Math 675: Analytic Theory of Numbers
Solutions to problem set # 6

April 23, 2024

1. (a) Let $T \geq x \geq 2$ with x not an integer and set $c = 1 + \frac{1}{\log x}$. As we saw in class, there exists a $T' \in [T, T+1]$ such that $|T' - \gamma| \gg \frac{1}{\log(qx)}$ for all nontrivial zeros of $L(s,\chi)$ by the pigeonhole principle. We also showed that

$$\psi(x,\chi) = -\frac{1}{2\pi i}\int_{c-iT'}^{c+iT'} \frac{L'}{L}(s,\chi)\frac{x^s}{s}\mathrm{d}s + O\left(\frac{x(\log x)^2}{T} + \frac{x\log x}{T\langle x\rangle}\right).$$

Shifting the contour to all the way to the left and using the estimates for $\frac{L'}{L}(s,\chi)$ obtained in class yields

$$\psi(x,\chi) = -\sum_{|\rho|\leq T}\frac{x^\rho}{\rho} - \frac{L'}{L}(0,\chi) + \sum_{m=1}^{\infty}\frac{x^{1-2m}}{2m-1} + O\left(\frac{x(\log^2(qx) + \log(qT))}{T} + \frac{x\log x}{T\langle x\rangle}\right)$$

since $\frac{L'}{L}(s,\chi)\frac{x^s}{s}$ has a simple pole at 0 with residue $\frac{L'}{L}(0,\chi)$ (recall that, since χ is odd, $L(0,\chi) \neq 0$ by the functional equation and the fact that $\Gamma(s)$ never vanishes), a simple pole at each trivial zero $1-2m$ of $L(s,\chi)$ with residue $\frac{x^{1-2m}}{1-2m}$, and a simple pole at each nontrivial zero ρ of $\frac{L'}{L}(s,\chi)$ with residue $m_\rho\frac{x^\rho}{\rho}$, where m_ρ denotes the multiplicity of ρ as a zero of $L(s,\chi)$. Taking $T \to \infty$ thus yields

$$\psi(x,\chi) = -\sum_{\rho}\frac{x^\rho}{\rho} - \frac{L'}{L}(0,\chi) + \sum_{m=1}^{\infty}\frac{x^{1-2m}}{2m-1}$$

whenever x is not an integer. If $n \in \mathbf{N}$ is not a prime power, then $\psi(n,\chi) = \psi(n+\varepsilon,\chi)$ and $\langle n+\varepsilon\rangle \geq \frac{\langle n\rangle}{2}$ (say) for all $\varepsilon \in (0,1/2)$. Thus,

$$\psi(n,\chi) = -\sum_{|\rho|\leq T}\frac{(n+\varepsilon)^\rho}{\rho} - \frac{L'}{L}(0,\chi) + \sum_{m=1}^{\infty}\frac{(n+\varepsilon)^{1-2m}}{2m-1} + O\left(\frac{n(\log^2(qn) + \log(nT))}{T} + \frac{n\log n}{T\langle n\rangle}\right)$$

for all $\varepsilon \in (0,1/2)$. Taking $\varepsilon \to 0$ and then $T \to \infty$ yields the desired result for integers that are not prime powers as well.

(b) We proceed analogously to above, with the only difference being that, since χ is even, $L(s,\chi)$ has trivial (simple) zeros at $0,-2,-4,\ldots$. This means that $\frac{L'}{L}(s,\chi)\frac{x^s}{s}$ has simple poles at $-2,-4,\ldots$ and the nontrivial zeros of $L(s,\chi)$ and a double pole at $s=0$. Since $L(s,\chi)$ has a simple zero at $s=0$, the resudue of $\frac{L'}{L}(s,\chi)$ at $s=0$ is 1. Thus, $\frac{L'}{L}(s,\chi)=\frac{1}{s}+\beta(\chi)+f(s)$, where $f(s)$ is analytic and vanishes at $s=0$. We can also easily compute that $x^s=1+s\log x+g_x(s)$, where $g_x(s)$ is an analytic function of s that vanishes to order at least 2 at $s=0$. It follows that

$$\frac{L'}{L}(s,\chi)\frac{x^s}{s}=\left(\frac{1}{s^2}+\frac{\beta(\chi)}{s}+\frac{\log x}{s}+h_x(s)\right)$$

where $h_x(s)$ is an analytic function. Thus, the residue of $\frac{L'}{L}(s,\chi)\frac{x^s}{s}$ at $s=0$ is $\beta(\chi)+\log x$. Now, arguing exactly as in the previous part, we obtain that

$$\psi(x,\chi)=-\sum_{\rho}\frac{x^\rho}{\rho}-\log x-b(\chi)+\sum_{n=1}^{\infty}\frac{x^{-2m}}{2m}$$

for all noninteger x. In a previous problem set, we showed that $\sum_{n=1}^{\infty}\frac{x^{-2m}}{2m}=-\frac{1}{2}\log\left(1-x^{-2}\right)$. We conclude that

$$\psi(x,\chi)=-\sum_{\rho}\frac{x^\rho}{\rho}-\log x-b(\chi)-\frac{1}{2}\log\left(1-x^{-2}\right).$$

2. Let χ_1 and χ_2 be primitive quadratic characters modulo q_1 and q_2, respectively, with $q_1\neq q_2$, and set

$$F(s):=\zeta(s)L(s,\chi_1)L(s,\chi_2)L(s,\chi_1\chi_2),$$

so that $F(s)$ is analytic in all of $\mathbf{C}$ except for a simple pole at $s=1$ with residue $\lambda:=L(1,\chi_1)L(1,\chi_2)L(1,\chi_1\chi_2)$.

(a) First of all, the Dirichlet series $\sum_{n=1}^{\infty}\frac{a_n}{n^s}=\zeta(s)L(s,\chi_1)L(s,\chi_2)L(s,\chi_1\chi_2)$ is absolutely convergent when $\sigma>1$ since each of $\zeta(s)$, $L(s,\chi_1)$, $L(s,\chi_2)$, and $L(s,\chi_1\chi_2)$ is absolutely convergent in that half-plane, and a_n is multiplicative. This immediately yields $a_1=1$. Since all of $\zeta(s)$, $L(s,\chi_1)$, $L(s,\chi_2)$, and $L(s,\chi_1\chi_2)$ are nonvanishing when $\sigma>1$, $F(s)$ is also nonvanishing in this half-plane, which means that we can take $\log F(s)$. Note that if $G(s)$ is a Dirichlet series with nonnegative coefficients, then $e^{G(s)}$ is also a Dirichlet series with nonnegative coefficients, since all of the power series coefficients of e^z are nonnegative. It thus suffices to check that $\log F(s)$ has nonnegative Dirichlet series coefficients when $\sigma>1$. Using the Euler product expansion, we have that $\log F(s)$ equals

$$-\sum_p\left(\log\left(1-p^{-s}\right)+\log\left(1-\chi_1(p)p^{-s}\right)+\log\left(1-\chi_2(p)p^{-s}\right)+\log\left(1-\chi_1\chi_2(p)p^{-s}\right)\right)$$

$$=\sum_p\sum_{k=1}^{\infty}\frac{1+\chi_1(p^k)+\chi_2(p^k)+\chi_1\chi_2(p^k)}{kp^{ks}}=\sum_p\sum_{k=1}^{\infty}\frac{(1+\chi_1(p^k))(1+\chi_2(p^k))}{kp^{ks}}$$

when $\sigma > 1$, where we have also used that $\log(1-z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}$ whenever $|z| < 1$. Since χ_1 and χ_2 are real, the above implies that the coefficients of the Dirichlet series for $\log F(s)$ are all nonnegative, and thus the same holds for $F(s)$ itself.

(b) Since $F(s)$ is analytic in $\sigma > 1$ and has a simple pole at $s = 1$, the radius of convergence of this power series centered at $s = 2$ will be 1 (i.e., the distance from 2 to the closest singularity), so it just suffices to compute the b_m and show that $b_0 \geq 1$ and $b_m \geq 0$ for all $m \in \mathbf{Z}_{\geq 0}$. First of all, since $F(s)$ has nonnegative Dirichlet series coefficients and $a_1 = 1$, $b_0 = F(2) = 1 + \sum_{n=2}^{\infty} \frac{a_n}{n^2} \geq 1$. More generally,

$$b_m = \frac{(-1)^m}{m!} F^{(m)}(2).$$

Since the Dirichlet series for $F(s)$ converges uniformly on compacts, we can differentiate $F(s)$ by differentiating its Dirichlet series term-wise. Thus,

$$F^{(m)}(2) = \sum_{n=1}^{\infty} (-\log n)^m \frac{a_n}{n^2} = (-1)^m \sum_{n=1}^{\infty} (\log n)^m \frac{a_n}{n^2},$$

so that

$$b_m = \frac{1}{m!} \sum_{n=1}^{\infty} (\log n)^m \frac{a_n}{n^2} \geq 0.$$

Finally, $F(s)$ is analytic everywhere in $\mathbf{C}$ except for a simple pole of residue λ at $s = 1$. Thus, $F(s) - \frac{\lambda}{s-1}$ is entire. The power series expansion of $\frac{\lambda}{s-1}$ centered at $s = 2$ is

$$\frac{\lambda}{s-1} = \sum_{m=0}^{\infty} \lambda(-1)^m (s-2),$$

where the radius of convergence is also 1. It follows that

$$F(s) - \frac{\lambda}{s-1} = \sum_{m=0}^{\infty} (b_m - \lambda)(2-s)^m$$

on all of $\mathbf{C}$, and, in particular, whenever $|s-2| < 2$.

(c) By the bounds for $L(s,\chi)$ from class, when $|s-2| = \frac{3}{2}$, we have

$$F(s) \ll q^{1/4} \log q_1 \cdot q_2^{1/4} \log q_2 \cdot (q_1 q_2)^{1/4} \log q_1 q_2 \ll \sqrt{q_1 q_2} \log^2 q_1 q_2 \ll q_1^2 q_2^2,$$

since, for such s, $|t| \leq \frac{3}{2} \ll 1$, as well as

$$\lambda \ll \log q_1 \log q_2 \log q_1 q_2 \ll q_1^2 q_2^2.$$

Thus, again for $|s-2| = \frac{3}{2}$, we have

$$\frac{\lambda}{s-1} \ll \frac{q_1^2 q_2^2}{1/2} \ll q_1^2 q_2^2,$$

since the closest such s to 1 is the point $1/2$. Now, from Cauchy's integral formula, we have the bound

$$|F^{(m)}(2)| \leq \frac{m!}{(3/2)^m} \max_{|s-2|=3/2} |F(s)| \ll \frac{m! q_1^2 q_2^2}{(3/2)^m}.$$

Thus,

$$|b_m - \lambda| = \frac{1}{m!} \left| F^{(m)}(2) \right| \ll q_1^2 q_2^2 \left(\frac{2}{3} \right)^m .$$

(d) Let $M \in \mathbf{N}$. We have, since $b_1 \geq 1$ and $b_m \geq 0$ for all $m \in \mathbf{N}$,

$$\begin{aligned} F(s) - \frac{\lambda}{s-1} &= \sum_{m=0}^{\infty} b_m (2-s)^m - \lambda \sum_{m=0}^{\infty} (2-s)^m \\ &\geq 1 + \sum_{m=M}^{\infty} b_m (2-s)^m - \lambda \sum_{m=0}^{\infty} (2-s)^m \\ &= 1 - \lambda \sum_{m=0}^{M-1} (2-s)^m + \sum_{m=M}^{\infty} (b_m - \lambda)(2-s)^m \\ &\geq 1 - \lambda \frac{(2-s)^M - 1}{1-s} - c_0 q_1^2 q_2^2 e^{-M/4} \end{aligned}$$

for some absolute constant $c_0 > 0$, since

$$\sum_{m=M}^{\infty} (b_m - \lambda)(2-s)^m \ll q_1^2 q_2^2 \sum_{m=M}^{\infty} \left(\frac{2}{3} \cdot \frac{9}{8} \right)^m \ll q_1^2 q_2^2 \left(\frac{3}{4} \right)^M \ll q_1^2 q_2^2 e^{-M/4}$$

because $e^{1/4} < \frac{4}{3}$.

(e) Let $M \in \mathbf{N}$ be such that $e^{-1/4}/2 \leq c_0 q_1^2 q_2^2 e^{-M/4} < 1/2$.
Then, by the previous part,

$$F(s) - \frac{\lambda}{s-1} \geq \frac{1}{2} - \lambda \frac{(2-s)^M - 1}{1-s}.$$

By the choice of M (which guarantees that $M \leq c_0' + 8 \log q_1 q_2$ for some absolute constant c_0'), we have

$$(2-s)^M = \exp(M \log(2-s)) \leq \exp(M(1-s)) \ll (q_1 q_2)^{8(1-s)},$$

using that $\log(2-s) = \log(1 - [s-1]) \leq 1 - s$ for all $s \in [7/8, 1]$. We conclude that

$$F(s) > \frac{1}{2} - \frac{c_1 \lambda}{1-s} (q_1 q_2)^{8(1-s)}$$

for some absolute constant $c_1 > 0$.

(f) In this case, we have $F(\beta_1) = 0$ since $L(\beta_1, \chi_1) = 0$, so that, by the previous part,

$$\frac{c_1\lambda}{1-\beta_1}(q_1q_2)^{8(1-\beta_1)} \geq \frac{1}{2}$$

rearranging yields

$$c_1\lambda > \frac{1}{2}(1-\beta_1)(q_1q_2)^{-8(1-\beta_1)}.$$

(g) We will first show that $F(\beta_1) \leq 0$. Indeed, $L(\sigma, \chi_1), L(\sigma, \chi_2), L(s, \chi_1\chi_2) > 0$ for all $\sigma \geq 1 - \frac{\varepsilon}{16}$ by the assumption that there are no zeros of $L(s, \chi)$ in $[1 - \frac{\varepsilon}{16}, 1]$ for any primitive quadratic character χ, the already-proven fact that $L(s, \chi)$ is always nonvanishing on $(1, \infty)$, the fact that clearly $L(\sigma, \chi) > 0$ for $\sigma \in (1, \infty)$ sufficiently large (as $L(\sigma, \chi) \to 1$ as $\sigma \to \infty$, and that every $L(s, \chi)$ is continuous on $[1 - \frac{\varepsilon}{16}, \infty)$. So, it suffices to show that $\zeta(\beta_1) \leq 0$. We have

$$\zeta(\beta_1) = \frac{\beta_1}{\beta_1 - 1} - \beta_1 \int_1^\infty \frac{\{t\}}{t^{\beta_1+1}} \mathrm{d}t < 0$$

since $\beta_1 > 0$ but $\beta_1 - 1 < 0$. Thus, as in the previous part, we have

$$\frac{c_1\lambda}{1-\beta_1}(q_1q_2)^{8(1-\beta_1)} \geq \frac{1}{2}$$

rearranging again yields

$$c_1\lambda > \frac{1}{2}(1-\beta_1)(q_1q_2)^{-8(1-\beta_1)}.$$

(h) By the previous two parts, we have no matter what that

$$c_1\lambda > \frac{1}{2}(1-\beta_1)(q_1q_2)^{-8(1-\beta_1)}.$$

for some $\beta_1 \in [1 - \varepsilon/16, 1)$ and fixed q_1 and χ_1. Thus, for all $q_2 \neq q_1$ and χ_2 (mod q_2), we obtain

$$L(1, \chi_2)L(1, \chi_1\chi_2) \gg q_2^{-\varepsilon/2}.$$

Since $L(1, \chi_1\chi_2) \ll \log q_1q_2$, it follows that

$$L(1, \chi_2) \gg q_2^{-3\varepsilon/4},$$

say. Thus, any real zero β of $L(s, \chi_2)$ must satisfy

$$1 - \beta \gg \frac{L(1, \chi_2)}{\log^2 q_2} \gg q_2^{-\varepsilon}$$

3. We will allow all implied constants to depend on a. Note that $\#\{p \leq x : p + a \text{ is squarefree}\}$ equals

$$\sum_{p \leq x} \mu(p+a)^2 = \sum_{p \leq x} \sum_{d^2 | p+a} \mu(d) = \sum_{d \leq \sqrt{x+a}} \mu(d)\pi(x; d^2, -a),$$

where we have used the fact that $\mu(m)^2 = \sum_{d^2|m} \mu(d)$ and then swapped the order of summation. The contribution to the sum on the right-hand side coming from $(d,a) > 1$ is $\ll \sqrt{x}$. We will split up the remaining portion of the sum according to the size of d. Set $y = \log^{2A} x$. Then, the Siegel–Walfisz theorem tells us that

$$\sum_{\substack{d \le y \\ (d,a)=1}} \mu(d)\pi(x; d^2, -a) = \mathrm{li}(x) \sum_{\substack{d \le y \\ (d,a)=1}} \frac{\mu(d)}{\phi(d^2)} + O\left(\frac{x}{\exp\left(c_A\sqrt{\log x}\right)}\right).$$

In class, we showed that

$$\sum_{\substack{d \le y \\ (d,a)=1}} \frac{\mu(d)}{\phi(d^2)} = \sum_{(d,a)=1} \frac{\mu(d)}{d\phi(d)} + O\left(\frac{1}{\sqrt{y}}\right) = c(a) + O\left(\frac{1}{\sqrt{y}}\right).$$

Thus,

$$\sum_{\substack{d \le y \\ (d,a)=1}} \mu(d)\pi(x; d^2, -a) = c(a)\,\mathrm{li}(x) + O_A\left(\frac{x}{\sqrt{y}\log x}\right).$$

To bound the contribution from $y < d \le \sqrt{x+a}$, we use the trivial bound $\pi(x; q, a) \ll 1 + \frac{x}{q}$ to obtain

$$\sum_{\substack{y < d \le \sqrt{x+a} \\ (d,a)=1}} \mu(d)\pi(x; d^2, -a) \ll \sum_{\substack{y < d \le \sqrt{x+a} \\ (d,a)=1}} \left(1 + \frac{x}{d^2}\right) \ll \sqrt{x} + \frac{x}{y}.$$

We thus conclude that

$$\#\{p \le x : p + a \text{ is squarefree}\} = c(a)\,\mathrm{li}(x) + O_A\left(\frac{x}{(\log x)^A}\right).$$