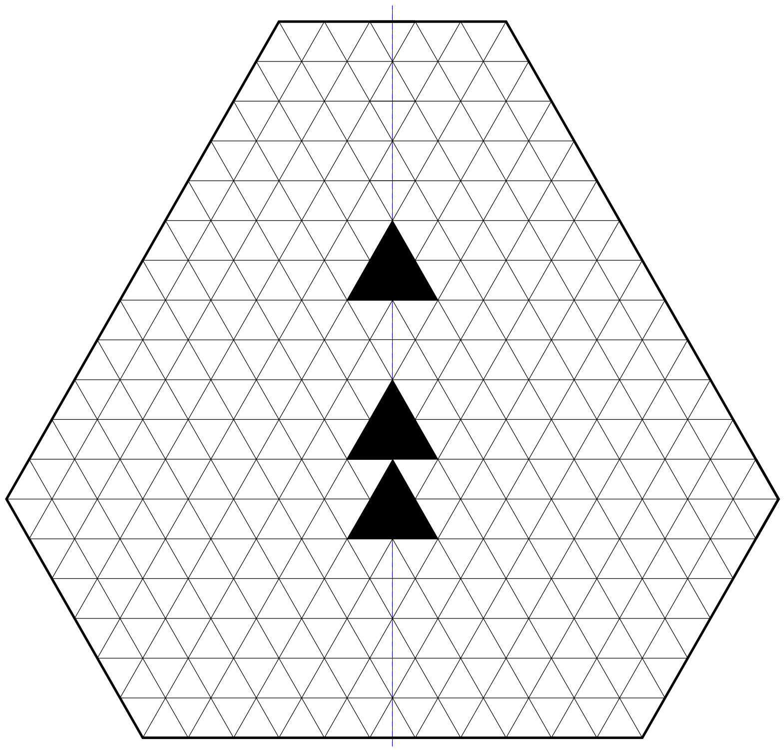

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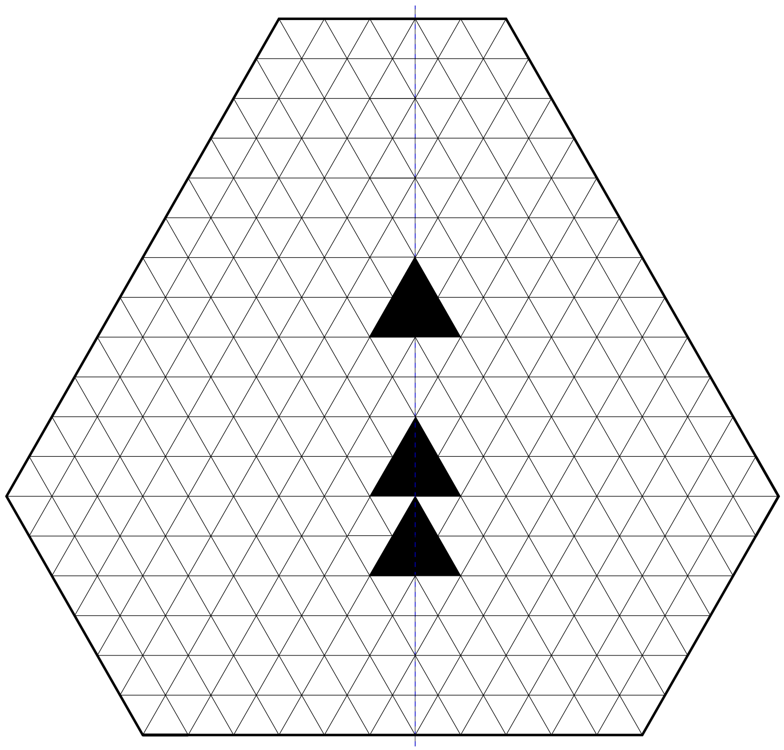
Theorem (C., 2005): Simple product formula.

Two open problems of Lai



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Open problem (Lai, 2022): Find simple product formula.

Second problem

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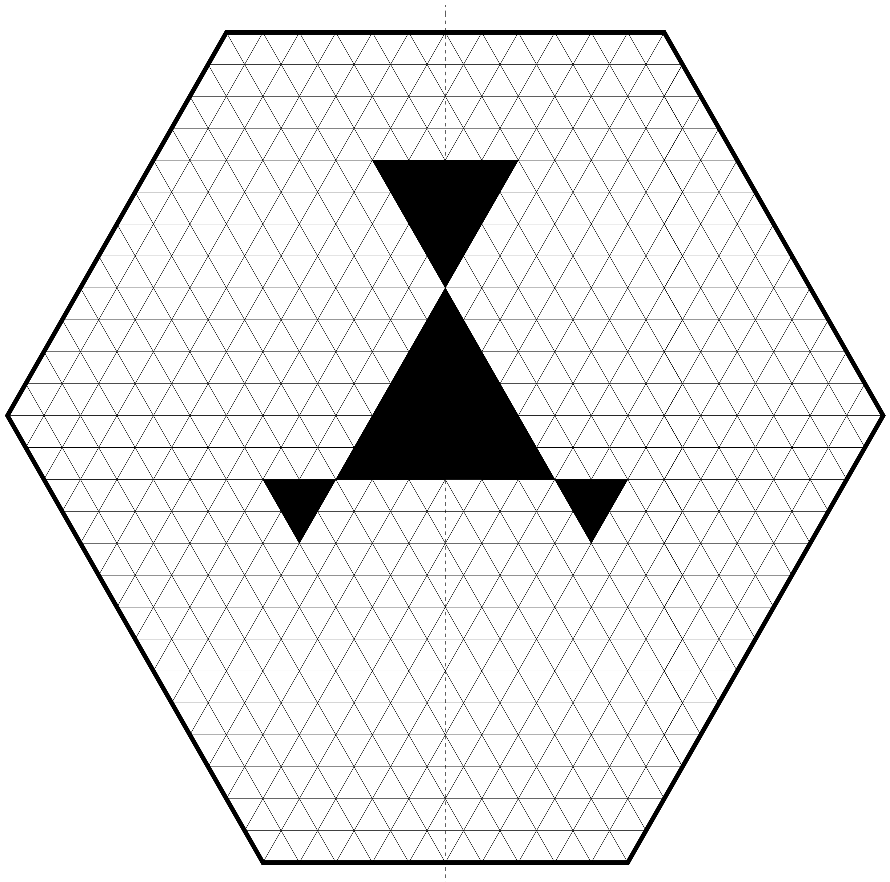

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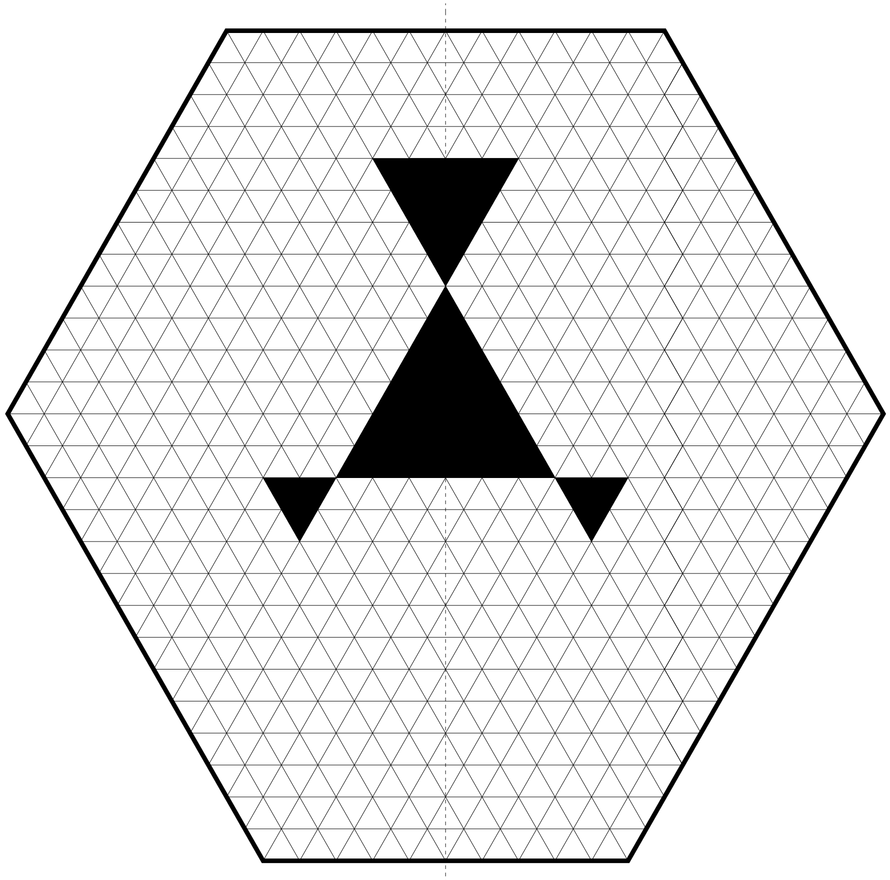

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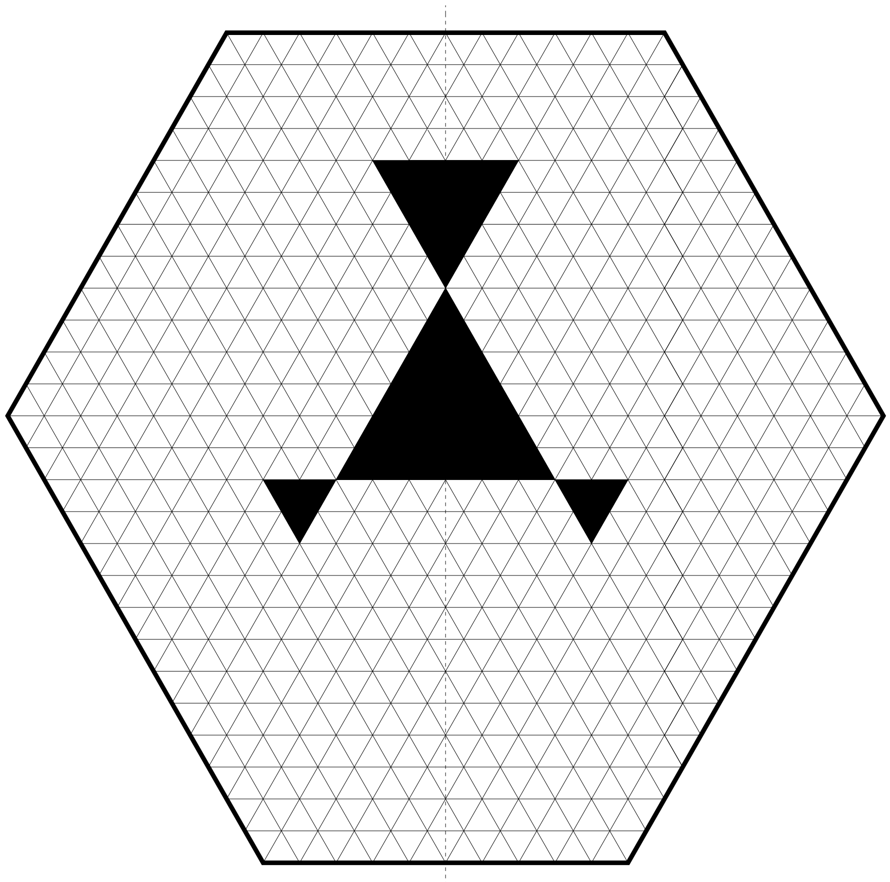
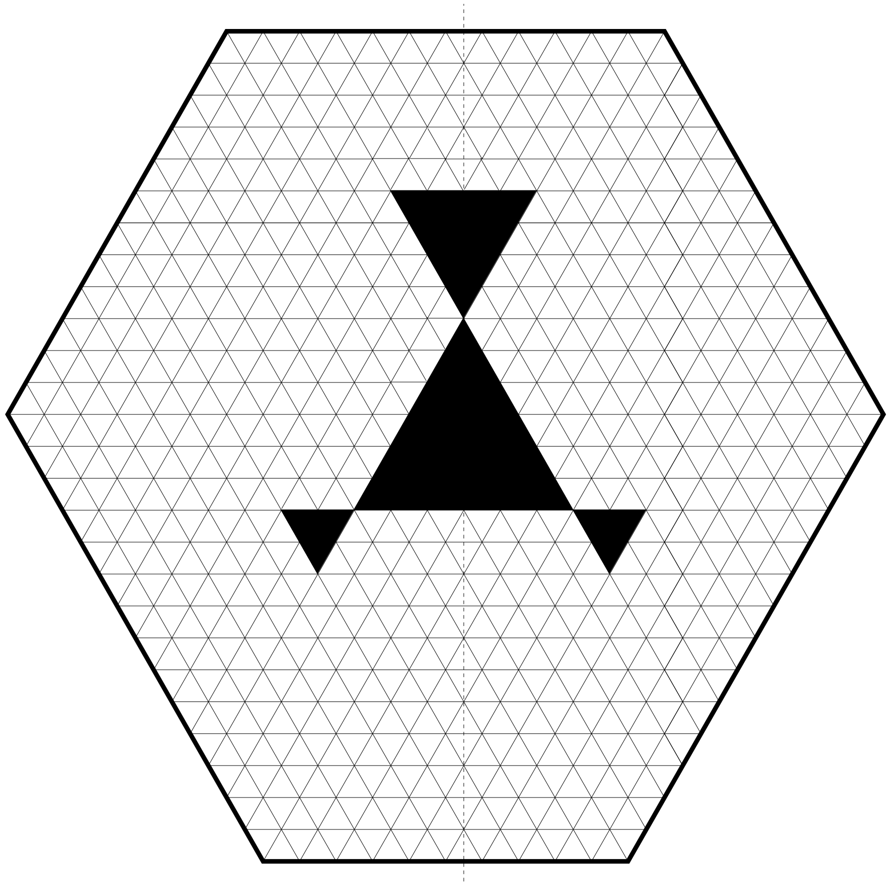


Q: How many lozenge tilings?

Theorem (C., 2016; Lai and Rohatgi, 2016): Simple product formula.

Second problem



Open problem (Lai, 2022): Find simple product formula.

A more general result

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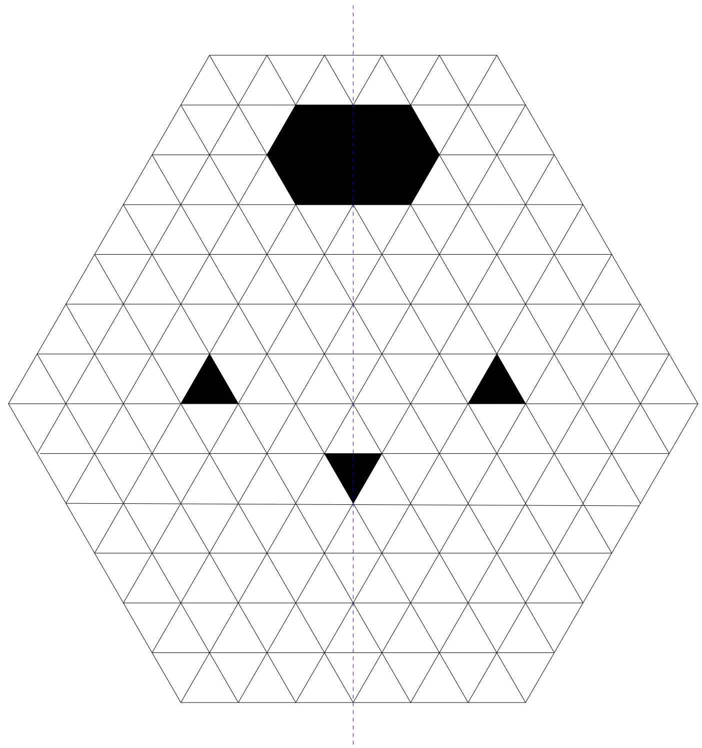

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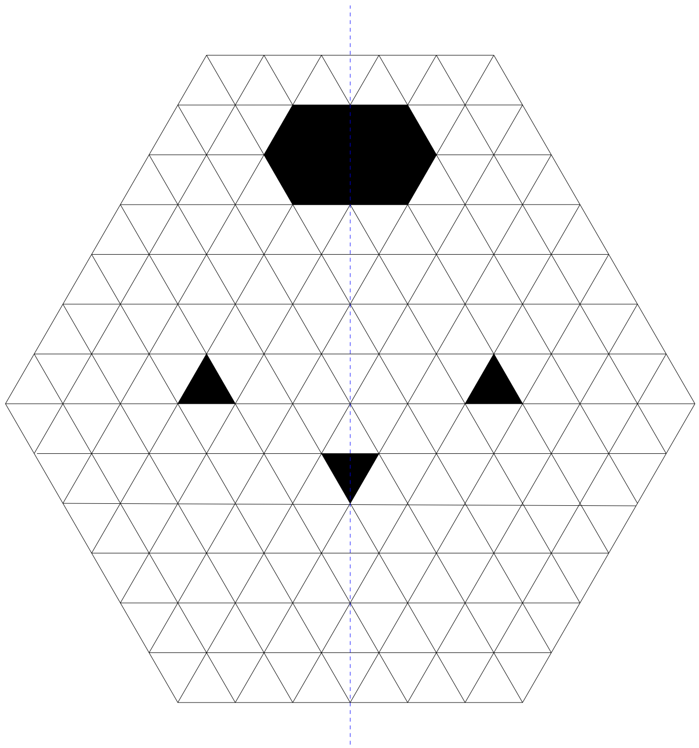

Symmetric hexagon with holes

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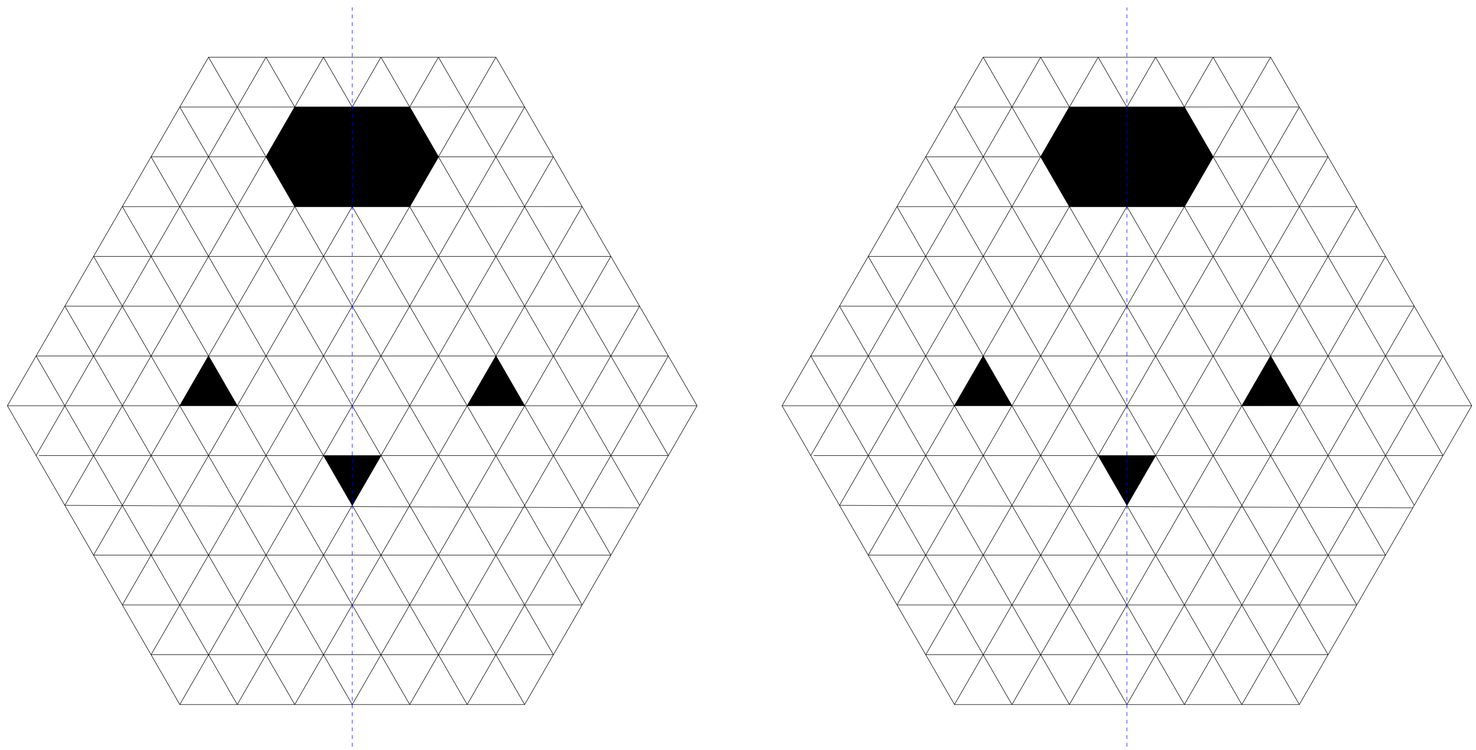

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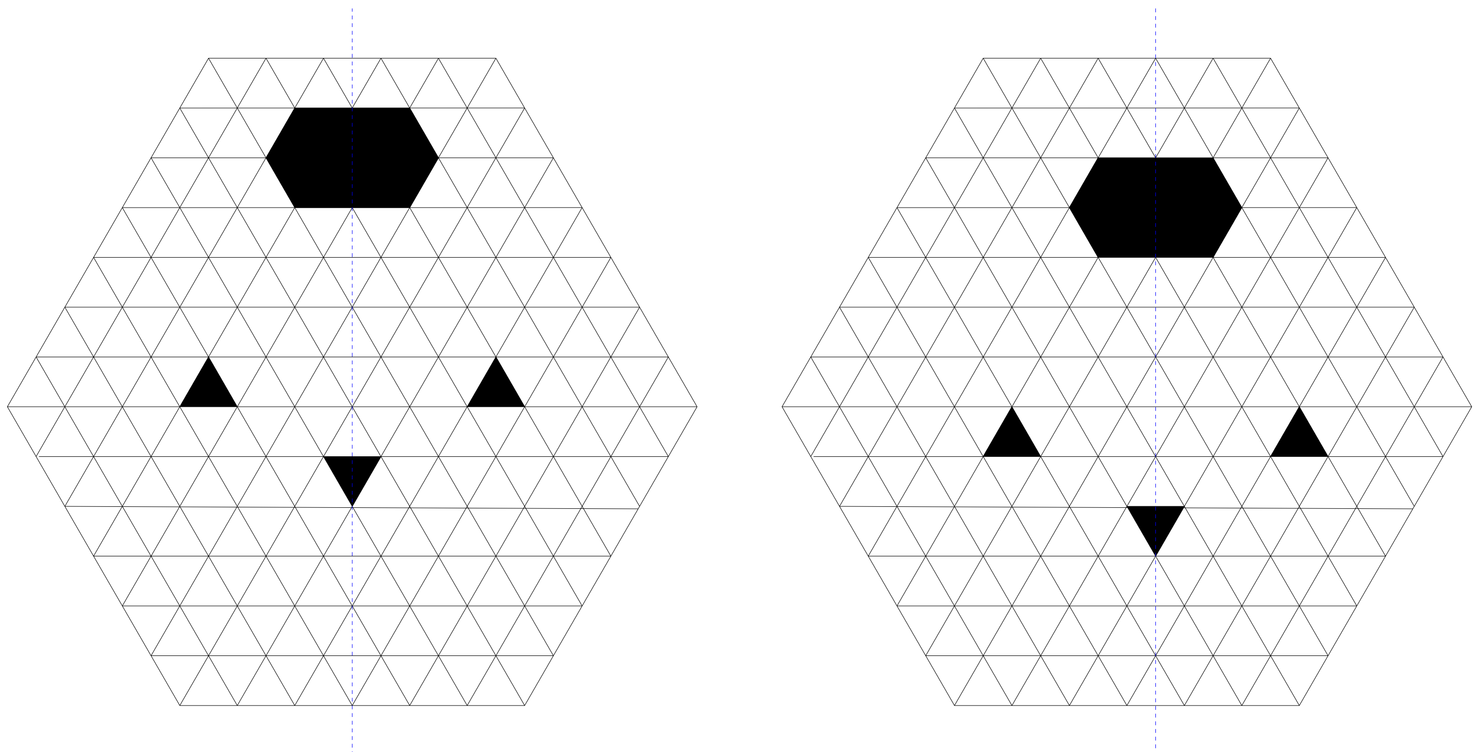

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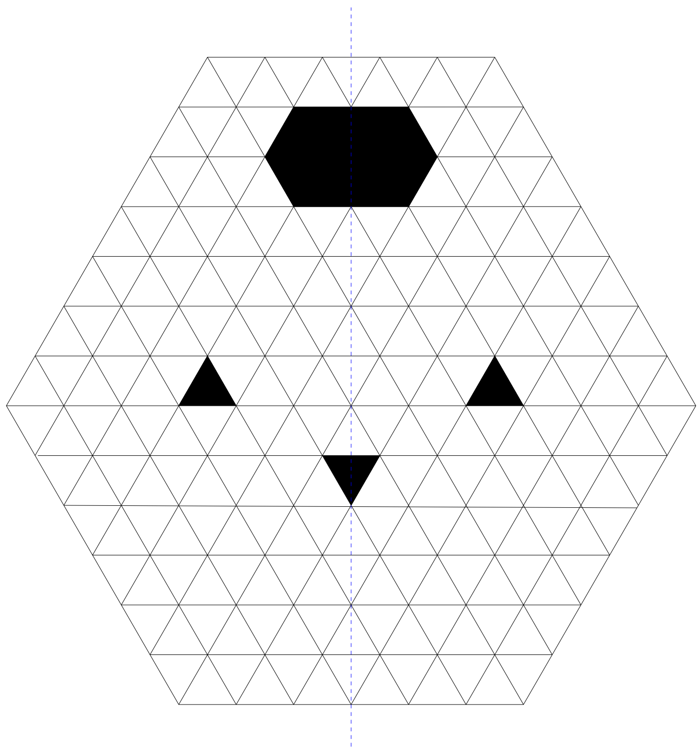

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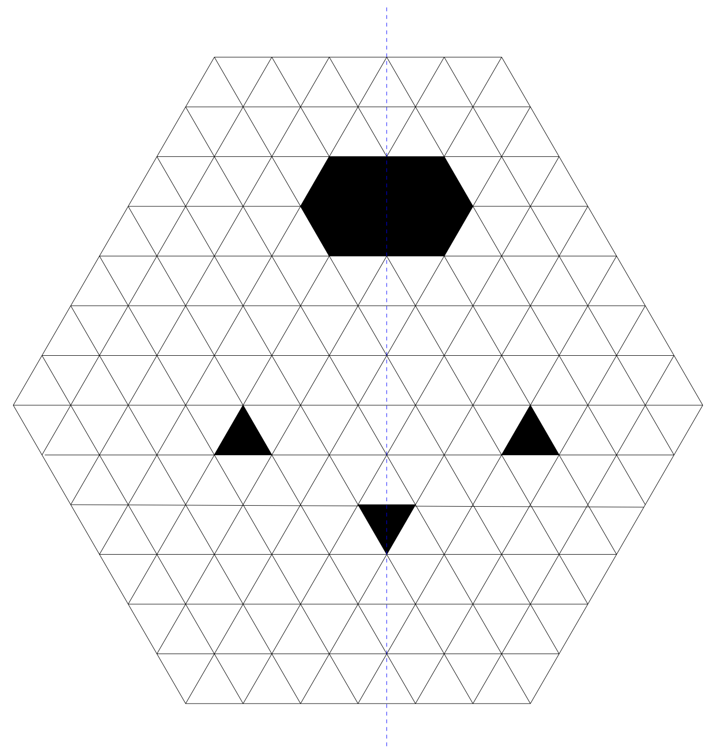

Nearly symmetric hexagon with holes

Factorization theorem (C., 1997) gives

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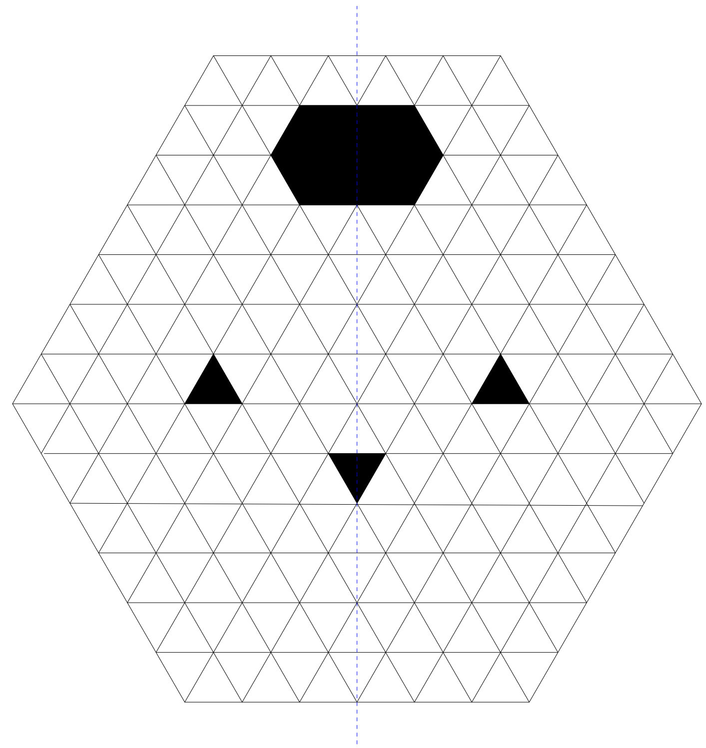

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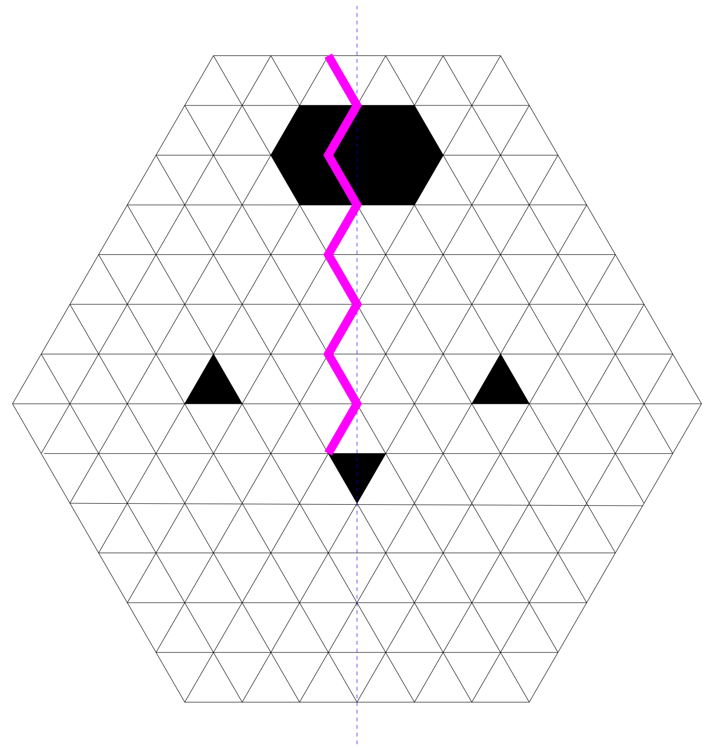

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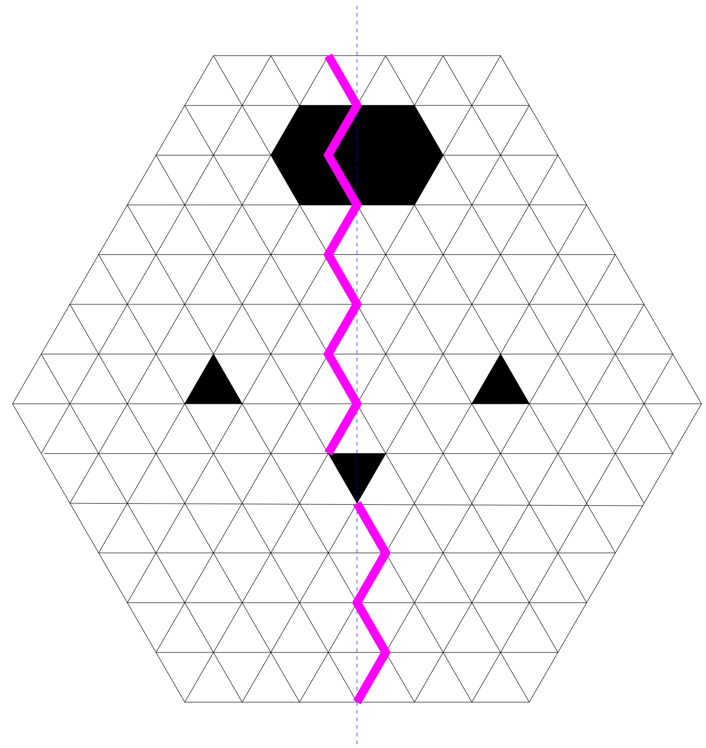

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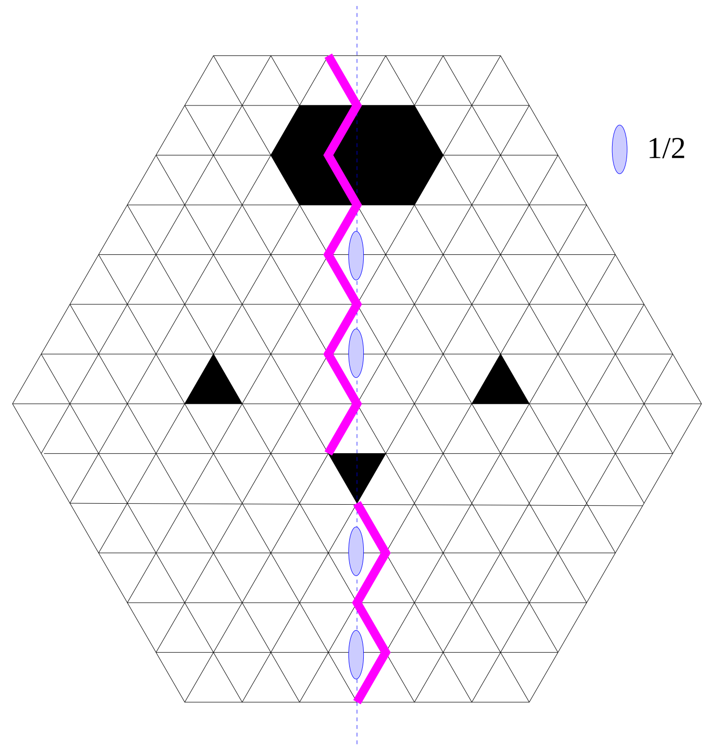

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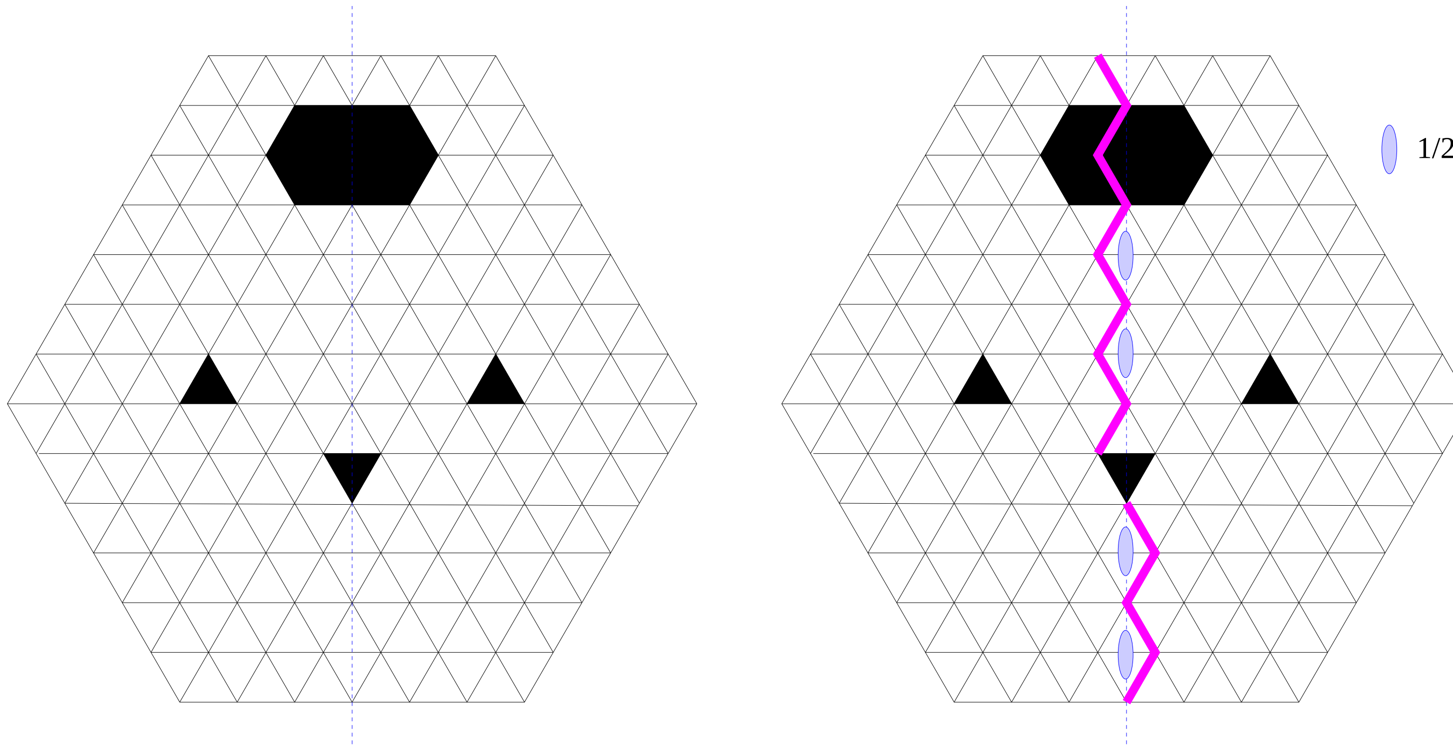

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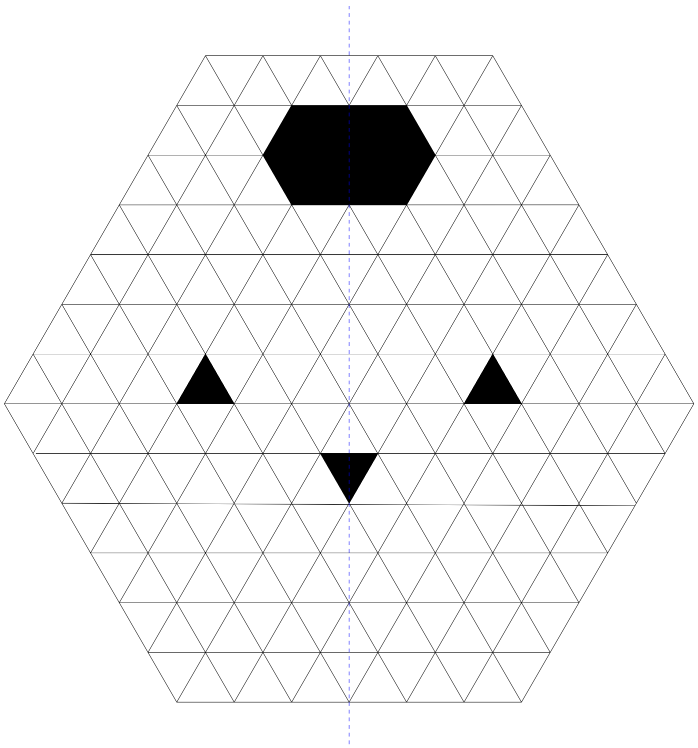

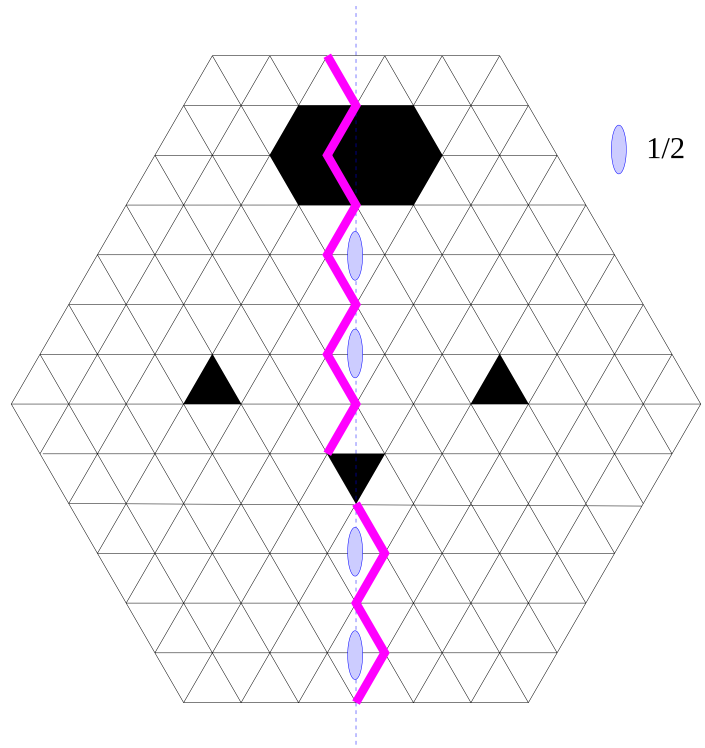

R

R^+ R^-

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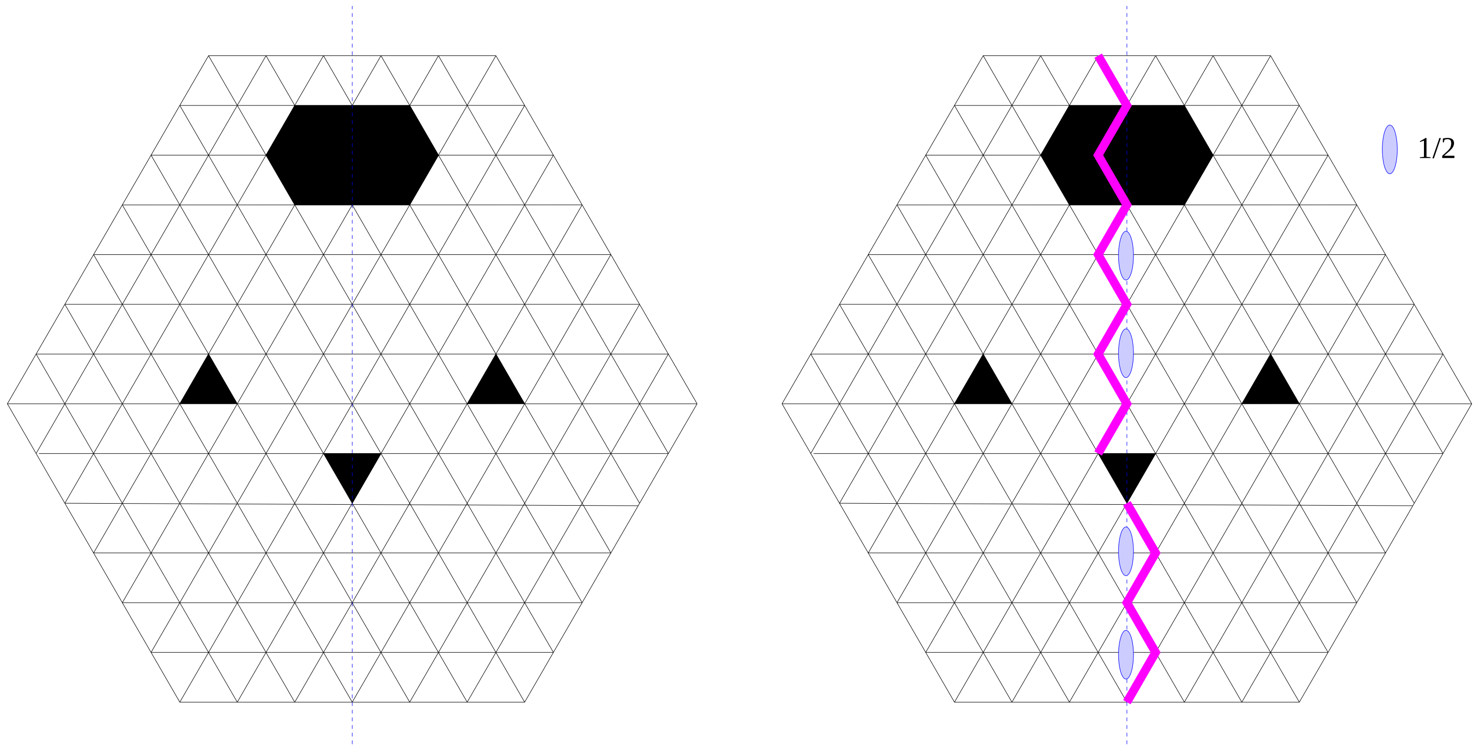

R $\qquad\qquad$ R^+ $\qquad$ R^-

$$\mathrm{M}(R) = 2^{\mathrm{w}(R)}\,\mathrm{M}(R^+)\,\mathrm{M}(R^-)$$

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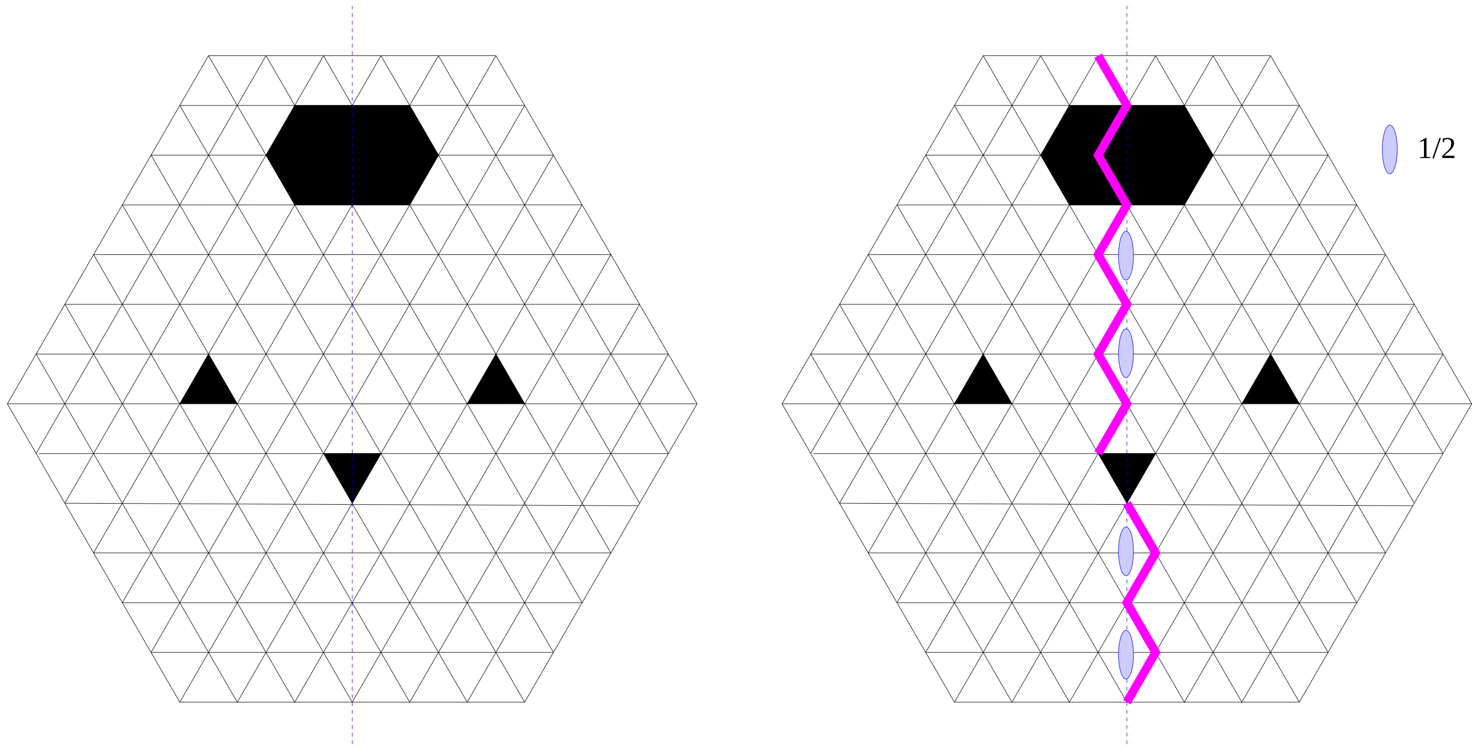
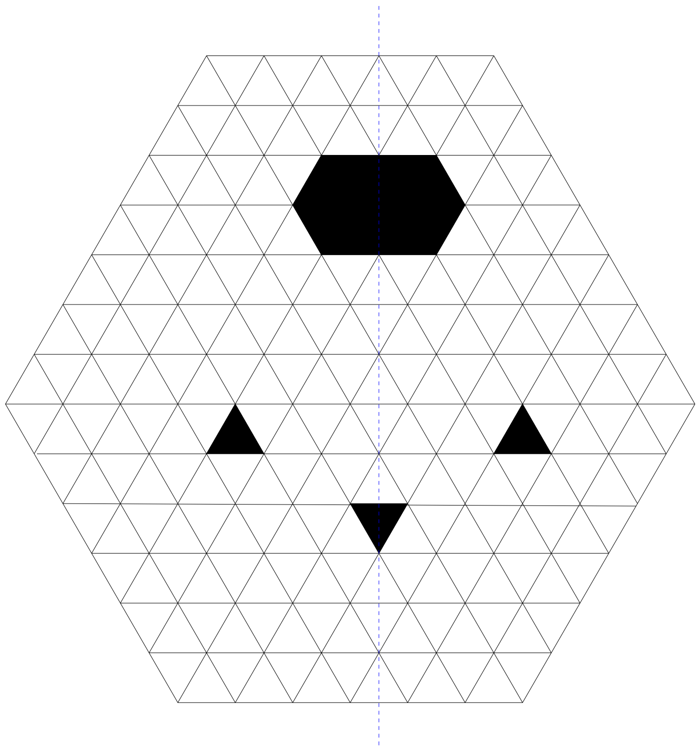


R $\qquad$ R^+ $\qquad$ R^-

$$\mathrm{M}(R) = 2^{\mathrm{w}(R)}\,\mathrm{M}(R^+)\,\mathrm{M}(R^-)$$

$\mathrm{w}(R) = 1/2$(# unit triangles along symmetry axis)

Semifactorization Theorem (Byun, C. and Lee, 2024)



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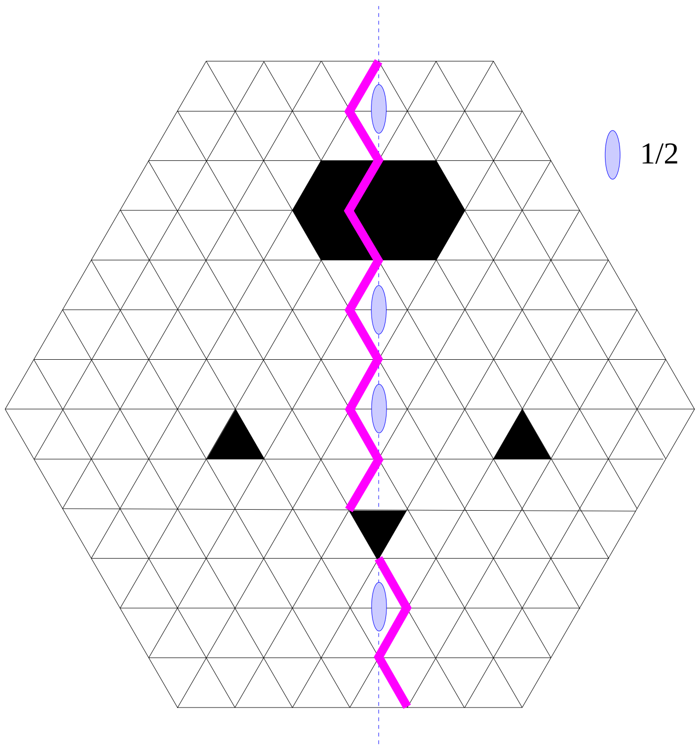

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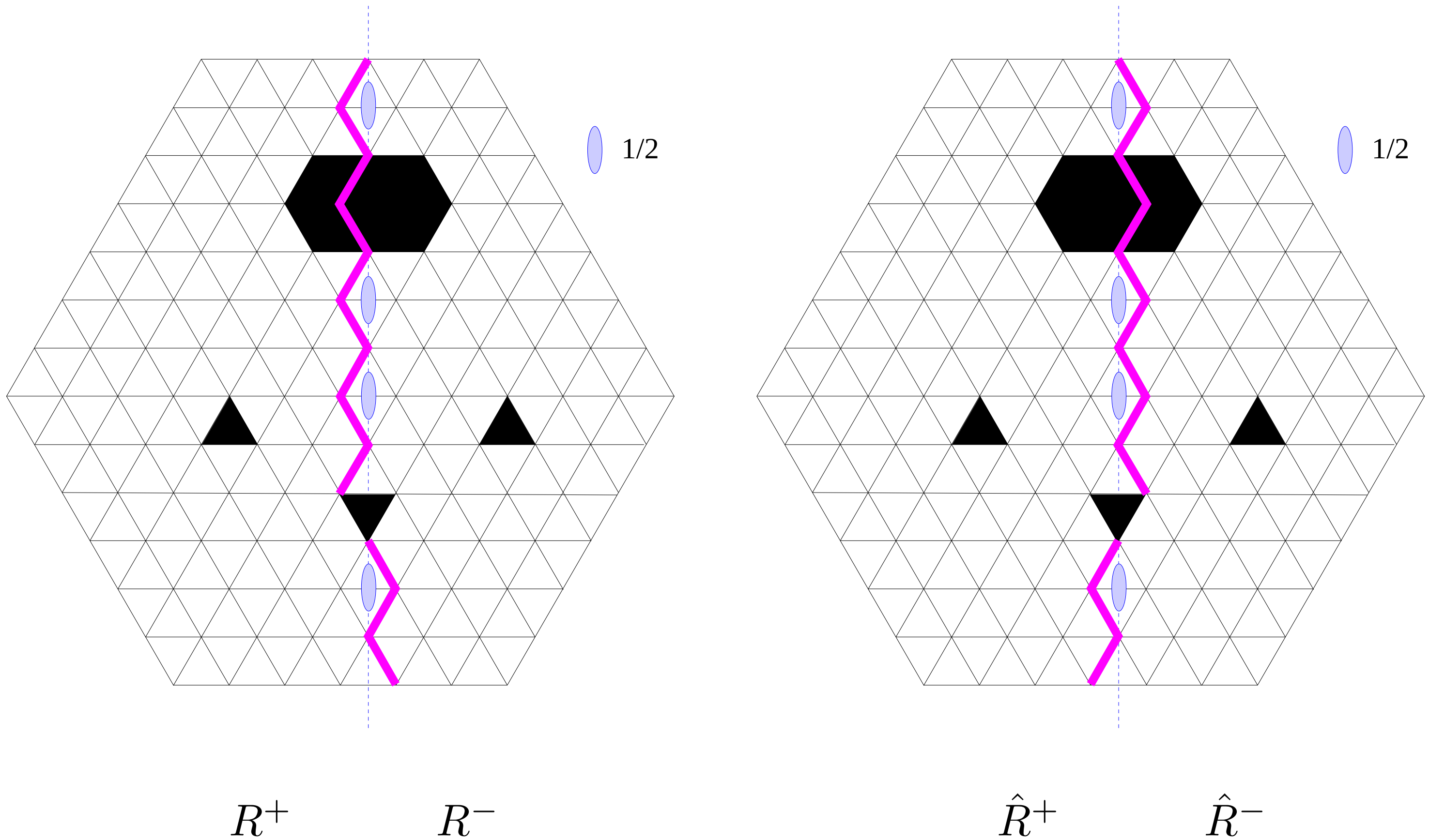

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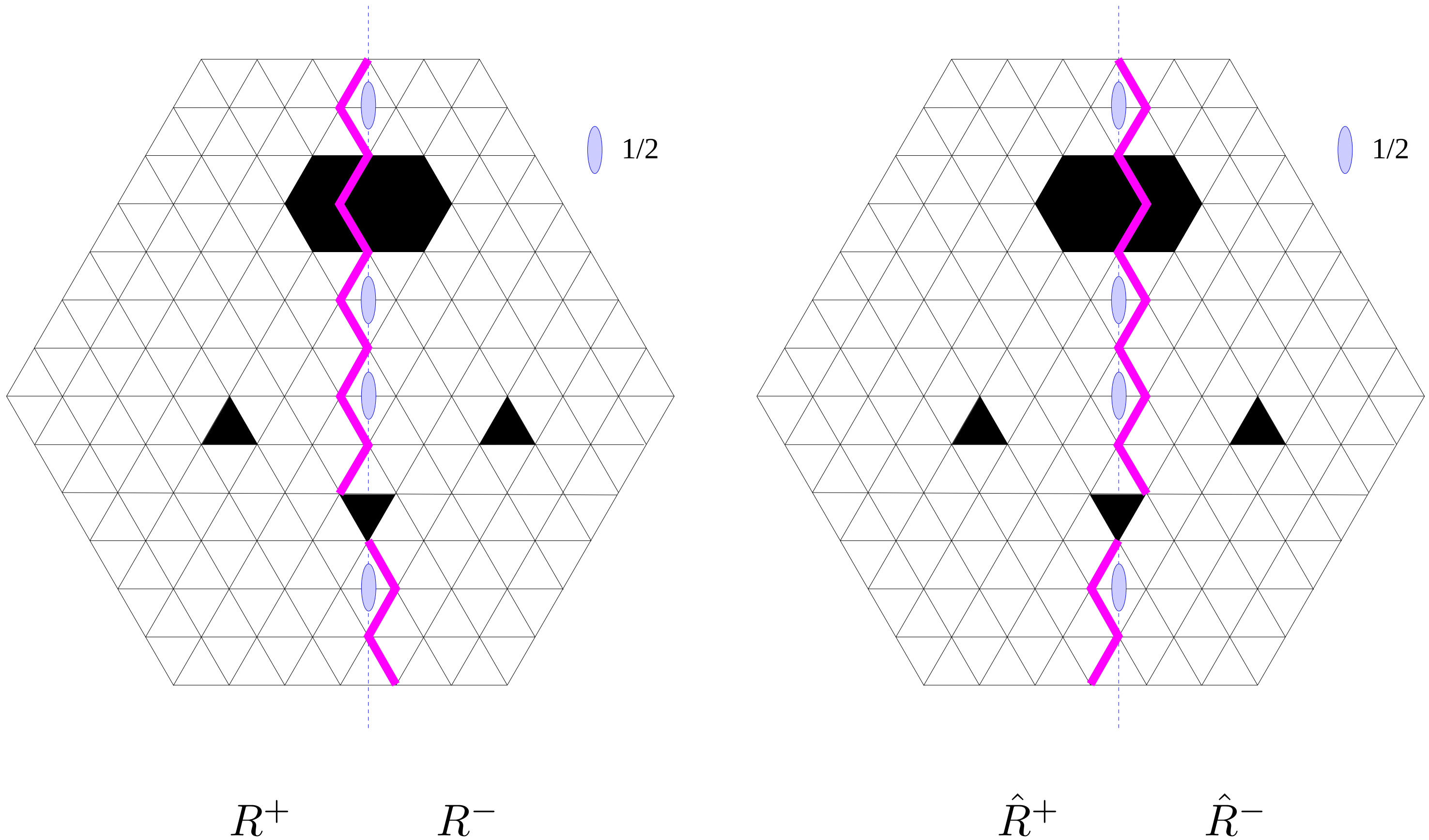

$$\mathrm{M}(R) = 2^{\mathrm{w}(R)-1}\left(\mathrm{M}(R^+)\,\mathrm{M}(R^-) + \mathrm{M}(\hat{R}^+)\,\mathrm{M}(\hat{R}^-)\right)$$

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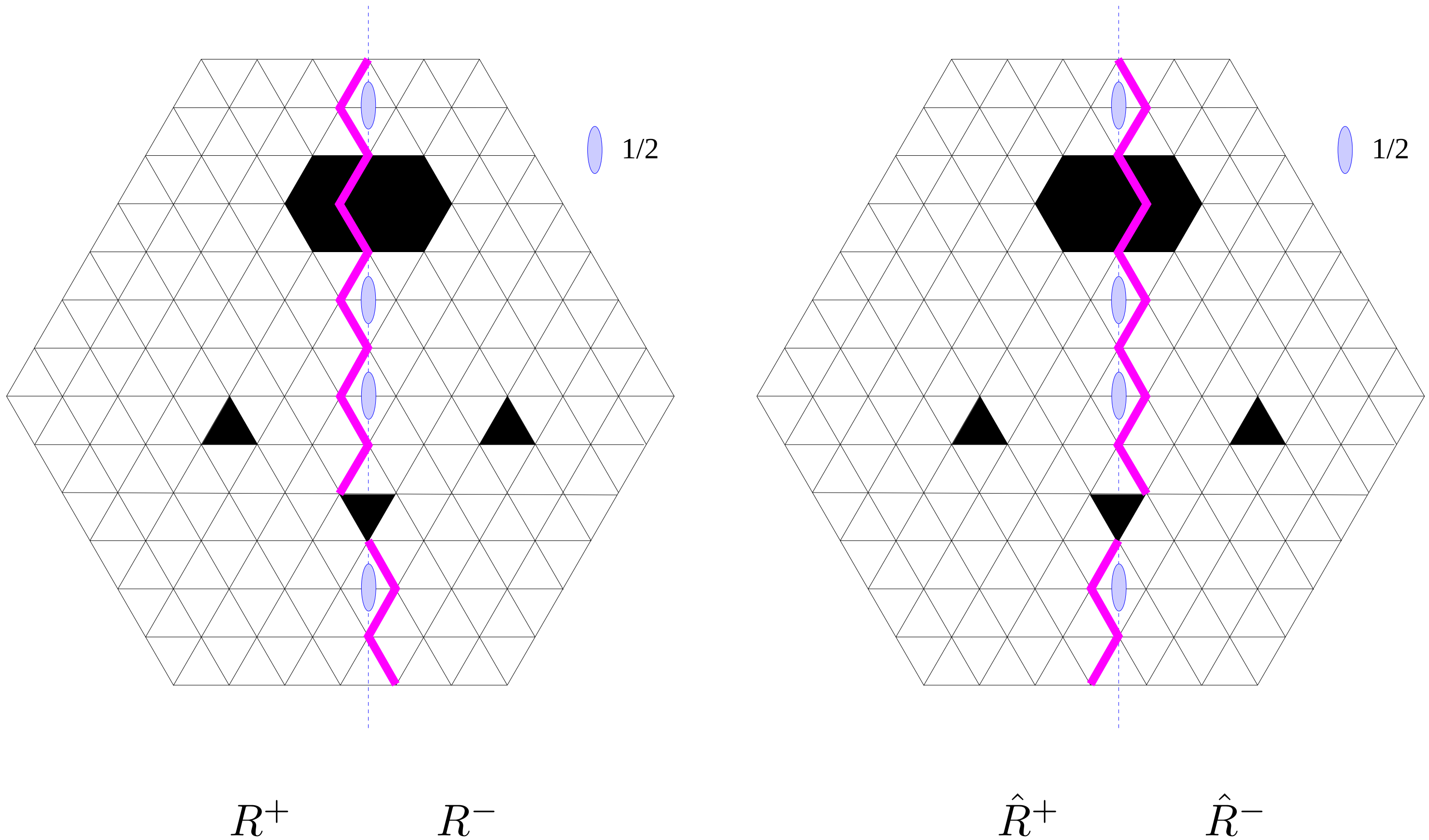

$$\mathrm{M}(R) = 2^{\mathrm{w}(R)-1}\left(\mathrm{M}(R^+)\,\mathrm{M}(R^-) + \mathrm{M}(\hat{R}^+)\,\mathrm{M}(\hat{R}^-)\right)$$

Note: RHS = average of the two different "applications" of factorization theorem!

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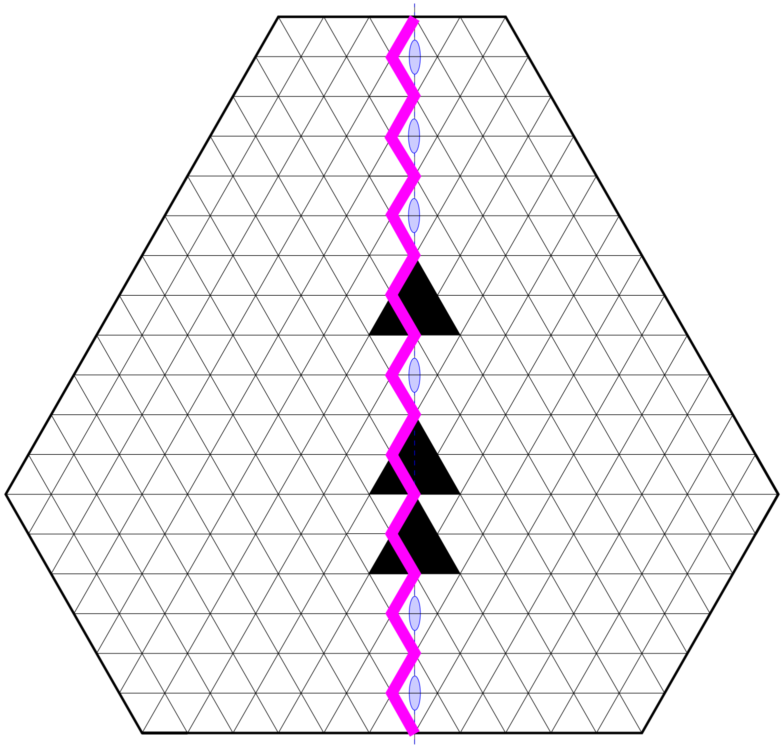

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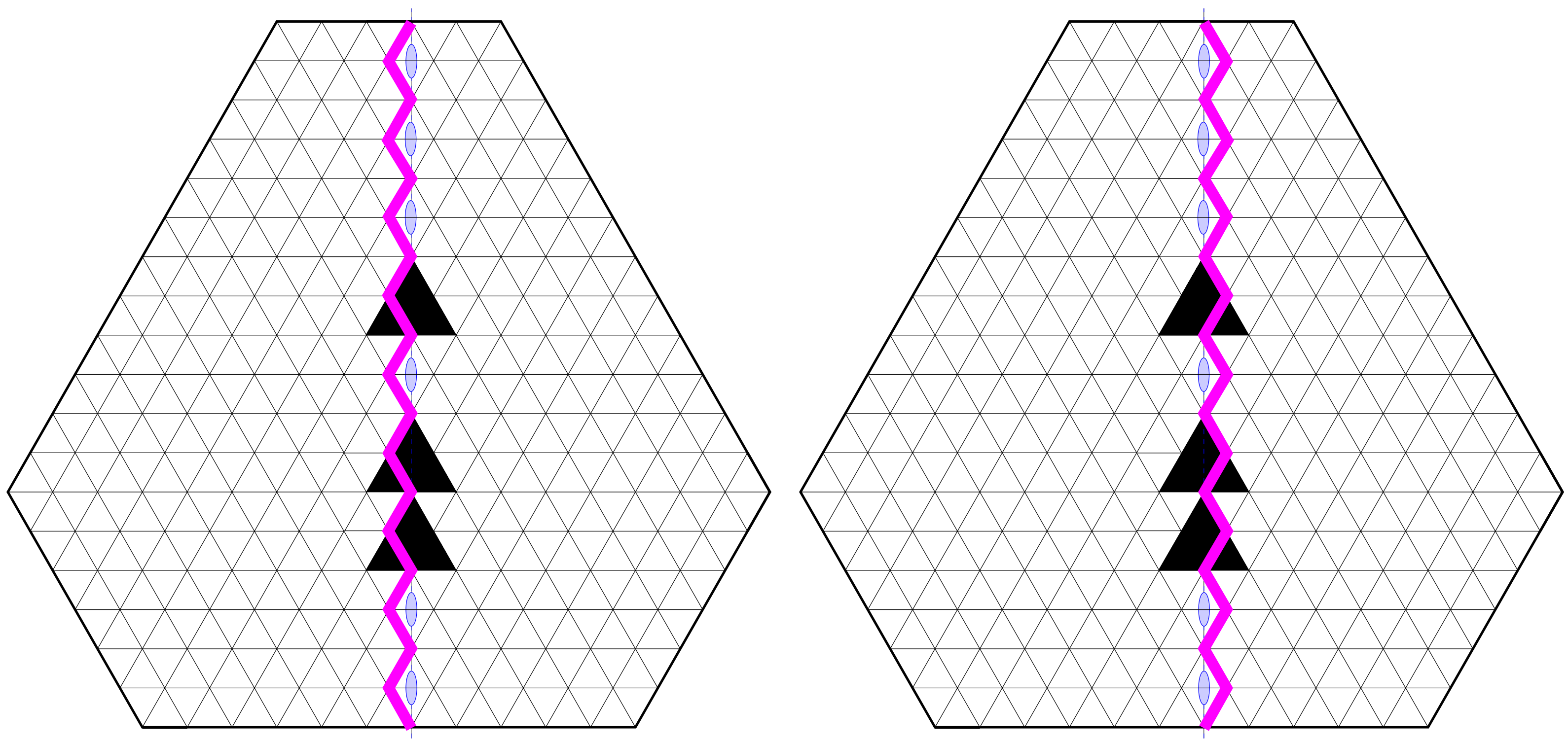

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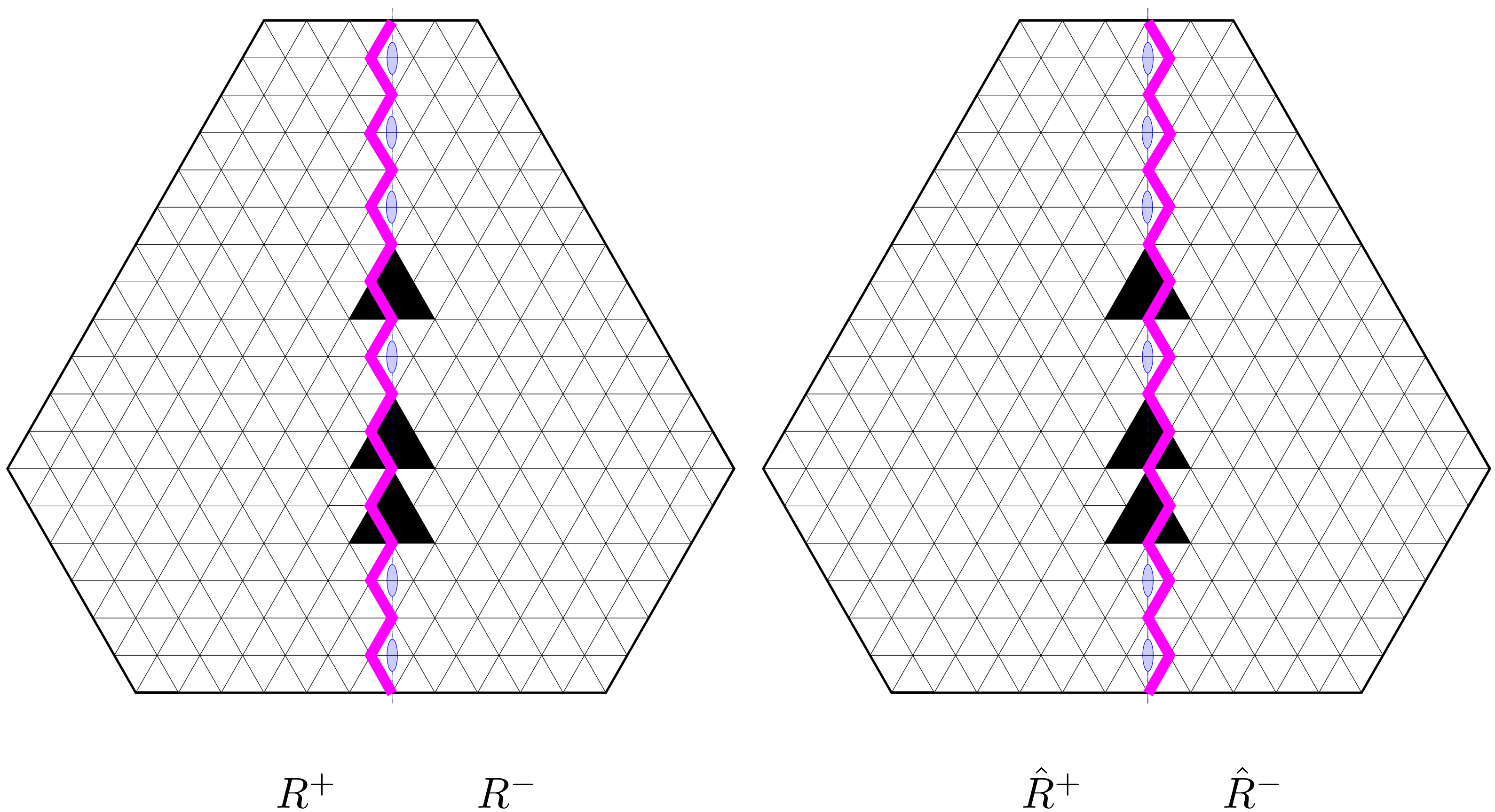
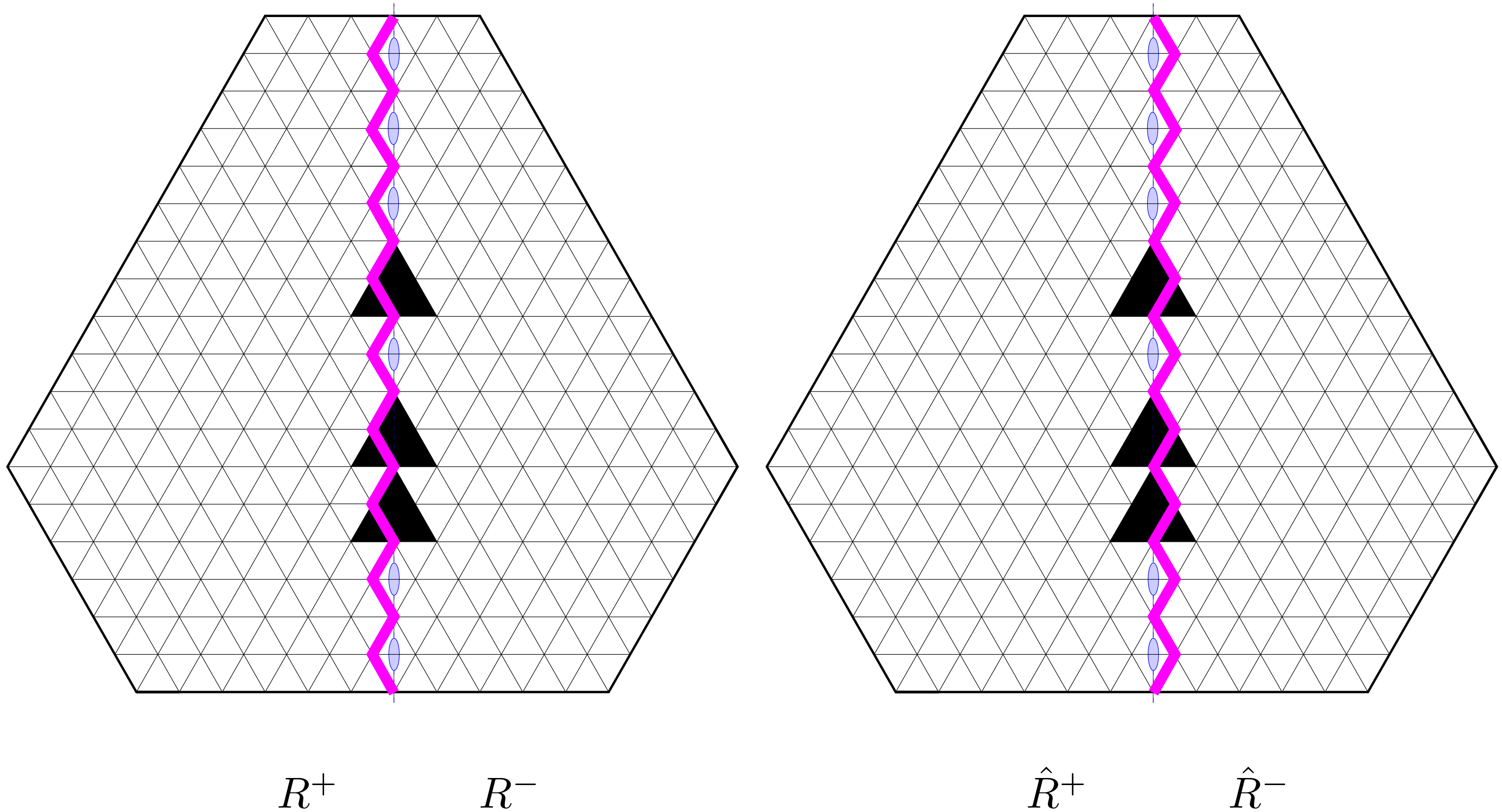
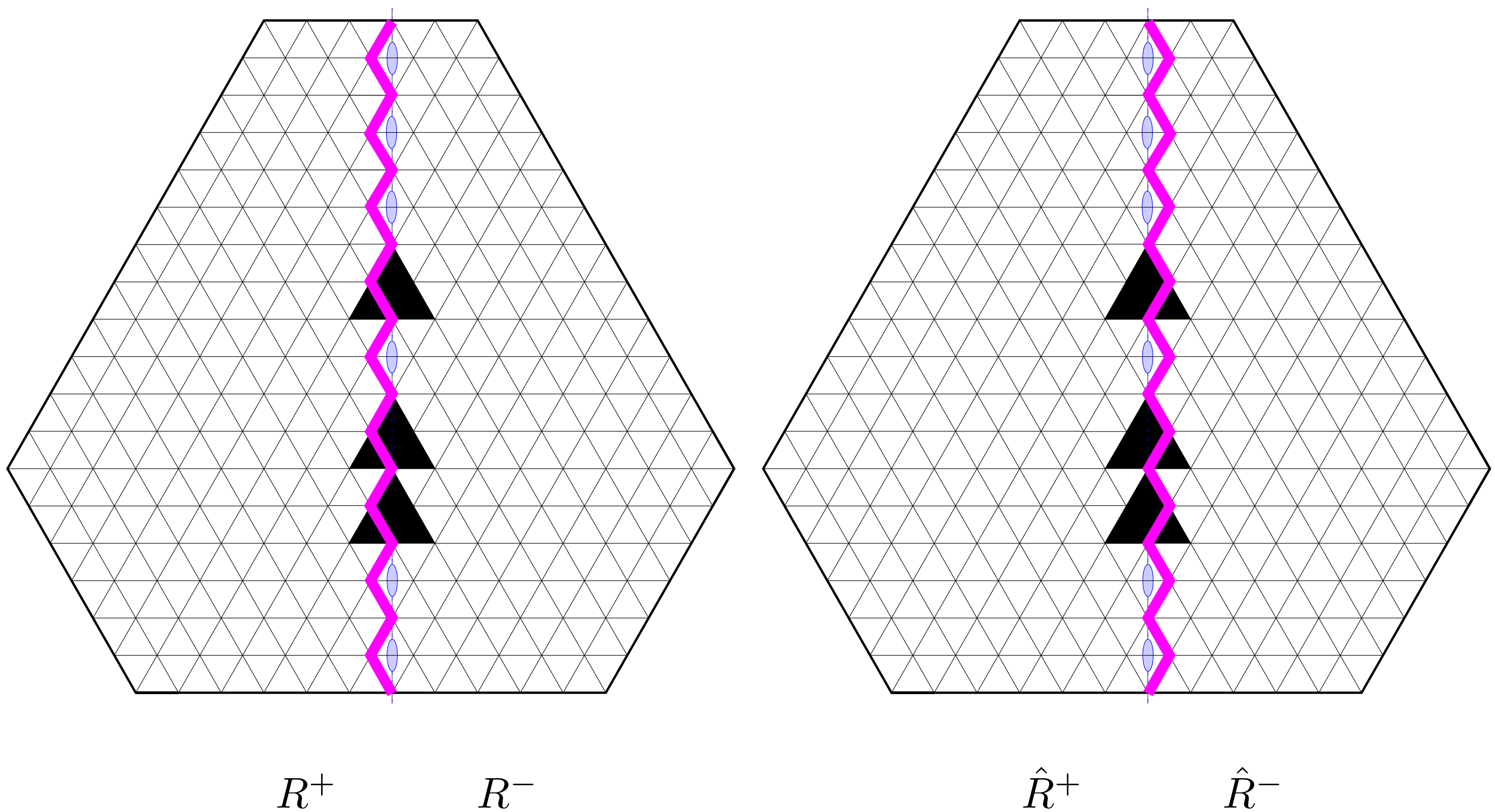


This solves Lai's two open problems because



R^+, $\hat{R}^-$ same family

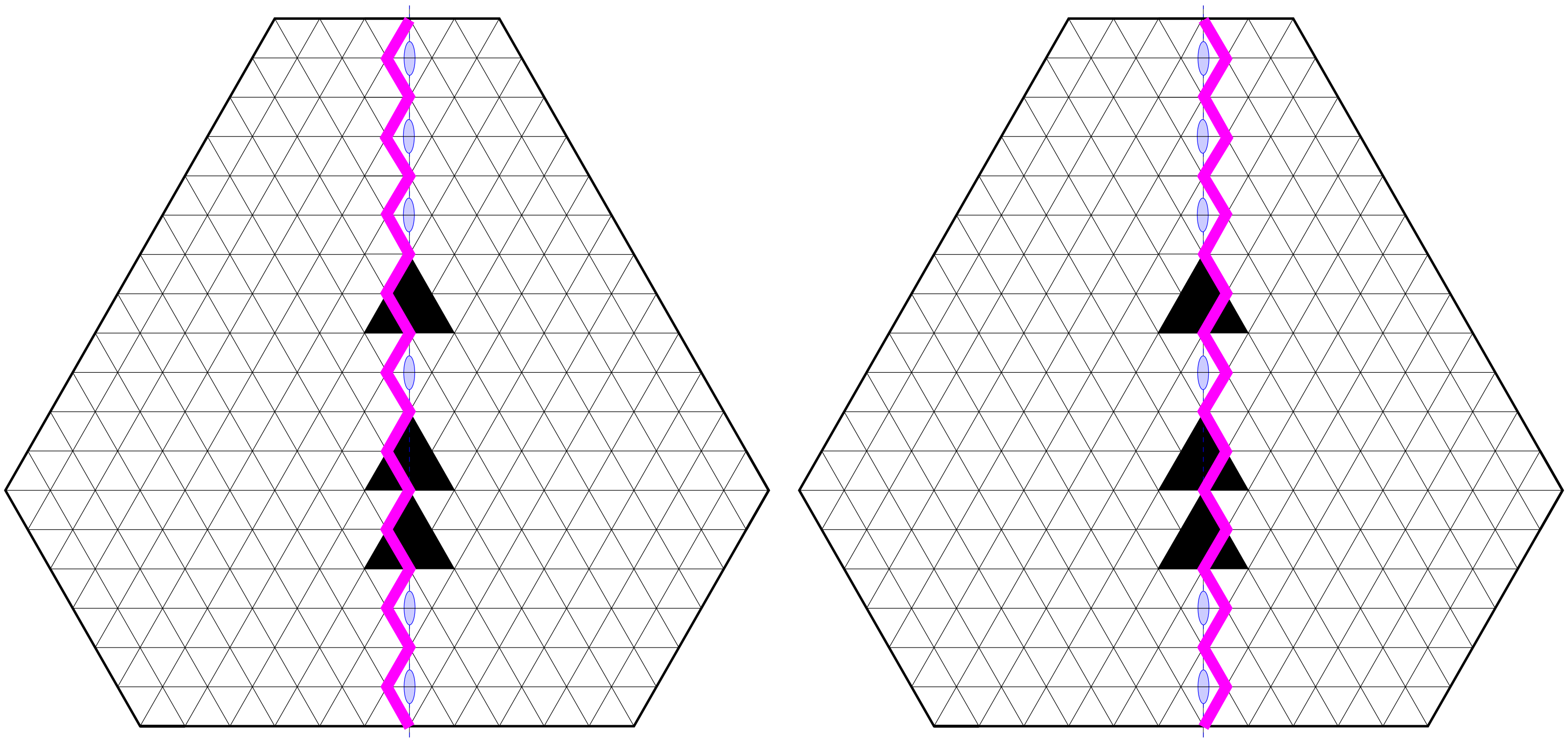
This solves Lai's two open problems because



R^+, $\hat{R}^-$ same family

R^-, $\hat{R}^+$ same family

This solves Lai's two open problems because



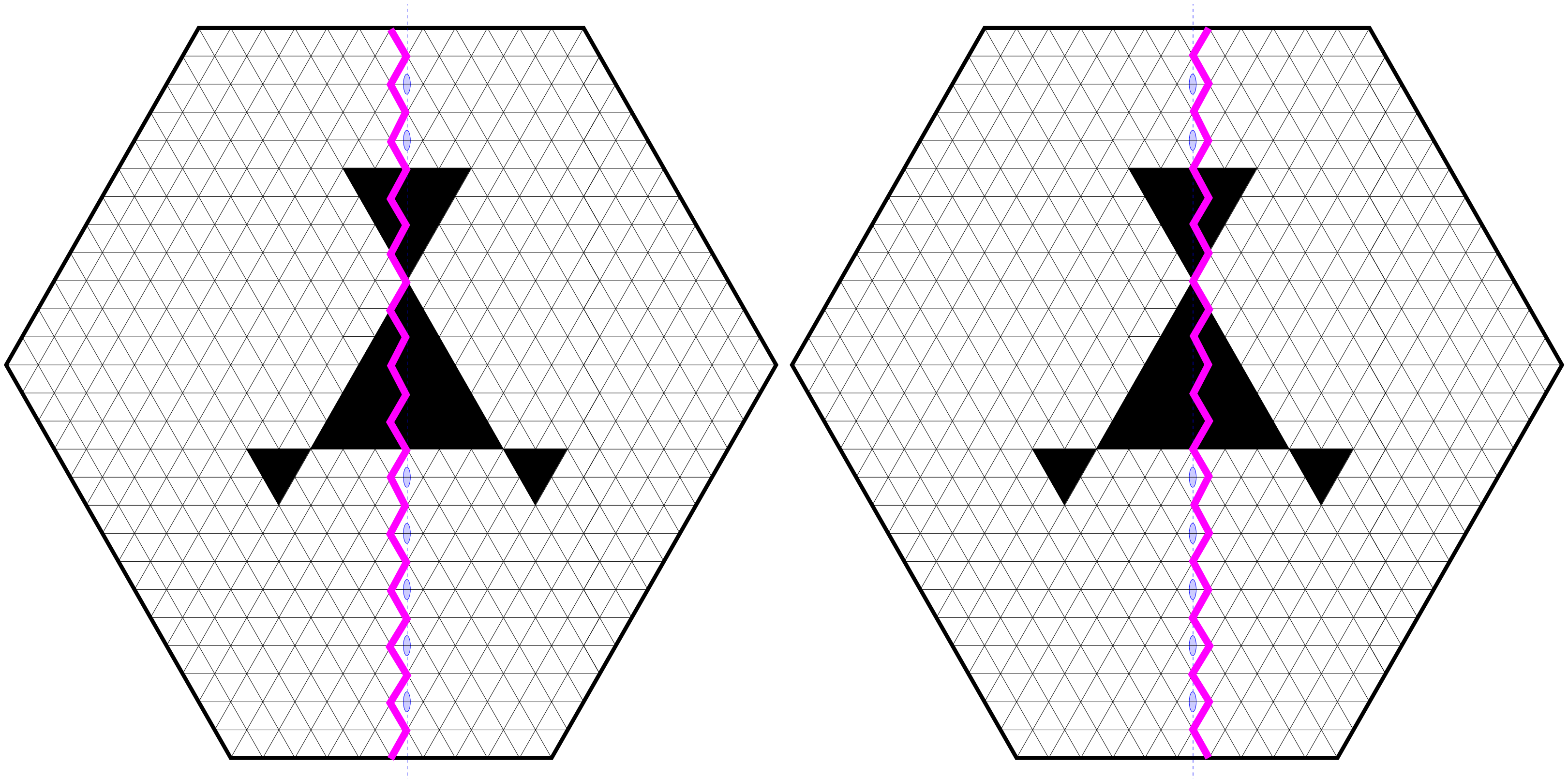
R^+ $\quad$ R^- $\qquad\qquad$ $\hat{R}^+$ $\quad$ $\hat{R}^-$

R^+, $\hat{R}^-$ same family

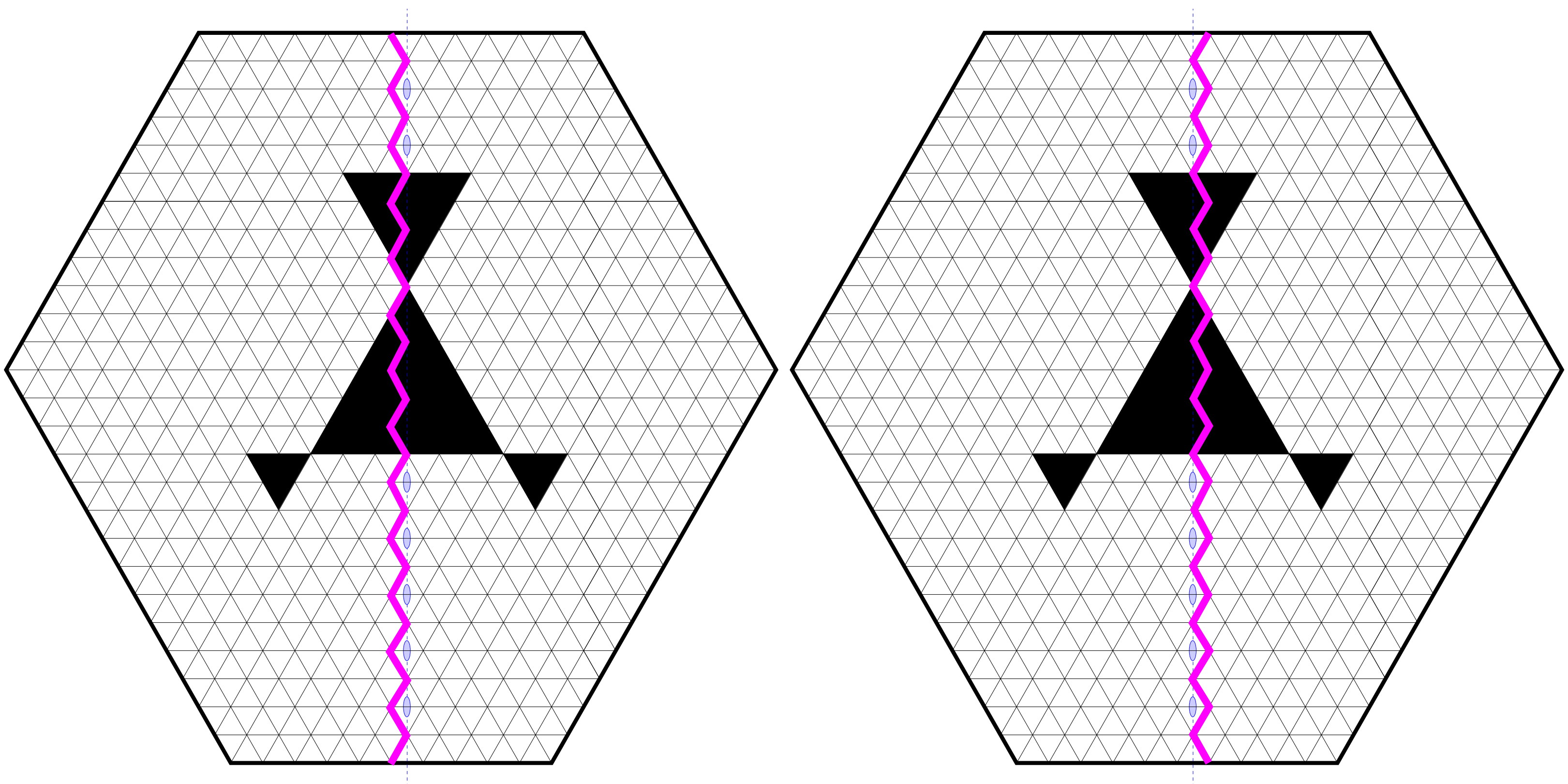
R^-, $\hat{R}^+$ same family

Theorem (C., 2005) Simple product formula for number of tilings of both families.

This solves Lai's two open problems because



This solves Lai's two open problems because



R^+ R^- $\hat{R}^+$ $\hat{R}^-$

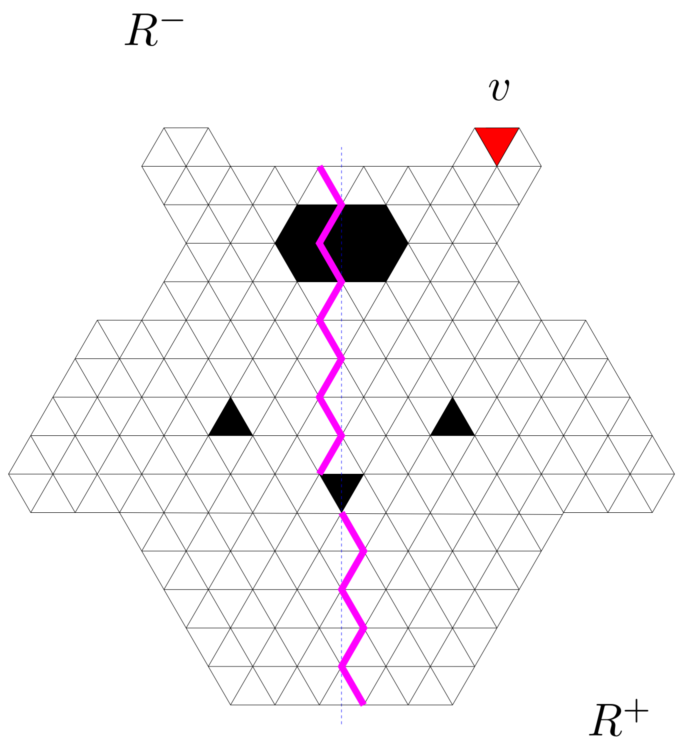
R^+, $\hat{R}^-$ same family; R^-, $\hat{R}^+$ same family

Theorem (C., 2016; Lai and Rohatgi, 2016) Simple product formula for number of tilings of both families.

Remark. In both cases, $\mathrm{M}(R^+)\,\mathrm{M}(R^-)+\mathrm{M}(\hat{R}^+)\,\mathrm{M}(\hat{R}^-)$ (by the above, a *sum of two simple product formulas*) turns out to simplify to a simple product formula.

Proof of semifactorization theorem

A variant of the factorization theorem: symmetric region with unit dent

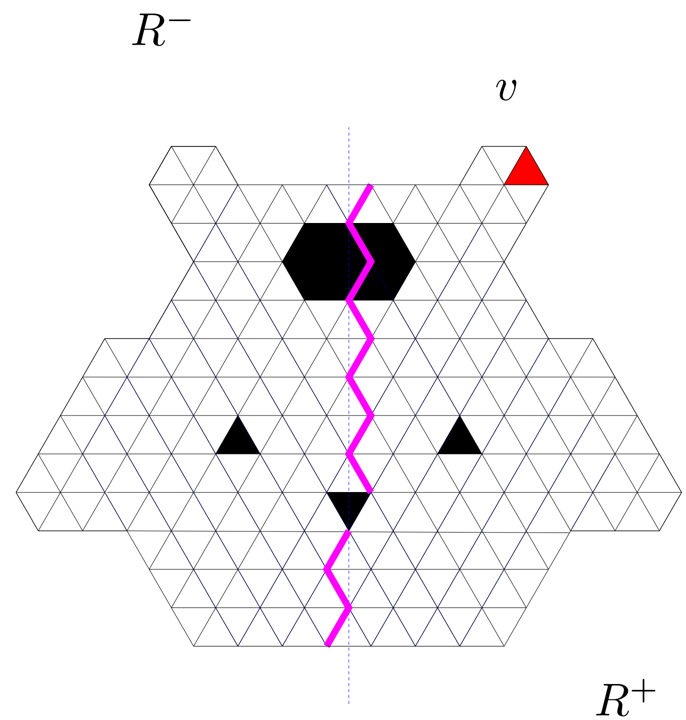


If v and top unit triangle along symmetry axis have *same* orientation:

$$\mathrm{M}(R) = 2^{n/2}\,\mathrm{M}(R^+ \setminus v)\,\mathrm{M}(R^-),$$

- n= # (unit squares on symmetry axis) -1
- zig-zag cut starts to the *left* of symmetry axis
- tile positions along symmetry axis weighted by 1/2

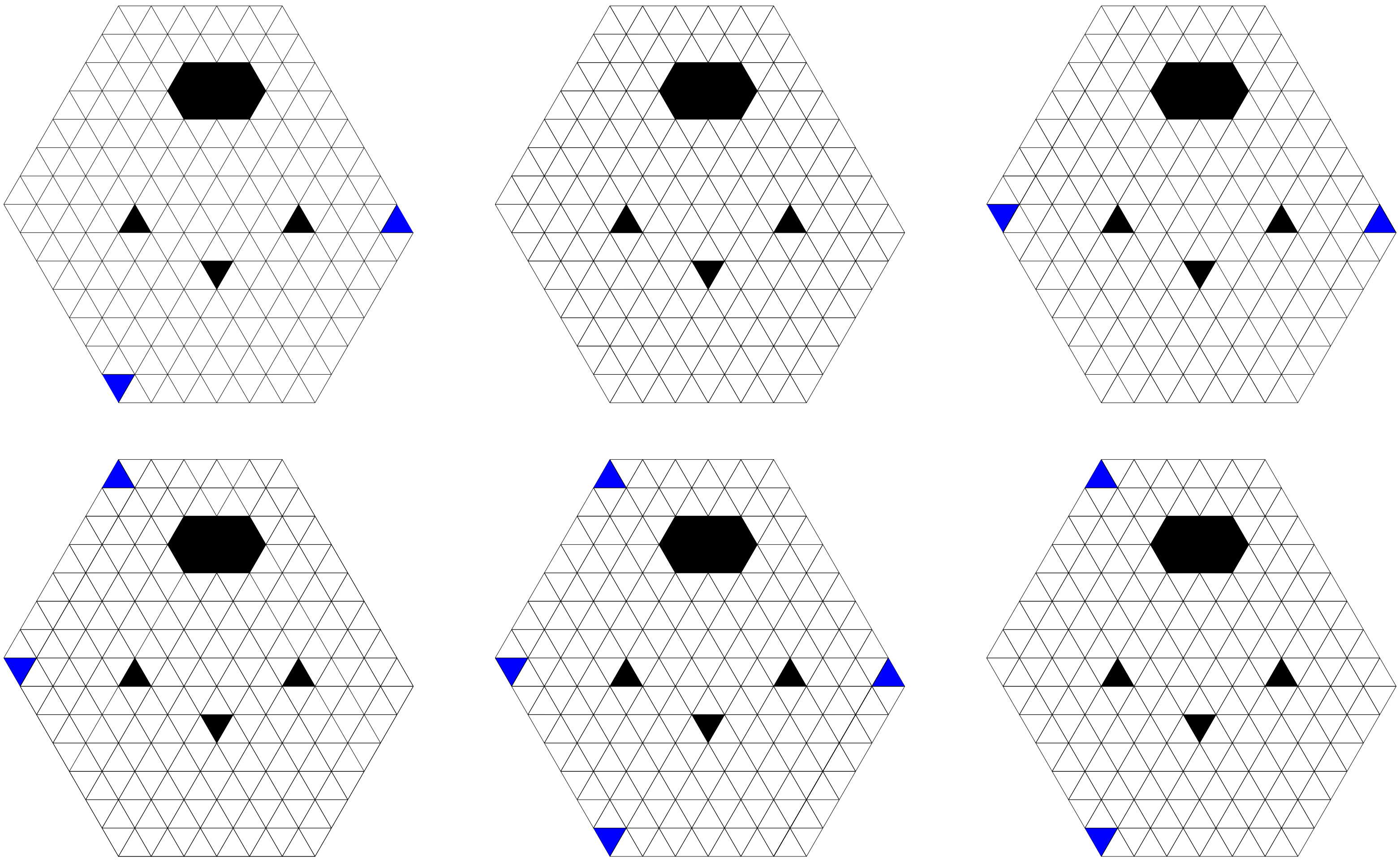
A variant of the factorization theorem: symmetric region with unit dent



If v and top unit triangle along symmetry axis have *opposite* orientation:

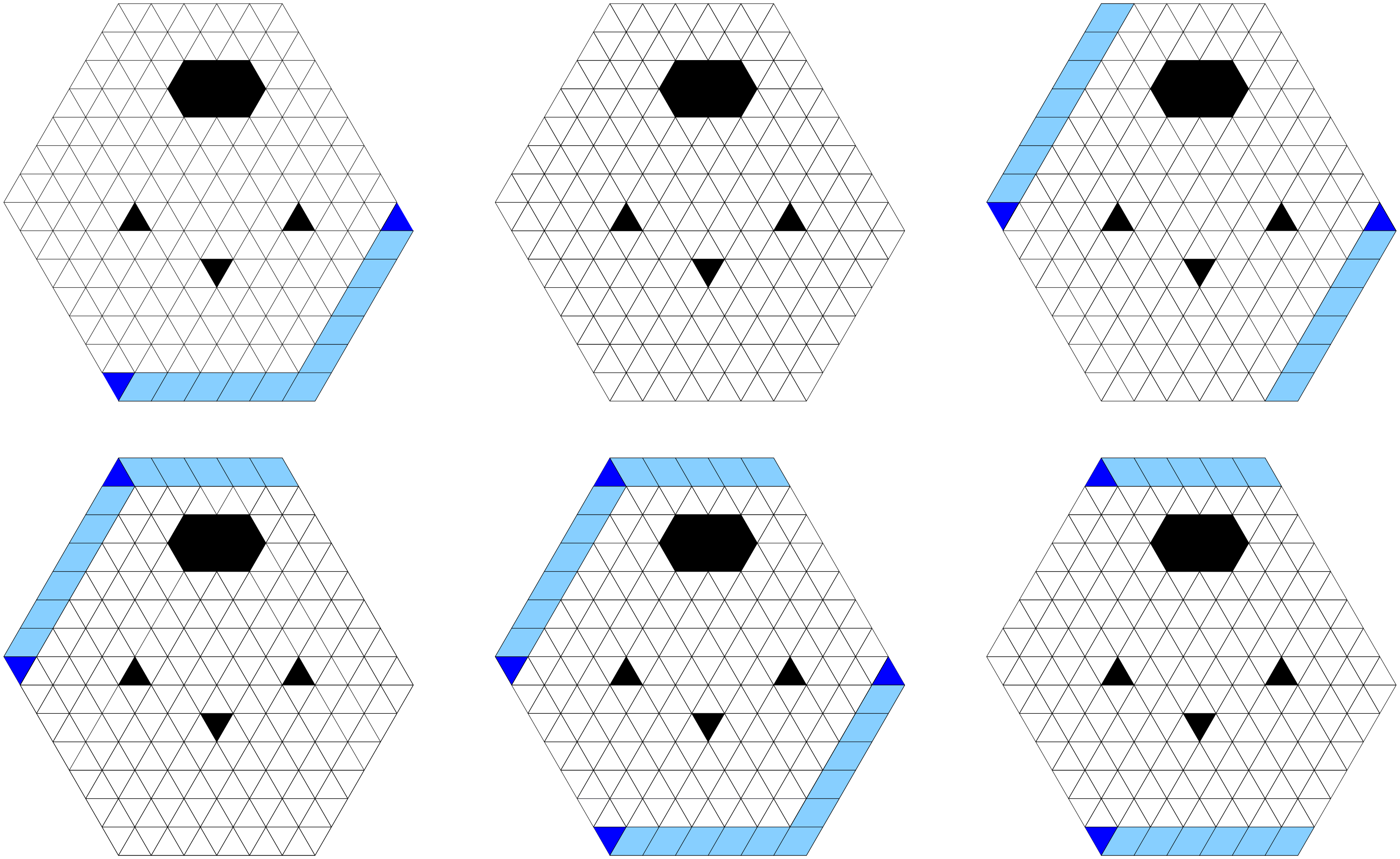
$$\mathrm{M}(R) = 2^{n/2}\,\mathrm{M}(R^+ \setminus v)\,\mathrm{M}(R^-),$$

- n= # (unit squares on symmetry axis) -1
- zig-zag cut starts to the *right* of symmetry axis
- tile positions along symmetry axis weighted by 1/2

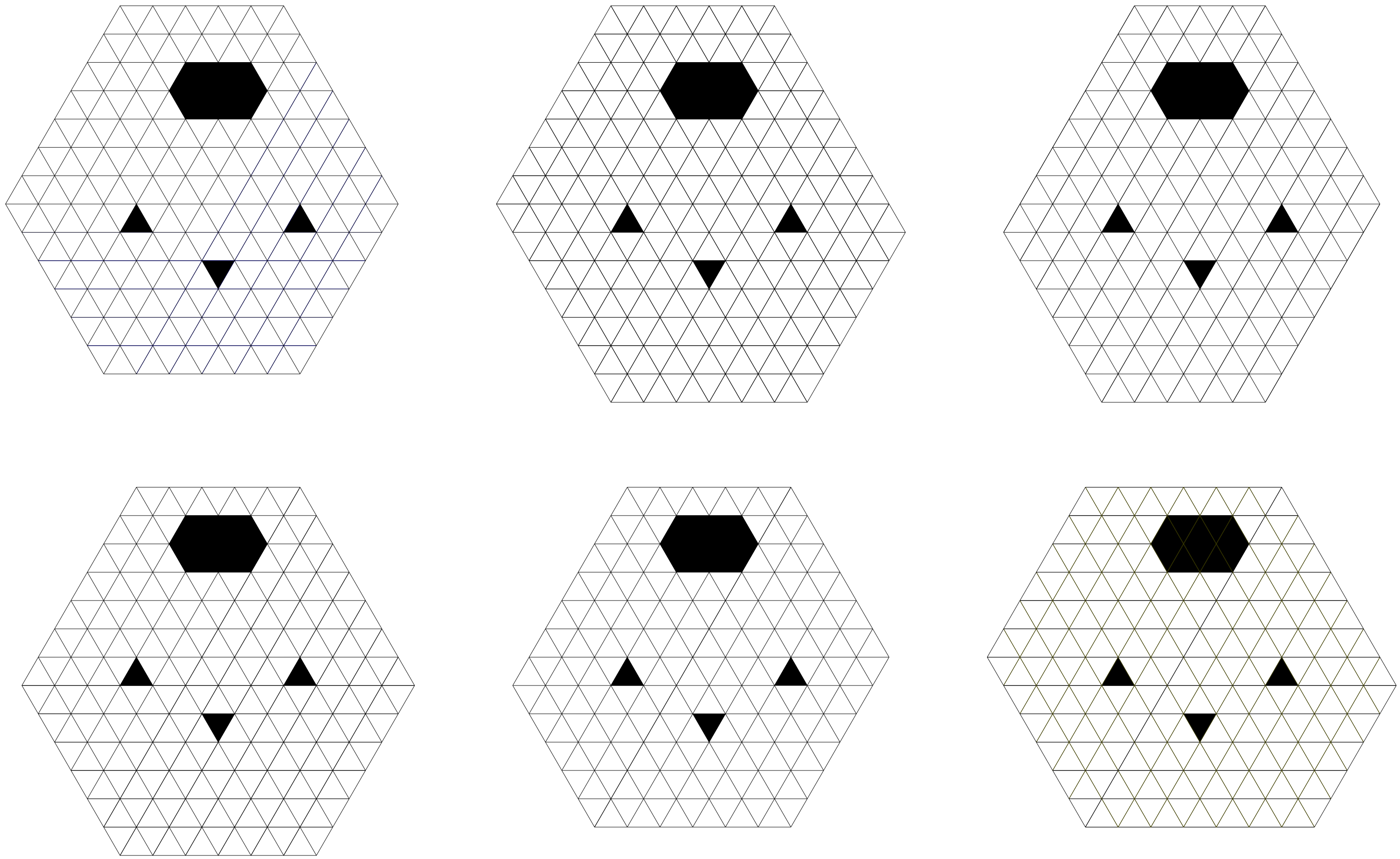


Kuo's graphical condensation method (Kuo, 2004):

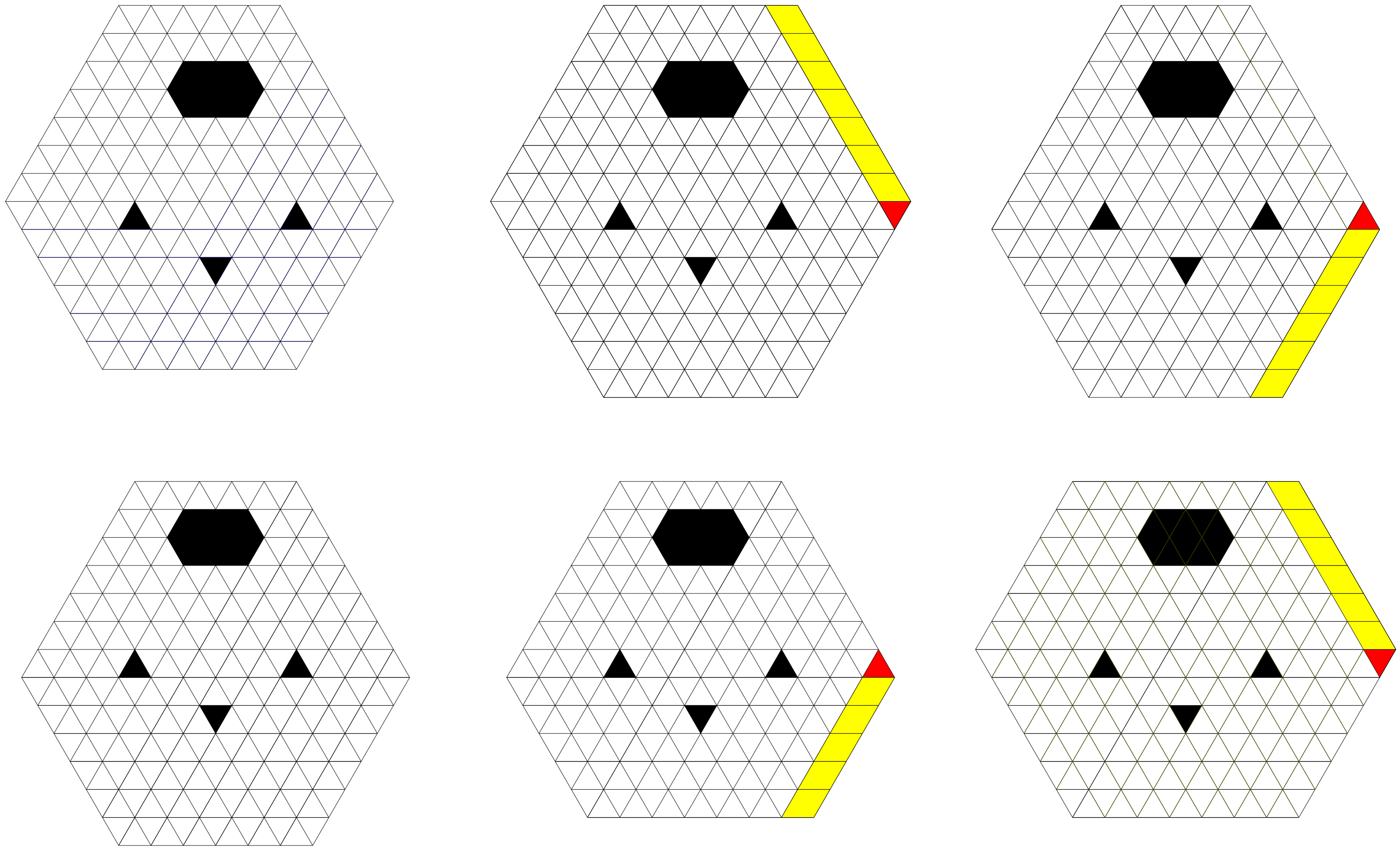
- blue triangles: a, b, c, d (counterclockwise from right)
- $\mathrm{M}(R \setminus a, d)\,\mathrm{M}(R \setminus b, c) = \mathrm{M}(R)\,\mathrm{M}(R \setminus a, b, c, d) + \mathrm{M}(R \setminus a, c)\,\mathrm{M}(R \setminus b, d)$



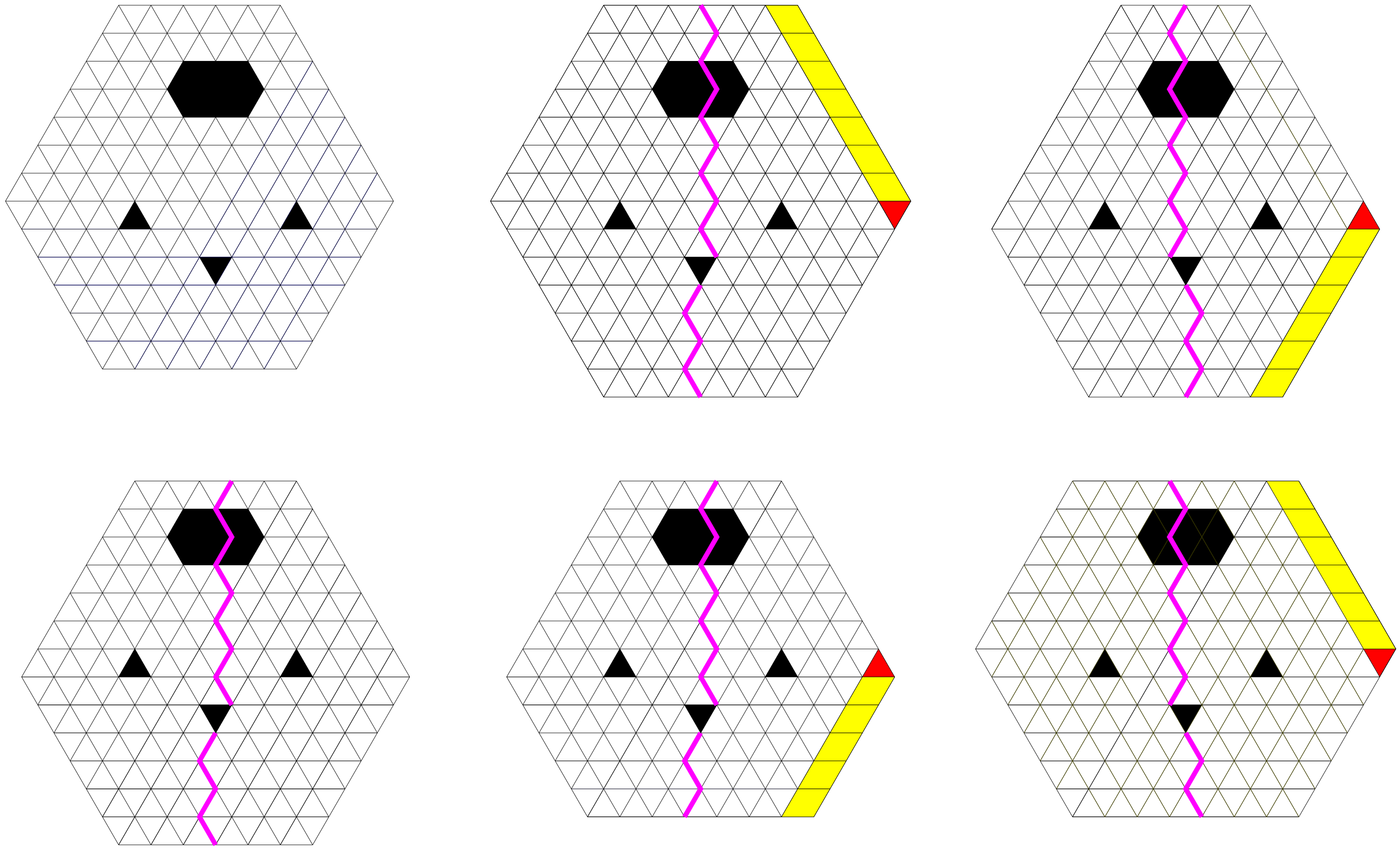
$$\mathrm{M}(R \setminus a, d)\,\mathrm{M}(R \setminus b, c) = \mathrm{M}(R)\,\mathrm{M}(R \setminus a, b, c, d) + \mathrm{M}(R \setminus a, c)\,\mathrm{M}(R \setminus b, d)$$



$$\mathrm{M(top)\,M(bottom) = M(top)\,M(bottom) + M(top)\,M(bottom)}$$

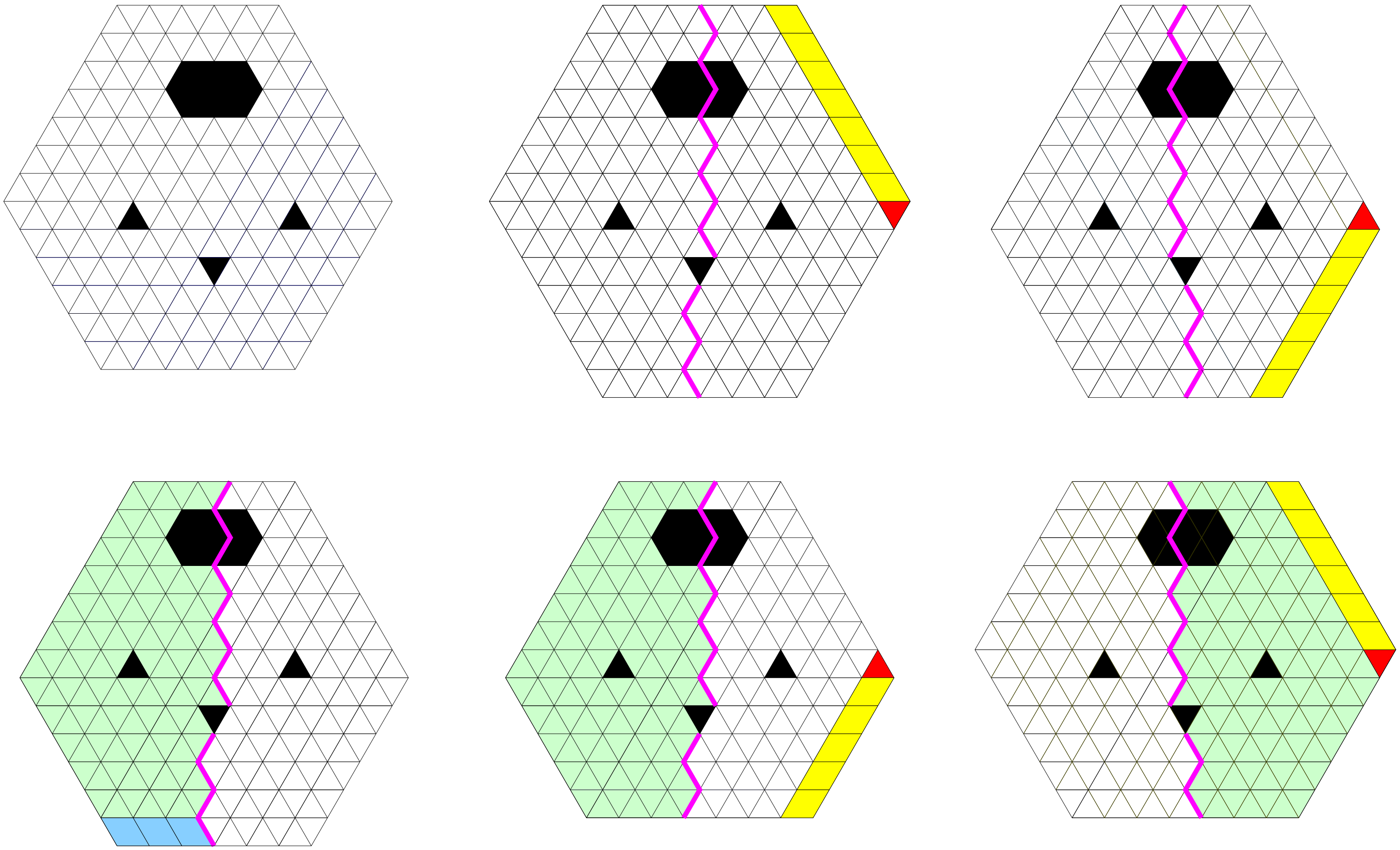


$$\mathrm{M(top)\,M(bottom) = M(top)\,M(bottom) + M(top)\,M(bottom)}$$



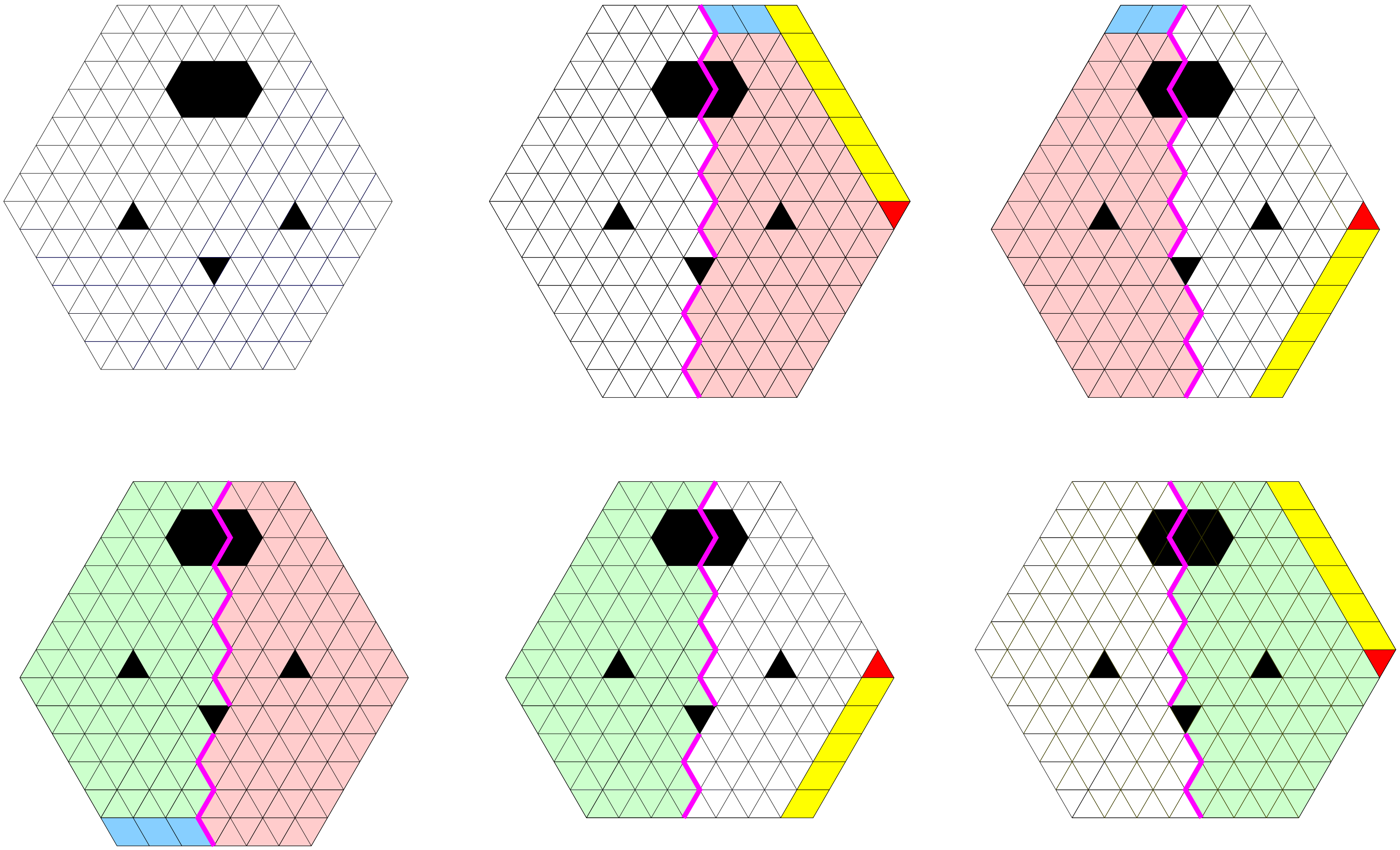
$$\mathrm{M(top)\,M(bottom)} = \mathrm{M(top)\,M(bottom)} + \mathrm{M(top)\,M(bottom)}$$

Apply FT or FT variant to the last 5 regions!



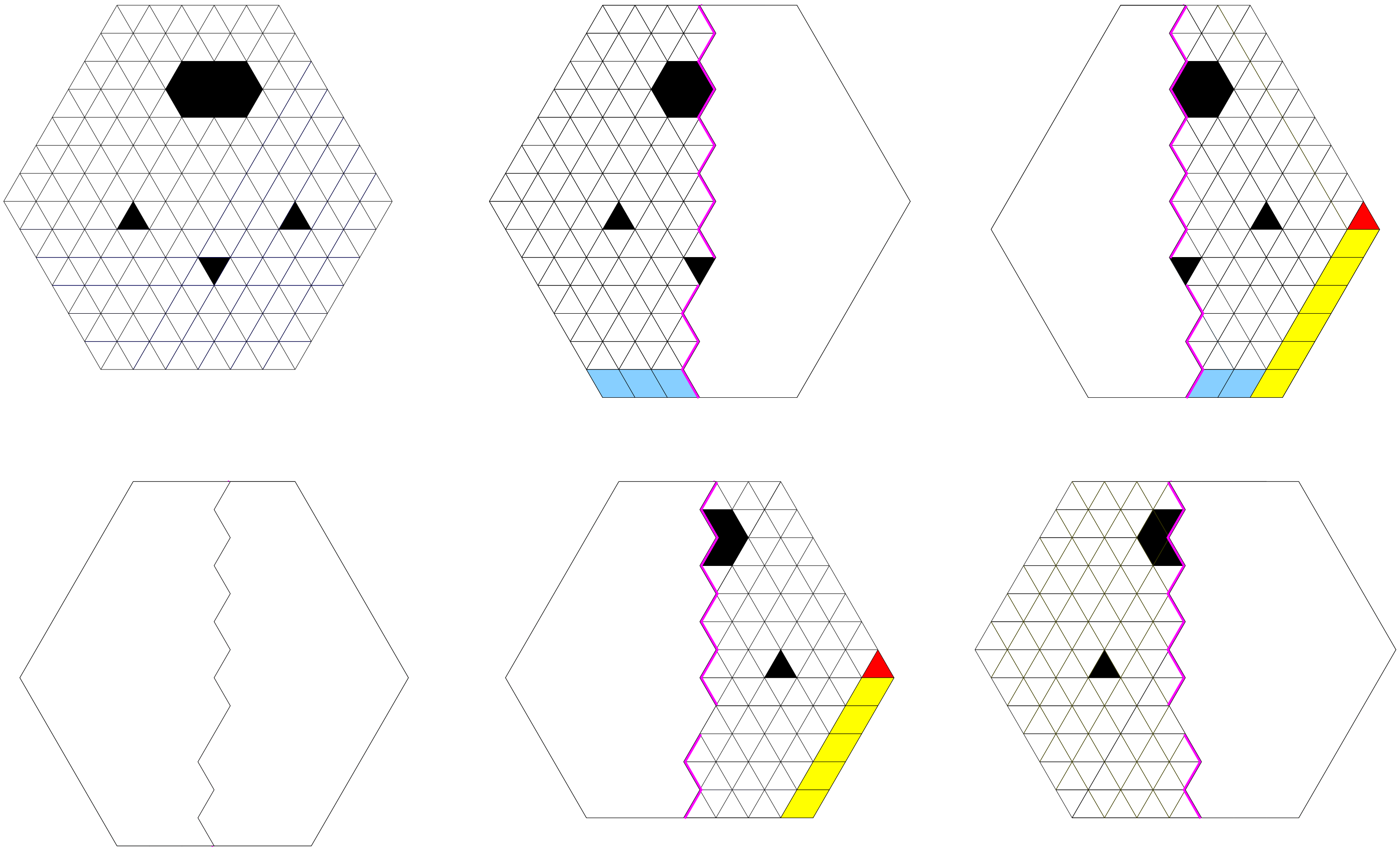
$$\mathrm{M(top)\,M(bottom) = M(top)\,M(bottom) + M(top)\,M(bottom)}$$

Green regions congruent



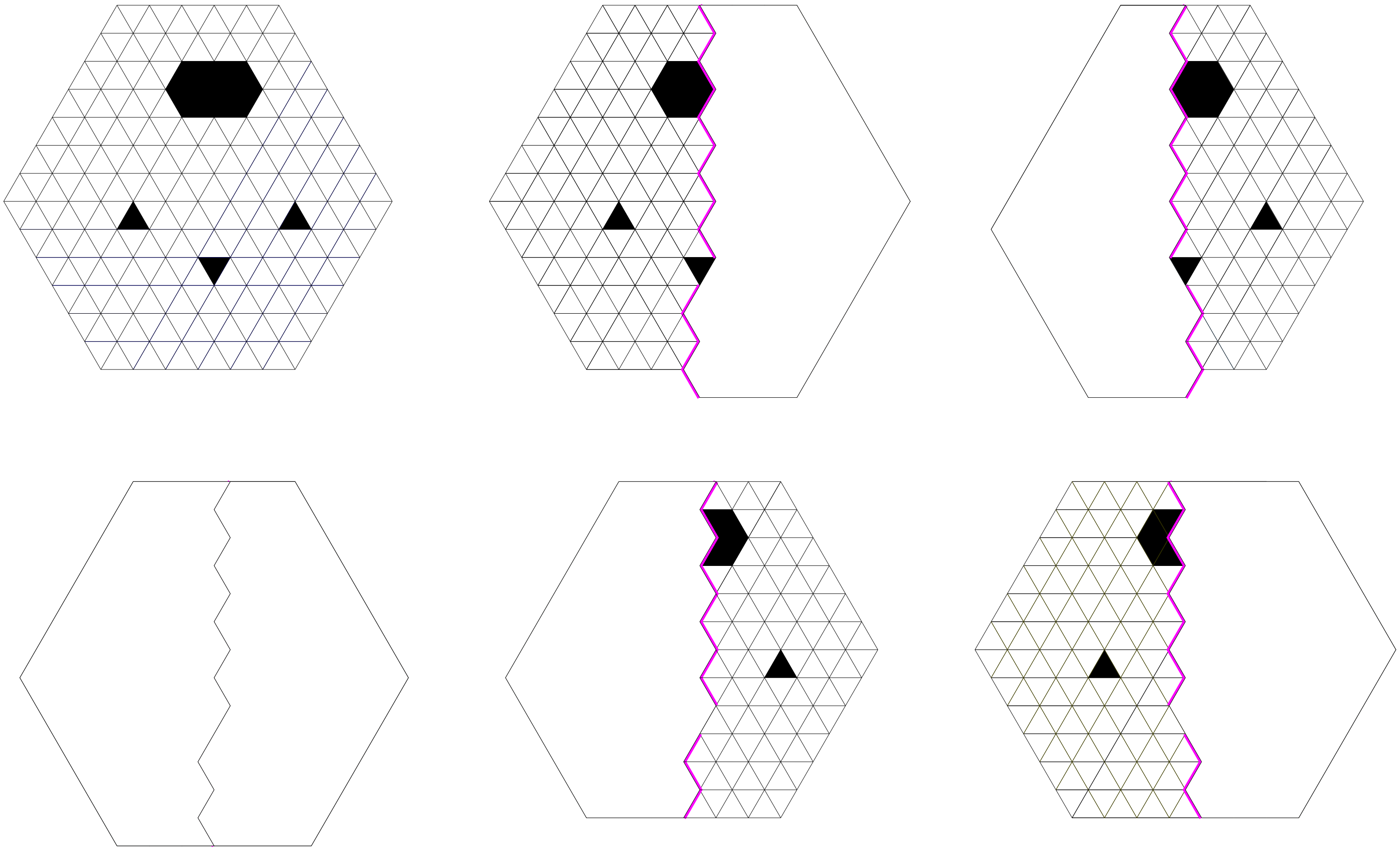
$$\mathrm{M(top)\,M(bottom)} = \mathrm{M(top)\,M(bottom)} + \mathrm{M(top)\,M(bottom)}$$

Green regions congruent, also red regions



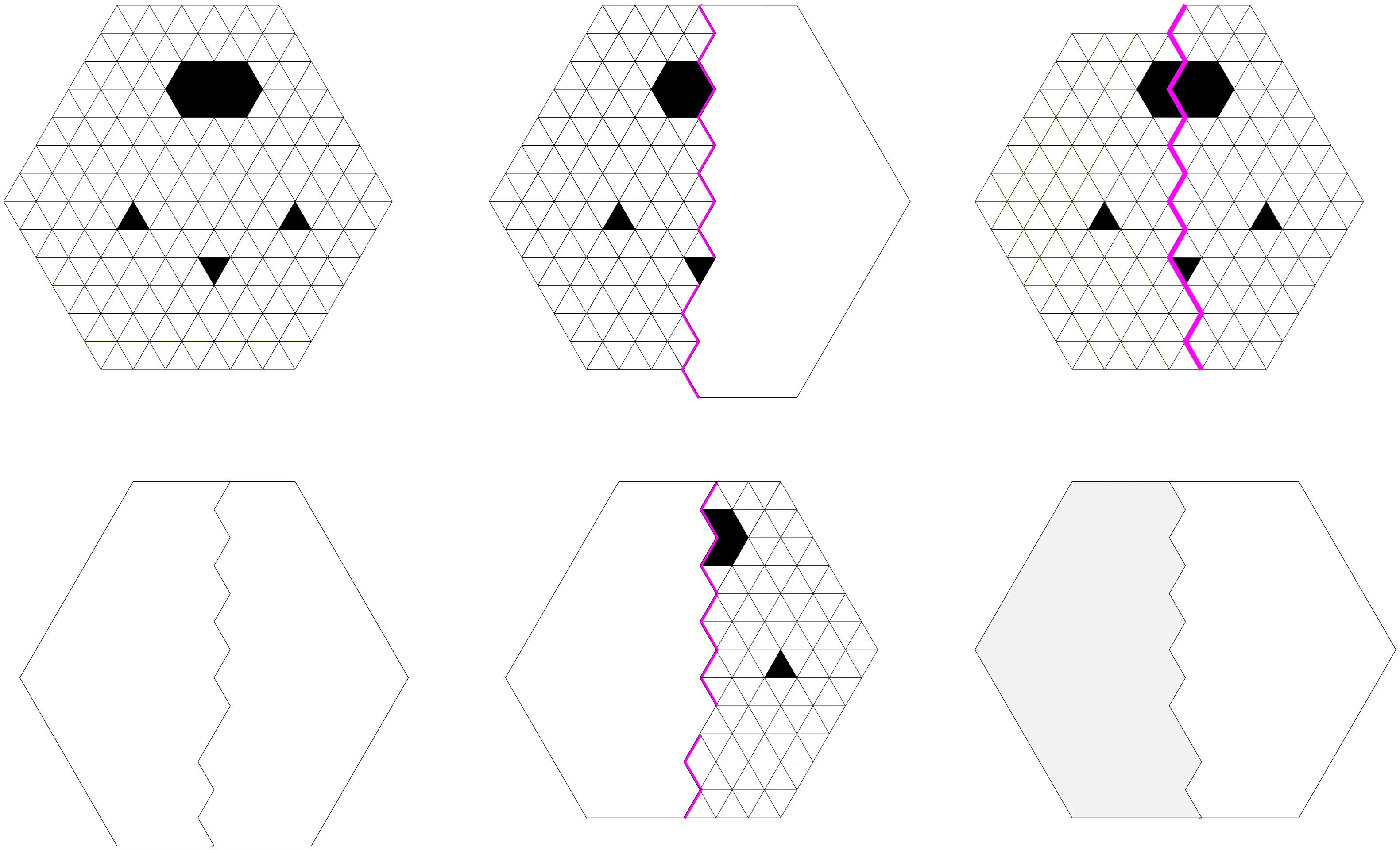
$$\mathrm{M}(\mathrm{top}) = 2^{\mathrm{w}(R)-1}\left(\mathrm{M}(\mathrm{top})\,\mathrm{M}(\mathrm{bottom}) + \mathrm{M}(\mathrm{top})\,\mathrm{M}(\mathrm{bottom})\right)$$

“Cancel” them out



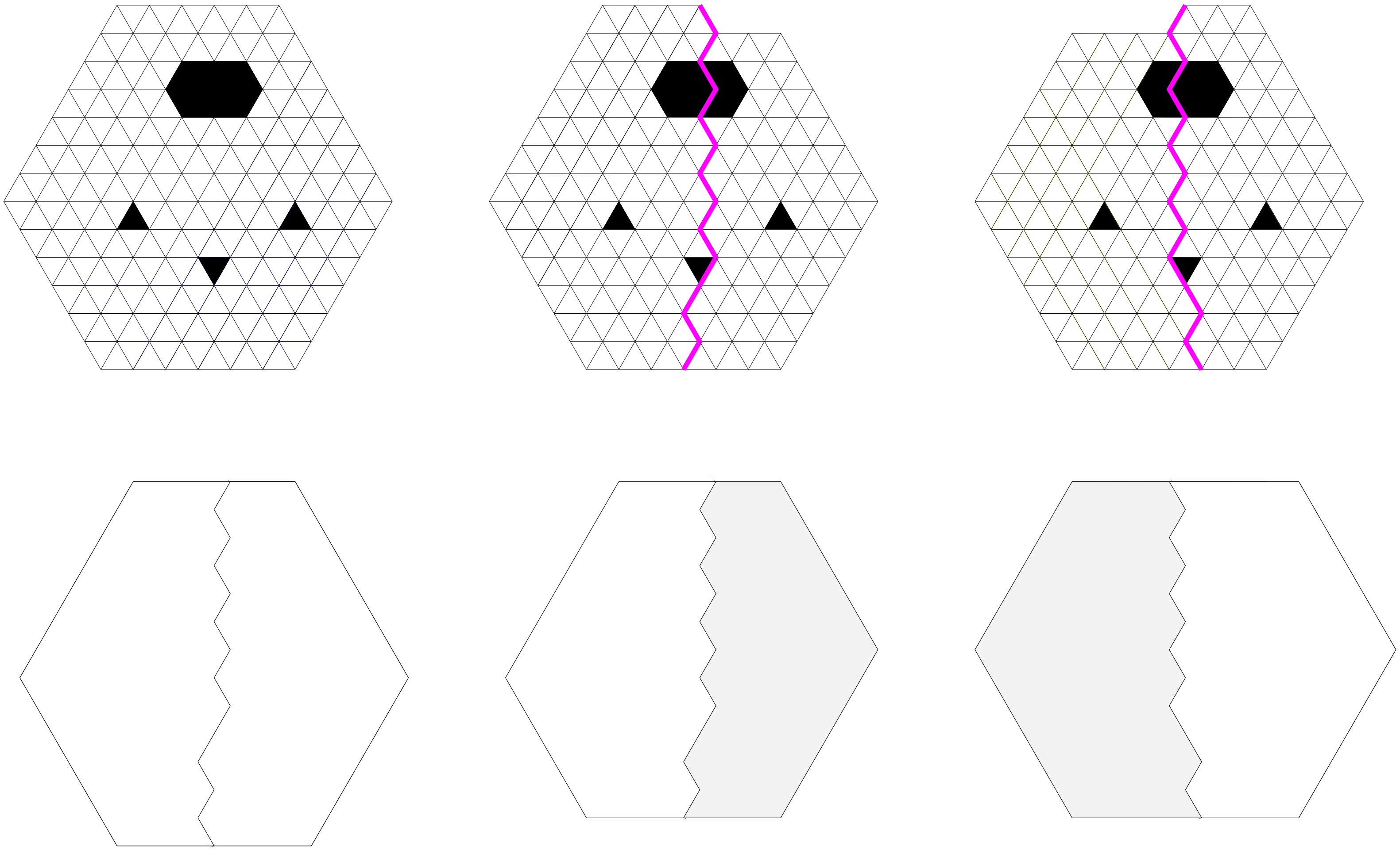
$$\mathrm{M}(\text{top}) = 2^{\mathrm{w}(R)-1}\left(\mathrm{M}(\text{top})\,\mathrm{M}(\text{bottom}) + \mathrm{M}(\text{top})\,\mathrm{M}(\text{bottom})\right)$$

Remove some forced lozenges



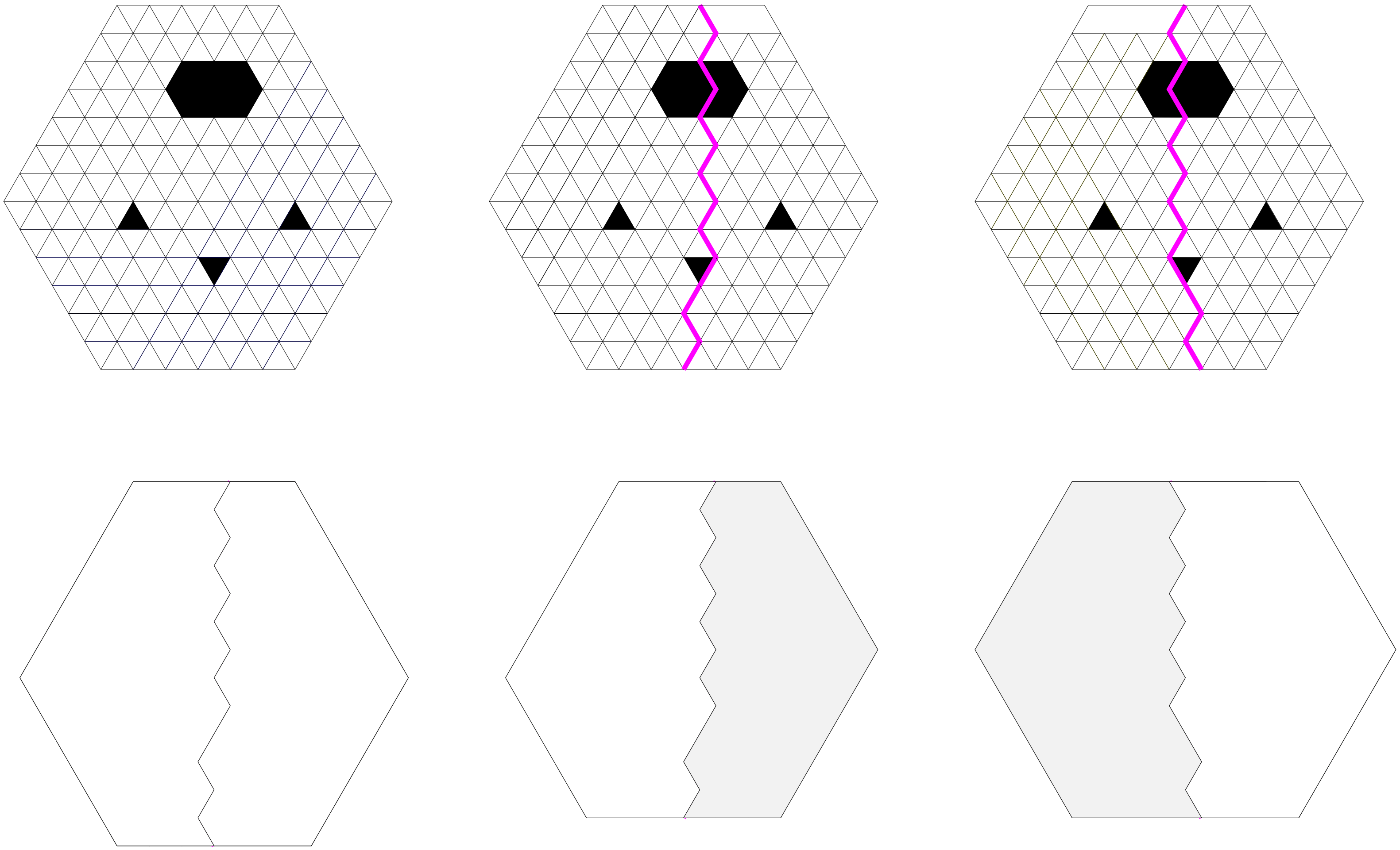
$$\mathrm{M(top)} = 2^{\mathrm{w}(R)-1}\left(\mathrm{M(top)}\,\mathrm{M(bottom)} + \mathrm{M(top)}\,\mathrm{M(bottom)}\right)$$

Move bottom right region to top right



$$\mathrm{M(top)} = 2^{\mathrm{w}(R)-1}\left(\mathrm{M(top)}\,\mathrm{M(bottom)} + \mathrm{M(top)}\,\mathrm{M(bottom)}\right)$$

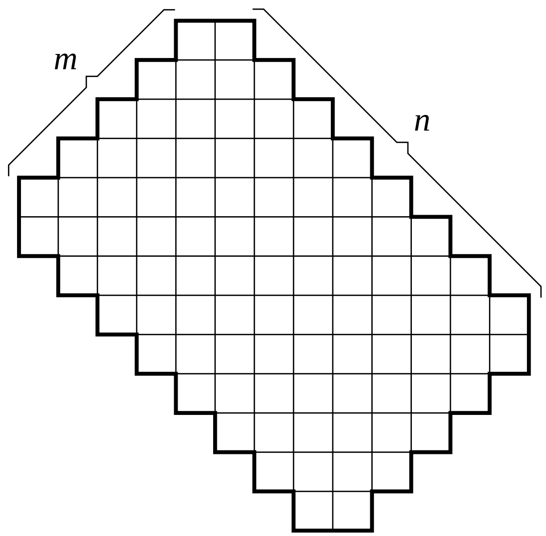
Move bottom right region to top right, bottom center to top center



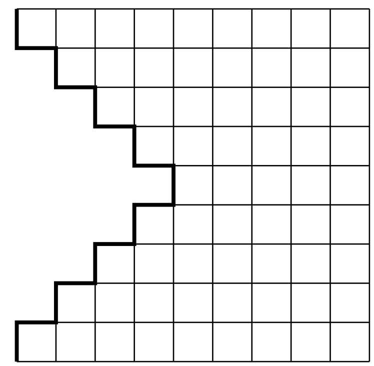
$$\mathrm{M}(\text{top}) = 2^{\mathrm{w}(R)-1}\,(\mathrm{M}(\text{top})\,\mathrm{M}(\text{bottom}) + \mathrm{M}(\text{top})\,\mathrm{M}(\text{bottom}))$$

The four regions on top center and right are precisely R^+, R^-, $\hat{R}^+$, $\hat{R}^-$

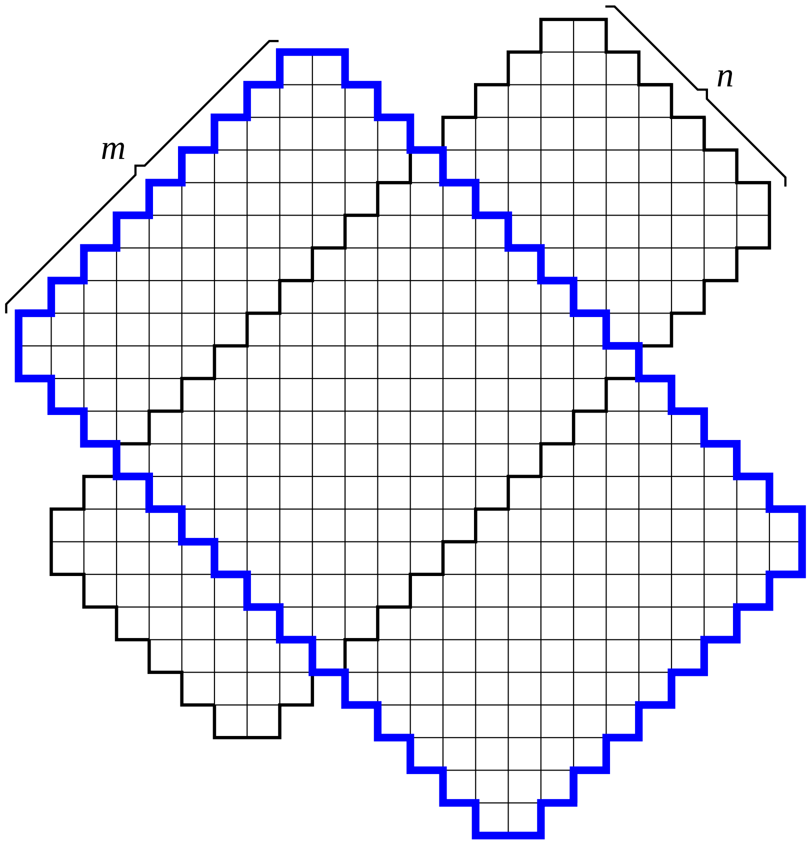
Cruciform regions

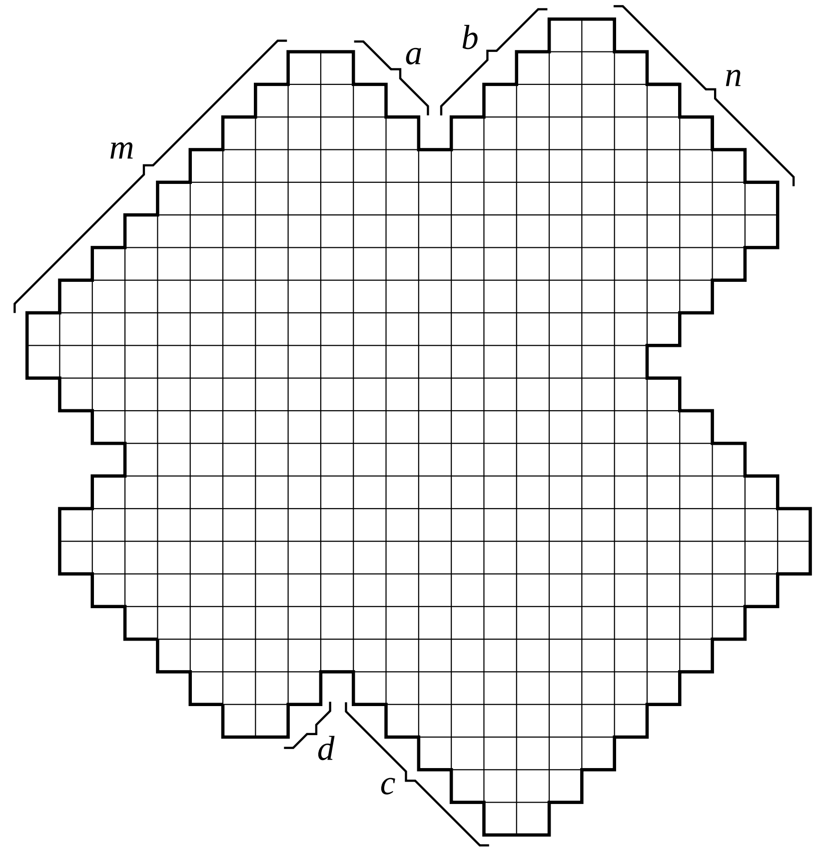


The Aztec rectangle region $AR_{m,n}$ for $m = 5$, $n = 8$



Allowed corner type when two AR's are superimposed





The cruciform region $C_{m,n}^{a,b,c,d}$ for
$m = 9$, $n = 6$, $a = 3$, $b = 4$, $c = 5$, $d = 2$.

Call the 4 protrusions *piers* (NW, NE, SE, SW).
They stick out a, b, c and d units.

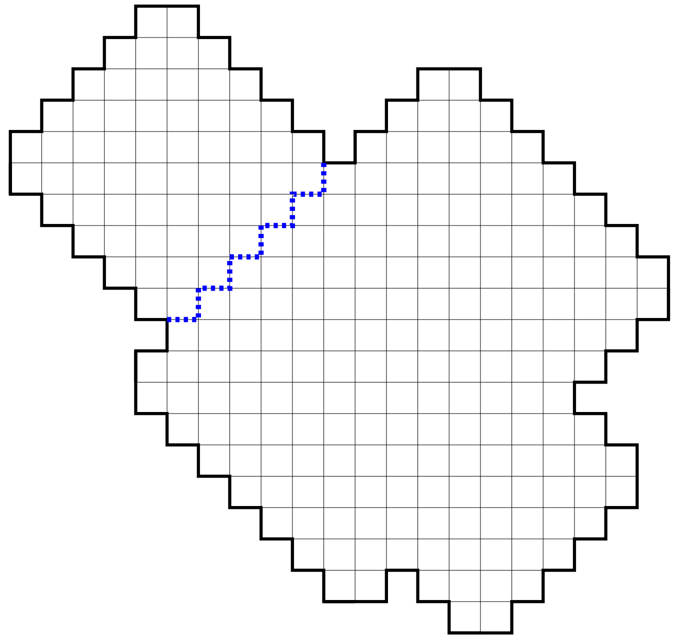
Theorem (C., 2022). *Let $C_{m,n}^{a,b,c,d}$ be a tileable cruciform region. Then*

$$\mathrm{M}(C_{m,n}^{a,b,c,d}) = 2^{\left\{\frac{1}{4}m(3m+1)+\frac{1}{4}n(3n+1)-\frac{1}{2}(a+c)(b+d)-\frac{1}{4}(m-n)(a-b+c-d)\right\}}$$

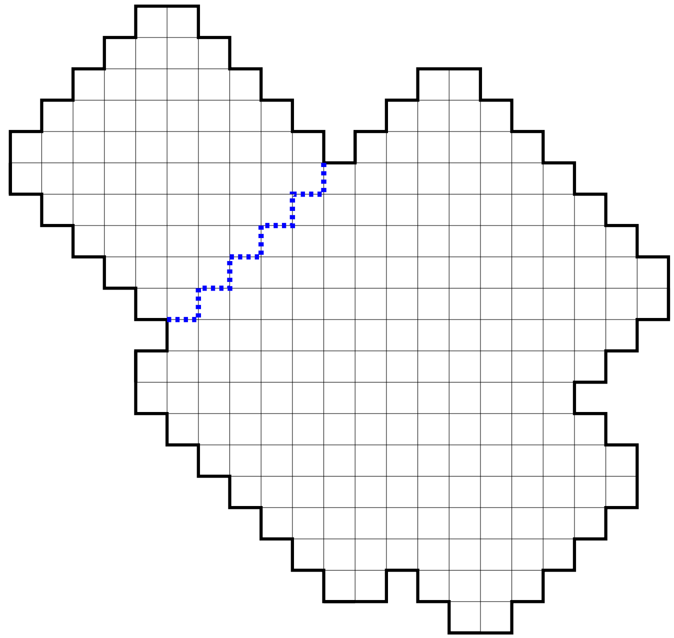
$$\times \frac{\mathrm{H}(m+n+1)^2\,\mathrm{H}(m-a)\,\mathrm{H}(n-b)\,\mathrm{H}(m-c)\,\mathrm{H}(n-d)}{\mathrm{H}(n+a+1)\,\mathrm{H}(m+b+1)\,\mathrm{H}(n+c+1)\,\mathrm{H}(m+d+1)},$$

where $H(n) = 0!\,1!\,\cdots\,(n-1)!$

Special case $a = n$:

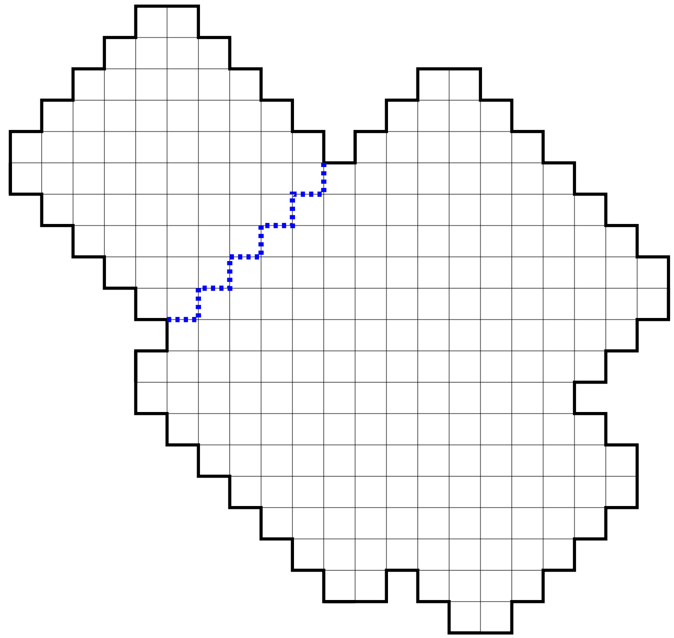


Special case $a = n$:



Region above blue zig-zag is AD_n, and must be internally tiled.

Special case $a = m$:



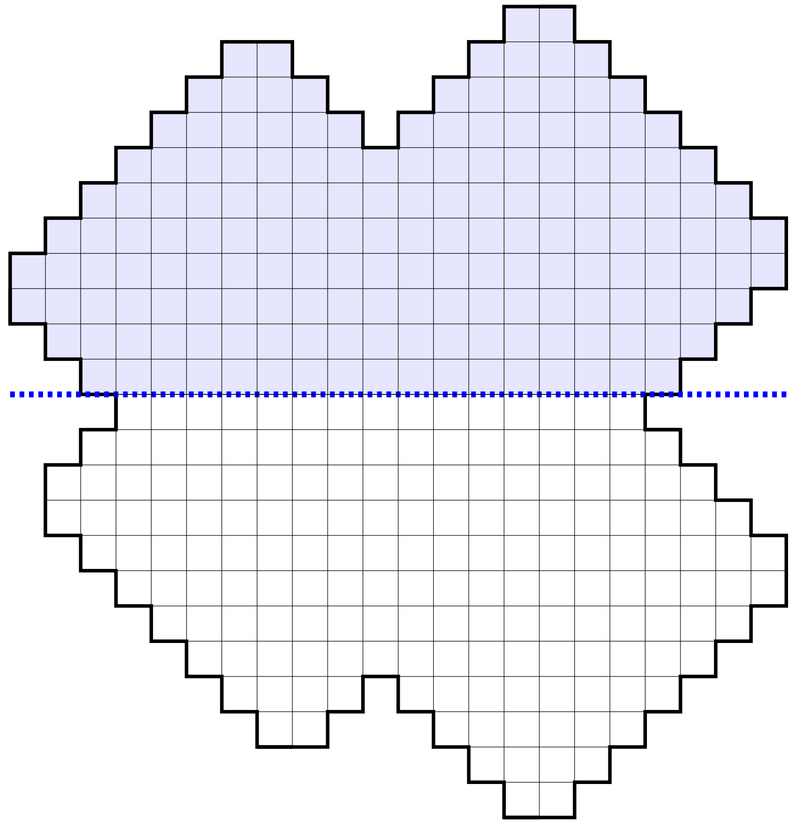
Region above blue zig-zag is AD_m, and must be internally tiled.

COROLLARY. *Let m, n, b, c, d be integers with $m, n, c \geq 0$ and $b + c + d = n - 1$. Then the number of domino tilings of the T-region $T_{m,n}^{b,c,d}$ is given by*

$$\mathrm{M}(T_{m,n}^{b,c,d}) = 2^{\left\{\frac{1}{4}m(m-1)+\frac{1}{4}n(3n+1)+\frac{1}{2}(m+c)(b+d)-\frac{1}{4}(m-n)(m-b+c-d)\right\}}$$

$$\times \frac{\mathrm{H}(m+n+1)\,\mathrm{H}(n-b)\,\mathrm{H}(m-c)\,\mathrm{H}(n-d)}{\mathrm{H}(m+b+1)\,\mathrm{H}(n+c+1)\,\mathrm{H}(m+d+1)}.$$

The elbow regions $E_n^{a,b}$



The elbow region $E_n^{a,b}$ consists of the portion of the cruciform region $C_{n,n}^{a,b,b,a-1}$ that is above its central horizontal row of unit squares (shown here is the case $n = 7$, $a = 3$, $b = 4$). Since $a + b + b + (a - 1) = n + n - 1$, we have $a + b = n$.

THEOREM 2 (C., 2022). *Let $E_n^{a,b}$ be a tileable elbow region. Then*

$$\mathrm{M}(E_n^{a,b}) = 2^{n(n+1)/2}\, n!\, \frac{\mathrm{H}(2n+1)\,\mathrm{H}(a)\,\mathrm{H}(b)}{\mathrm{H}(n+a+1)\,\mathrm{H}(n+b+1)}.$$

Happy Birthday, Jim!