Math 675: Analytic Theory of Numbers
Solutions to problem set # 5

April 23, 2024

1. (a) Note first that $c_1(n) = 1$ for all $n \in \mathbf{N}$. Let $q_1, q_2 \in \mathbf{N}$ be relatively prime. We have that $c_{q_1}(n)c_{q_2}(n)$ equals

$$\sum_{\substack{a \in (\mathbf{Z}/q_1\mathbf{Z})^\times \\ b \in (\mathbf{Z}/q_2\mathbf{Z})^\times}} e\left(\frac{an}{q_1} + \frac{bn}{q_2}\right) = \sum_{\substack{a \in (\mathbf{Z}/q_1\mathbf{Z})^\times \\ b \in (\mathbf{Z}/q_2\mathbf{Z})^\times}} e\left(\frac{(aq_2 + bq_1)n}{q_1 q_2}\right) = \sum_{c \in (\mathbf{Z}/q_1 q_2\mathbf{Z})^\times} e_{q_1 q_2}(cn)$$

by the Chinese remainder theorem.

(b) Prove that

$$\sum_{d \mid q} c_d(n) = \begin{cases} q & q \mid n \\ 0 & \text{otherwise} \end{cases}.$$

Since both $c_q(n)$ and $q1_{q|n}$ are multiplicative function of q, it suffices to prove the result when q is a prime power. Note first that, for $k \geq 1$,

$$\begin{aligned} c_{p^k}(n) = \sum_{\substack{1 \leq m \leq p^k \\ (m,p)=1}} e_{p^k}(mn) &= \sum_{m=1}^{p^k} e_{p^k}(mn) - \sum_{m=1}^{p^{k-1}} e_{p^{k-1}}(mn) \\ &= \begin{cases} p^k & p^k \mid n \\ 0 & \text{otherwise} \end{cases} - \begin{cases} p^{k-1} & p^{k-1} \mid n \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

by orthogonality of additive characters. In particular, $c_{p^k}(n) = 0$ if $p^{k-1} \nmid n$. Now, if $p^\ell \| n$, then

$$\sum_{d \mid p^a} c_q(n) = \sum_{k=0}^{a} c_{p^k}(n) = \sum_{k=0}^{\min(a,\ell+1)} c_{p^k}(n) = \begin{cases} p^a & \ell \geq a \\ 0 & \ell < a \end{cases} = \begin{cases} p^a & p^a \mid n \\ 0 & \text{otherwise} \end{cases}$$

by telescoping.

(c) It again suffices, by the multiplicativity of $c_q(n)$ as a function of q, to check that the desired identity holds on prime powers. First note that, for all $n \in \mathbf{N}$,

$\frac{\mu(1/(1,n))}{\phi(1/(1,n))}\phi(1) = 1 = c_1(n)$. By the computation above, for each $k \geq 1$,

$$c_{p^k}(n) = \begin{cases} \phi(p^k) & p^k \mid n \\ -p^{k-1} & p^{k-1} \| n \\ 0 & \text{otherwise} \end{cases}.$$

On the other hand, we have

$$\frac{\mu(p^k/(p^k,n))}{\phi(p^k/(p^k,n))}\phi(p^k) = \begin{cases} \frac{\mu(1)}{\phi(1)}\phi(p^k) & p^k \mid n \\ \frac{\mu(p)}{\phi(p)}\phi(p^k) & p^{k-1} \| n \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \phi(p^k) & p^k \mid n \\ -p^{k-1} & p^{k-1} \| n \\ 0 & \text{otherwise} \end{cases},$$

so the identity does, indeed, hold on prime powers.

2. (a) When p is odd, $(\mathbf{Z}/p^k\mathbf{Z})^\times$ is cyclic, say generated by $n \in (\mathbf{Z}/p^k\mathbf{Z})^\times$. Since χ is nonprincipal, $\chi(n) = -1$, which completely determines the values of χ on all elements of $(\mathbf{Z}/p^k\mathbf{Z})^\times$. Thus, there is only one possibility for χ. Observe that $\left(\frac{n}{p}\right)$ is real-valued and, since it is completely multiplicative and depends only on the value of $n \pmod p$, it defines a real nonprincipal Dirichlet character on $(\mathbf{Z}/p^k\mathbf{Z})^\times$. Thus, we must have $\chi(n) = \left(\frac{n}{p}\right)$, which has conductor p.

   (b) We have $(\mathbf{Z}/8\mathbf{Z})^\times \cong (\mathbf{Z}/2\mathbf{Z})^2$, and $(\mathbf{Z}/8\mathbf{Z})^\times$ is generated by 3 and 5. Thus, χ is determined by its values on 3 and 5, and there are three possibilities for χ: $\chi(3) = 1$ and $\chi(5) = -1$, $\chi(3) = -1$ and $\chi(5) = 1$, and $\chi(3) = \chi(5) = -1$. The first possibility yields $\chi(1) = \chi(3) = 1$ and $\chi(5) = \chi(7) = -1$, which has conductor 8. The second possibility yields $\chi(1) = 1$, $\chi(3) = -1$, $\chi(5) = 1$, and $\chi(7) = -1$, which has conductor 4. Finally, the third possibility yields $\chi(1) = 1$, $\chi(3) = -1$, $\chi(5) = -1$, and $\chi(7) = 1$, which has conductor 8.

   (c) When $k > 3$, $(\mathbf{Z}/2^k\mathbf{Z})^\times \cong (\mathbf{Z}/2\mathbf{Z}) \times (\mathbf{Z}/2^{k-2}\mathbf{Z})$. Thus, $(\mathbf{Z}/2^k\mathbf{Z})^\times$ can be generated by two elements a and b, and so χ is completely determined by its values on a and b. Since χ is real, $\chi(a), \chi(b) \in \{-1, 1\}$, and so there are three possibilities for χ. Each Dirichlet character ψ constructed in the previous part induces a Dirichlet character modulo 2^k, and so these must be exactly the real nonprincipal Dirichlet characters modulo 2^k.

3. (a) First of all, note that, for any $(a, b) \in \mathbf{Z}^2$ for which $a^2 + b^2 = n$, we can write $d^2[(a/d)^2 + (b/d)^2] = n$ where $d = \gcd(a, b)$, so that $\gcd(a/d, b/d) = 1$ and $d^2 \mid n$. Thus, we immediately obtain

$$R(n) = \sum_{d^2 \mid n} r(n/d^2).$$

   Now recall that if $m \in \mathbf{N}$ has prime factorization $m = 2^a p_1^{b_1} \cdots p_k^{b_k} q_1^{c_1} \cdots q_\ell^{c_\ell}$ with each $p_i \equiv 1 \pmod 4$ and each $q_j \equiv 3 \pmod 4$, then $R(n) = 0$ unless each of

$c_1, \ldots, c_\ell$ is even, in which case

$$R(n) = \prod_{i=1}^{k} (b_i + 1).$$

Thus, $R(n)$ is multiplicative, and satisfies $R(2^a) = 1$ for all $a \in \mathbf{Z}_{\geq 0}$, $R(p^b) = 2^b$ for all $p \equiv 1 \pmod 4$ and $b \in \mathbf{Z}_{\geq 0}$, and $R(q^c) = \frac{1+(-1)^c}{2}$ for all $q \equiv 3 \pmod 4$ and $c \in \mathbf{Z}_{\geq 0}$. On the other hand, $\sum_{d|2^a} \chi_{-4}(d) = 1$,

$$\sum_{d|p^b} \chi_{-4}(d) = \sum_{d|p^b} 1 = b + 1,$$

when $p \equiv 1 \pmod 4$,

$$\sum_{d|q^c} \chi_{-4}(d) = \sum_{t=0}^{c} (-1)^t = \frac{1 + (-1)^c}{2}$$

when $q \equiv 3 \pmod q$. The second equality thus follows from the multiplicativity of R.

(b) By the previous part, we have $R = 1 \star \chi_{-4}$. Since both $\zeta(s)$ and $L(s, \chi_{-4})$ are absolutely convergent for $\sigma > 1$, we thus have

$$\sum_{n=1}^{\infty} \frac{R(n)}{n^s} = \zeta(s) L(s, \chi_{-4})$$

when $\sigma > 1$.

(c) Let $f : \mathbf{N} \to \mathbf{C}$ denote the indicator functions of the squares, so that f is multiplicative. Then the first part says that $f \star r = R$. Note that $r(n) \leq \tau(n)$, and thus $\mathcal{D}r(s)$ converges absolutely for $\sigma > 1$. Thus, since $\mathcal{D}f(s) = \zeta(2s)$, we have

$$\zeta(2s)\mathcal{D}r(s) = \zeta(s) L(s, \chi_{-4})$$

when $\sigma > 1$. It follows that

$$\sum_{n=1}^{\infty} \frac{r(n)}{n^s} = \frac{\zeta(s) L(s, \chi_{-4})}{\zeta(2s)}$$

when $\sigma > 1$.

4. (a) Since f_1 and f_2 are 1-bounded, each of $\mathcal{D}f_1(s)$, $\mathcal{D}f_2(s)$, $\mathcal{D}(f_1 f_2)(s)$, and $\mathcal{D}([1 \star f_1][1 \star f_2])(s)$ converge absolutely for $\sigma > 1$, and since f_1 and f_2 (and thus, also, their product) are completely multiplicative functions, we have

$$\frac{\zeta(s)\mathcal{D}f_1(s)\mathcal{D}f_2(s)\mathcal{D}(f_1 f_2)(s)}{\mathcal{D}(f_1 f_2)(2s)} = \prod_p \frac{(1 - f_1(p) f_2(p) p^{-2s})}{(1 - p^{-s})(1 - f_1(p) p^{-s})(1 - f_2(p) p^{-s})(1 - f_1(p) f_2(p) p^{-s})},$$

when $\sigma > 1$, which gives the second desired equality. Note that $1 \star f_1$ and $1 \star f_2$ are both multiplicative, so that their product is also multiplicative. To prove the first equality, we just need to check that

$$\sum_{k=0}^{\infty} \frac{(1 \star f_1)(p^k)(1 \star f_2)(p^k)}{p^{ks}} = \frac{(1 - f_1(p) f_2(p) p^{-2s})}{(1-p^{-s})(1-f_1(p)p^{-s})(1-f_2(p)p^{-s})(1-f_1(p)f_2(p)p^{-s})} \tag{1}$$

for all $\sigma > 1$. Assuming $\sigma > 1$, we then have that $[(1-p^{-s})(1-f_1(p)p^{-s})(1-f_2(p)p^{-s})(1-f_1(p)f_2(p)p^{-s})]^{-1}$ equals

$$\left(\sum_{a=0}^{\infty} \frac{1}{p^{as}}\right)\left(\sum_{b=0}^{\infty} \frac{f_1(p)^b}{p^{bs}}\right)\left(\sum_{c=0}^{\infty} \frac{f_2(p)^c}{p^{cs}}\right)\left(\sum_{d=0}^{\infty} \frac{f_1(p)^d f_2(p)^d}{p^{ds}}\right) = \sum_{a,b,c,d \geq 0} \frac{f_1(p)^{b+d} f_2(p)^{c+d}}{p^{(a+b+c+d)s}},$$

so that the right-hand side of (1) equals

$$\begin{aligned} &\sum_{a,b,c,d\geq 0} \frac{f_1(p)^{b+d} f_2(p)^{c+d}}{p^{(a+b+c+d)s}} - \sum_{a,b,c,d\geq 0} \frac{f_1(p)^{b+d+1} f_2(p)^{c+d+1}}{p^{(a+b+c+d+2)s}} \\ &= \sum_{a,c,d\geq 0} \frac{f_1(p)^{d} f_2(p)^{c+d}}{p^{(a+c+d)s}} + \sum_{a,b,d\geq 0} \frac{f_1(p)^{b+d} f_2(p)^{d}}{p^{(a+b+d)s}} - \sum_{a,d\geq 0} \frac{f_1(p)^{d} f_2(p)^{d}}{p^{(a+d)s}}, \end{aligned}$$

which equals

$$\sum_{a,b,c\geq 0} \frac{f_1(p)^b f_2(p)^c}{p^{(a+\max\{b,c\})s}}.$$

On the other hand, the left-hand side of (1) equals

$$\sum_{k=0}^{\infty}\left(\sum_{b=0}^{k} f_1(p)^b\right)\left(\sum_{c=0}^{k} f_2(p)^c\right)p^{-ks} = \sum_{\substack{b,c,k\geq 0\\ b,c\leq k}} \frac{f_1(p)^b f_2(p)^c}{p^{ks}} = \sum_{a,b,c\geq 0} \frac{f_1(p)^b f_2(p)^c}{p^{(a+\max\{b,c\})s}},$$

completing the proof.

(b) Fix $t \in \mathbf{R}$ and define $f_1(n) = \chi(d)d^{-it}$ and $f_2(n) = \overline{f_1(n)}$ for all $n \in \mathbf{N}$, so that both f_1 and f_2 are completely multiplicative (since both $\chi(d)$ and d^{-it} are) and 1-bounded. Assume by way of contradiction that $L(1+it,\chi) = 0$. Note that $F(s) = \sum_{n=1}^{\infty} \frac{(1\star f_1)(n)(1\star f_2)(n)}{n^s}$. Thus, by the previous part, we have that

$$F(s) = \frac{\zeta(s)^2 L(s+it,\chi)L(s-it,\overline{\chi})}{\zeta(2s)} \prod_{p|q} \left(1+p^{-s}\right)^{-1} \tag{2}$$

whenever $\sigma > 1$. The assumption that $L(1+it,\chi) = 0$ implies that $L(1-it,\overline{\chi}) = 0$ as well. Thus, the right-hand side of (2) is analytic at $s = 1$. Since $\zeta(2s)$ is nonvanishing for $\sigma \geq 1/2$, it follows that the right-hand side of (2) is analytic in a neighborhood of the closed half-plane $\sigma \geq 1/2$. Since the right-hand side is a meromorphic function on $\mathbf{C}$, this gives a meromorphic continuation of $F(s)$ to $\mathbf{C}$, and we also get that $F(s)$ is analytic in a neighborhood of $\sigma \geq 1/2$. Since $F(s)$ has nonnegative Dirichlet series coefficients, Landau's lemma implies that its abscissa of convergence is strictly less than $1/2$. In particular, $F(1/2) = \sum_{n=1}^{\infty} \frac{\left|\sum_{d|n} \chi(d) d^{-it}\right|^2}{n^{1/2}}$, so that $F(1/2) \geq 1$. On the other hand, (2) implies that $F(1/2) = 0$, since $\zeta(s)^2 L(s+it,\chi) L(s-it,\overline{\chi}) \prod_{p|q}(1+p^{-s})^{-1}$ is analytic in a neighborhood of $s = 1/2$ and $\zeta(2s)$ has a pole at $s = 1/2$. This gives a contradiction.

5. (a) Note first that since $\max_{t\in(0,1)} |\psi(x,\chi) - \psi(x+t,\chi)| \ll \log x$, it suffices to prove the result for $x \in \frac{1}{2} + \mathbf{N}$. The explicit formula then tells us that if $x \geq T \geq 2$,

$$\psi(x,\chi) = 1_{\chi=\chi_0} x - \sum_{|\gamma|\leq T} \frac{x^\rho - 1}{\rho} + O\left(\frac{x \log^2(xq)}{T}\right).$$

On GRH, the sum can be bounded by

$$\ll \sum_{|\gamma|\leq T} \frac{x^{1/2}}{\rho} \ll x^{1/2} \log^2(qT).$$

Thus,

$$\psi(x,\chi) = 1_{\chi=\chi_0} x + O\left(\frac{x \log^2(xq)}{T} + x^{1/2} \log^2(qT)\right).$$

Picking $T = \sqrt{x}$ then yields that $\psi(x,\chi) = 1_{\chi=\chi_0} x + O\left(x^{1/2} \log^2(qx)\right)$.

(b) By orthogonality of Dirichlet characters,

$$\psi(x;q,a) = \frac{1}{\phi(q)} \sum_{\chi \pmod q} \psi(x,\chi)\overline{\chi}(a)$$

whenever $(a,q) = 1$. Plugging in the approximation for $\psi(x,\chi)$ from the previous part then yields $\psi(x;q,a) = \frac{x}{\phi(q)} + O\left(x^{1/2}\log^2(qx)\right)$.