# Math 675: Analytic Theory of Numbers
# Solutions to problem set # 4

March 31, 2024

1. (a) We just need to be a bit more careful proving the explicit formula than we were in class. We showed that, when $|T-\gamma| \gg \frac{1}{\log T}$ with $T \geq 2$ and $x \geq 2$ is not an integer,

$$\psi(x) = x - \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} + \sum_{k=1}^{\infty} \frac{x^{-2k}}{2k} - \frac{\zeta'}{\zeta}(0) + O\left(\frac{x}{T}\left((\log xT)^2 + \frac{\log x}{\langle x \rangle}\right)\right)$$

for any odd positive integer $N$. To compute $\frac{\zeta'}{\zeta}(0)$, recall from the previous homework that, for $\sigma > -1$,

$$\begin{aligned}\zeta(s) &= \frac{s}{s-1} + \sum_{0 \leq r \leq 1} \frac{B_{r+1}}{r+1}\binom{s+r-1}{r} - \binom{s+1}{2}\int_1^\infty B_2(t) t^{-s-2}\mathrm{d}t \\ &= \frac{1}{2} + \frac{1}{s-1} + \frac{s}{12} - \frac{s^2+s}{2}\int_1^\infty \frac{B_2(t)}{t^{s+2}}\mathrm{d}t.\end{aligned}$$

This immediately yields $\zeta(0) = -\frac{1}{2}$. We will also use this expression to compute $\zeta'(0)$. By the above, we have, since $\int_1^\infty \frac{B_2(t)}{t^{s+2}}\mathrm{d}t$ is analytic in $\sigma > -1$,

$$\zeta'(s) = -\frac{1}{(s-1)^2} + \frac{1}{12} - \frac{s^2+s}{2}\left(\frac{\mathrm{d}}{\mathrm{d}s}\int_1^\infty \frac{B_2(t)}{t^{s+2}}\mathrm{d}t\right) - \frac{2s+1}{2}\int_1^\infty \frac{B_2(t)}{t^{s+2}}\mathrm{d}t$$

for $\sigma > -1$. Thus,

$$\zeta'(0) = -1 + \frac{1}{12} - \frac{1}{2}\int_1^\infty \frac{B_2(t)}{t^2}\mathrm{d}t.$$

We can compute $\frac{1}{2}\int_1^\infty \frac{B_2(t)}{t^2}\mathrm{d}t$ by Euler–Maclaurin summation and Stirling's formula. Indeed, for integer $N$, the former yields

$$\sum_{n \leq N} \log n = N\log N - N + 1 + \frac{\log N}{2} + \frac{\frac{1}{N} - 1}{12} - \frac{1}{2}\int_1^N \frac{B_2(t)}{t^2}\mathrm{d}t$$

and the latter yields

$$\sum_{n \leq N} \log n = N\log N - N + \frac{\log N}{2} + \frac{1}{2}\log 2\pi + O\left(\frac{1}{N}\right).$$

Combining these, we get

$$\frac{1}{2}\int_1^N \frac{B_2(t)}{t^2}\mathrm{d}t = 1 - \frac{1}{12} + \frac{1}{2}\log 2\pi + O\left(\frac{1}{N}\right),$$

and taking $N \to \infty$ shows that $\frac{1}{2}\int_1^N \frac{B_2(t)}{t^2}\mathrm{d}t = 1 - \frac{1}{12} + \frac{1}{2}\log 2\pi$. Thus, $\zeta'(0) = -\frac{1}{2}\log 2\pi$, and we conclude that $\frac{\zeta'}{\zeta}(0) = \log 2\pi$.

Now, as in class, we can bound $\sum_{k=1}^\infty \frac{x^{-2k}}{2k} \le x^{-2}\sum_{k=1}^\infty \frac{x^{-k}}{2k} \ll x^{-2}$ since $x \ge 2$. Thus,

$$\psi(x) = x - \sum_{|\gamma|\le T} \frac{x^\rho}{\rho} - \log 2\pi + O\left(\frac{x}{T}\left((\log xT)^2 + \frac{\log x}{\langle x\rangle}\right) + x^{-2}\right)$$

when $|T-\gamma| \gg \frac{1}{\log T}$ for all nontrivial zeros $\rho = \beta + i\gamma$ with $x, T \ge 2$ and $x$ not an integer. Since, for any $T \ge 2$, there are $\ll \log T$ nontrivial zeros of zeta with imaginary part in $[T, T+1]$, we may select a $T' \in [T, T+1]$ such that $|T' - \gamma| \gg \frac{1}{\log T}$. As in class, $\left|\sum_{T\le|\gamma|\le T'} \frac{x^\rho}{\rho}\right| \ll \frac{x\log T}{T}$, so that we have, in fact, that

$$\psi(x) = x - \sum_{|\gamma|\le T} \frac{x^\rho}{\rho} - \log 2\pi + O\left(\frac{x}{T}\left((\log xT)^2 + \frac{\log x}{\langle x\rangle}\right) + x^{-2}\right)$$

for all $x, T \ge 2$ with $x$ not an integer. Now, let $N \in \mathbf{N}$ be not a prime power, and set $x = N + \frac{1}{N^{10}}$, say, and $T = N^2$. Then $\psi(x) = \psi(N)$, and the above yields

$$\begin{aligned}\psi(N) &= N - \sum_{|\gamma|\le N^2} \frac{\left(N + \frac{1}{N^{10}}\right)^\rho}{\rho} - \log 2\pi + o(1) \\ &= N - \sum_{|\gamma|\le N^2} \frac{N^\rho}{\rho} - \log 2\pi + o(1),\end{aligned}$$

as desired.

(b) Suppose by way of contradiction that $\zeta(s)$ has only finitely many nontrivial zeros, and denote the set of such zeros by $S$. Note that, since $\log(1-x) = -\sum_{n=1}^\infty \frac{x^n}{n}$ for all $|x| < 1$, we have $\sum_{k=1}^\infty \frac{x^{-2k}}{2k} = -\frac{1}{2}\log\left(1 - x^{-2}\right)$. Thus, being a bit more careful in the argument above, we have

$$\psi(x) = x - \sum_{|\gamma|\le T} \frac{x^\rho}{\rho} - \frac{1}{2}\log\left(1 - x^{-2}\right) - \log 2\pi + O\left(\frac{x}{T}\left((\log xT)^2 + \frac{\log x}{\langle x\rangle}\right)\right)$$

for all $x, T \ge 2$ with $x$ not an integer. For each fixed noninteger $x \ge 2$, we get, by taking $T \to \infty$, that

$$\psi(x) = x - \sum_{\rho\in S} \frac{x^\rho}{\rho} - \frac{1}{2}\log\left(1 - x^{-2}\right) - \log 2\pi.$$

Set $f(x) = x - \sum_{\rho\in S}\frac{x^\rho}{\rho} - \frac{1}{2}\log\left(1-x^{-2}\right) - \log 2\pi$ for $x \ge 2$, and note that $f(x)$ is a continuous function of $x$. We have $\psi(5+\varepsilon) = \psi(5) = \log 5 + \psi(4) = \log 5 + \psi(5-\varepsilon)$ for all $\varepsilon \in (0, 1/2)$, say. Thus, $\lim_{x\to 5^-} f(x) = \lim_{\varepsilon\to 0^+}\psi(5-\varepsilon) \ne \lim_{\varepsilon\to 0^+}\psi(5+\varepsilon) = \lim_{x\to 5^+} f(x)$, which contradicts that $f(x)$ is continuous. We conclude that $\zeta(s)$ must have infinitely many nontrivial zeros.

(c) Recall that, by the functional equation for $\zeta(s)$, if $\rho = \beta + i\gamma$ is a nontrivial zero, then so is $1 - \beta - i\gamma$. Thus, since there is at least one nontrivial zero $\rho$ by the previous part of the problem, at least one of $\rho$ or $1-\rho$ will be a nontrivial zero with real part at least $\frac{1}{2}$, which is, obviously, greater than $\frac{1}{2} - \varepsilon$ for all $\varepsilon > 0$.

Now let $\varepsilon > 0$. Then the above tells us that $\zeta(s)$ has a nontrivial zero $\rho$ with $\beta \geq \frac{1}{2} - \frac{\varepsilon}{2}$, so that $\frac{\zeta'}{\zeta}(s)$ must have at least two poles in the half-plane $\sigma > \frac{1}{2} - \varepsilon$: at $s = 1$ and at $s = \rho$. Suppose by way of contradiction that $\psi(x) = x + O(x^{1/2-\varepsilon})$, i.e., $\psi(x) - x \ll x^{1/2-\varepsilon}$. Define the Dirichlet series

$$F(s) := \sum_{n=1}^{\infty} \frac{\Lambda(n) - 1}{n^s},$$

which converges absolutely for $\sigma > 1$, and note that $-\frac{\zeta'}{\zeta}(s) = F(s) + \zeta(s)$ for $\sigma > 1$. By partial summation, we have

$$\sum_{n=1}^{N} \frac{\Lambda(n) - 1}{n^s} = \frac{\psi(N) - N}{N^s} + s \int_1^N \frac{\psi(t) - t}{t^{s+1}} \mathrm{d}t$$

for all $N \in \mathbf{N}$ and $s \in \mathbf{C}$. The assumption that $\psi(x) - x \ll x^{1/2-\varepsilon}$ thus implies that $F(s)$ converges whenever $\sigma > \frac{1}{2} - \varepsilon$, and so $F(s)$ is analytic in this half-plane. Since $\zeta(s)$ is analytic except for a simple pole at $s = 1$, it follows that $F(s) + \zeta(s)$ is analytic in the half-plane $\sigma > \frac{1}{2} - \varepsilon$ except for a simple pole at $s = 1$. But, since $-\frac{\zeta'}{\zeta}(s) = F(s) + \zeta(s)$ whenever $\sigma > \frac{1}{2} - \varepsilon$ by the principle of analytic continuation, this contradicts that $-\frac{\zeta'}{\zeta}(s)$ has at least two poles in the half-plane $\sigma > \frac{1}{2} - \varepsilon$. Thus, we cannot have $\psi(x) = x + O(x^{1/2-\varepsilon})$.

2. (a) In class, we showed that $\frac{\zeta'}{\zeta}(s) \ll 1$ when $\sigma \geq 2$, so it suffices to consider $\sigma \in [1 - \delta_t/2, 2]$. By the lemma from class, we have

$$-\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} - \sum_{|\gamma - t| \leq 1} \frac{1}{s - \rho} + O\left(\log(|s| + 2)\right),$$

so that, by the assumption that $|t| \geq 3$ and $\sigma - \beta \gg \frac{1}{\log(|t|+2)}$ for all nontrivial zeros of $\zeta(s)$, we have

$$\frac{\zeta'}{\zeta}(s) \ll \log(|s| + 2) + \sum_{|\gamma - t| \leq 1} \log(|t| + 2) \ll \log^2 |t|,$$

since there are $\ll \log(|t| + 2) \ll \log |t|$ (as $|t| \geq 3$) nontrivial zeros of $\zeta(s)$, counted with multiplicity, satisfying $|\gamma - t| \leq 1$ and $|s| + 2 \asymp |t|$ for $s$ with $\sigma \in [1 - \delta_t/2, 2]$ and $|t| \geq 3$.

(b) First of all, note that when $\sigma \geq 1 + \delta_t$, we have

$$\left|\frac{\zeta'}{\zeta}(s)\right| \leq \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1+\delta_t}} \leq \sum_{n=2}^{\infty} \frac{\log n}{n^{1+\delta_t}} \ll 1 + \int_2^{\infty} \frac{\log x}{x^{1+\delta_t}} \mathrm{d}t \ll \frac{1}{\delta_t} \ll \log |t|$$

since $\frac{\log n}{n^{1+\delta}}$ is decreasing for $n \geq 3$. So, it remains to prove the desired bound when $\sigma \in [1 - \delta_t/2, 1 + \delta_t)$. For such $\sigma$ and for all $|t| \geq 3$, we have, again by the lemma in class, that

$$\begin{aligned}
\frac{\zeta'}{\zeta}(\sigma + it) - \frac{\zeta'}{\zeta}(1 + \delta_t + it) &= \sum_{|\gamma - t| \leq 1} \left(\frac{1}{\sigma + it - \rho} - \frac{1}{1 + \delta_t + it - \rho}\right) + O\left(\log |t|\right) \\
&= \sum_{|\gamma - t| \leq 1} \frac{1 + \delta_t - \sigma}{(\sigma + it - \rho)(1 + \delta_t + it - \rho)} + O\left(\log |t|\right).
\end{aligned}$$

Thus, for all $\sigma \in [1-\delta_t/2, 1+\delta_t)$ and $|t| \geq 3$, since $|1+\delta_t+it-\rho| \asymp |\sigma+it-\rho|$ for all nontrivial zeros $\rho$, we have

$$\left|\frac{\zeta'}{\zeta}(\sigma+it)\right| \ll \log|t| + \frac{1}{\log|t|}\sum_{|\gamma-t|\leq 1}\frac{1}{|1+\delta_t+it-\rho|^2}.$$

Since $\frac{1}{|z|^2} = \frac{1}{\mathrm{Re}\{z\}}\mathrm{Re}\{\frac{1}{z}\}$ for all $z \in \mathbf{C}$, we have

$$\sum_{|\gamma-t|\leq 1}\frac{1}{|1+\delta_t+it-\rho|^2} = \sum_{|\gamma-t|\leq 1}\frac{1}{1+\delta_t-\beta}\mathrm{Re}\left\{\frac{1}{1+\delta_t+it-\rho}\right\} \leq \frac{\delta_t}{2}\mathrm{Re}\left\{\sum_{|\gamma-t|\leq 1}\frac{1}{1+\delta_t+it-\rho}\right\}.$$

Since

$$\mathrm{Re}\left\{\sum_{|\gamma-t|\leq 1}\frac{1}{1+\delta_t+it-\rho}\right\} = \mathrm{Re}\left\{\frac{\zeta'}{\zeta}(1+\delta_t)\right\} + O(\log|t|) \ll \log|t|,$$

we conclude that $\frac{\zeta'}{\zeta}(\sigma+it) \ll \log|t|$.

(c) Let $s = \sigma+it$ with $\sigma \geq 1-\frac{\delta_t}{2}$ and $|t| \geq 3$. First of all, since $\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = \frac{\zeta'}{\zeta}(s)$, we have

$$\log\zeta(1+\delta_t+it) - \log\zeta(\sigma+it) = \int_{\sigma}^{1+\delta_t}\frac{\zeta'}{\zeta}(x+it)\mathrm{d}x.$$

Thus, whenever $\sigma \leq 1+\delta_t$, we have

$$|\log\zeta(1+\delta_t+it) - \log\zeta(\sigma+it)| \ll \frac{1}{\log|t|}\cdot\log|t| \ll 1$$

by the previous part of the problem. Since we know from class that $\log\zeta(s) = \sum_p\sum_{k=1}^{\infty}\frac{1}{kp^{ks}} = \sum_{n=2}^{\infty}\frac{\Lambda(n)}{(\log n)n^s}$ for $\sigma > 1$, we can bound

$$|\log\zeta(1+\delta_t+it)| \leq \sum_{n=2}^{\infty}\frac{\Lambda(n)}{(\log n)n^{1+\delta_t}} = \log\zeta(1+\delta_t) \leq \log\frac{1}{\delta_t} + O(1) \leq \log\log|t| + O(1).$$

Thus, $|\log\zeta(s)| \leq \log\log|t| + O(1)$ whenever $\sigma \in [1-\frac{\delta_t}{2}, 1+\delta_t]$ and $|t| \geq 3$. This bound trivially follows from the bound $\log\zeta(s) \leq \log\left(\frac{1}{\sigma-1}\right) + O(1)$ when $\sigma > 1+\delta_t$ and $|t| \geq 3$, completing the proof in general.

(d) Note that, like in our proof of the prime number theorem, it suffices to prove the result for $x \in \frac{1}{2}+\mathbf{Z}$. By Perron's formula, for such $x \geq 2$ with and $x \geq T \geq 2$, we have

$$\sum_{n\leq x}\mu(n) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{1}{\zeta(s)}x^s\frac{\mathrm{d}s}{s} + O\left(\frac{x\log x}{T}\right)$$

where $c = 1+\frac{1}{\log x}$, so that $x^c \asymp x$. Set $\alpha = 1-\frac{\delta_T}{2}$ and let $R_T$ denote the rectangular contour (traversed counterclockwise) with vertices at $c \pm iT$ and $\alpha \pm iT$. Note that $R_T$

is completely contained within the region $\sigma \geq 1 - \frac{\delta_t}{2}$. Since $\frac{x^s}{s\zeta(s)}$ is analytic in an open neighborhood of $R_T$, we have $\frac{1}{2\pi i}\int_{R_T} \frac{x^s}{s\zeta(s)}\mathrm{d}s = 0$. Thus,

$$\int_{c-iT}^{c+iT} \frac{x^s}{s\zeta(s)}\mathrm{d}s \ll \left|\int_{\alpha-iT}^{c-iT} \frac{x^s}{s\zeta(s)}\mathrm{d}s\right| + \left|\int_{\alpha+iT}^{c+iT} \frac{x^s}{s\zeta(s)}\mathrm{d}s\right| + \left|\int_{\alpha-iT}^{\alpha+iT} \frac{x^s}{s\zeta(s)}\mathrm{d}s\right|.$$

To bound each of these, we will use that

$$\left|\frac{1}{\zeta(s)}\right| = |\exp(-\log\zeta(s))| = \exp(-\operatorname{Re}\{\log\zeta(s)\}) \ll \log|t|$$

on $R_T$ by the previous part of the problem. For the integrals over the horizontal lines, we thus have

$$\left|\int_{\alpha\pm iT}^{c\pm iT} \frac{x^s}{s\zeta(s)}\mathrm{d}s\right| \ll \frac{\log T}{T}\int_\alpha^c x^\sigma \mathrm{d}\sigma \ll \frac{x}{T},$$

and for the integral along the vertical strip, we have

$$\left|\int_{\alpha-iT}^{\alpha+iT} \frac{x^s}{s\zeta(s)}\mathrm{d}s\right| \ll x^\alpha \int_{-T}^{T} \frac{\log t}{|t|+1}\mathrm{d}t \ll x^\alpha(\log T)^2.$$

Hence,

$$\sum_{n\leq x}\mu(n) \ll \frac{x\log x}{T} + x^\alpha(\log T)^2.$$

We can write $x^\alpha$ as $\frac{x}{\exp(\log x/2\log T)}$. Thus, in order to get the desired bound, we will want to select $T$ such that $\log T \asymp \sqrt{\log x}$. So, we take $T = \exp\big(\sqrt{\log x}\big)$, which is smaller than $x$ for $x$ sufficiently large. This yields

$$\sum_{n\leq x}\mu(n) \ll \frac{x}{\exp\big(c\sqrt{\log x}\big)}$$

for some absolute constant $c > 0$, as desired.

3. (a) These bounds (or stronger) were proven for $\sigma \geq 1$ unconditionally in the previous problem. So, we may assume that $\sigma \in [1/2+\varepsilon, 1]$. Let $s = \sigma + it$ for such $\sigma$ and for $|t| \geq 2$. Then, by the lemma from class,

$$\left|\frac{\zeta'}{\zeta}(s)\right| \leq \frac{1}{|s-1|} + \sum_{|\gamma-t|\leq 1}\frac{1}{|s-\rho|} + O\left(\log(|t|+2)\right) \ll_\varepsilon \log|t|$$

since the distance from $s$ to any nontrivial zero is at least $\varepsilon$ by the assumption of RH. For the other bound, we have $\log\zeta(1+it) \ll \log|t|$ from the previous problem, so that

$$|\log\zeta(s)| \leq |\log\zeta(1+it)| + \int_\sigma^1 \left|\frac{\zeta'}{\zeta}(x+it)\right| \mathrm{d}x \ll_\varepsilon \log|t|$$

as well.

(b) One way to do this is to re-prove the version of the Phragmén–Lindelöf principle from class, modifying the argument for logarithmic growth. Or, we can just use the three circles theorem from complex analysis:

**Theorem 1.** *Let $r_1, r_2 \in \mathbf{R}$ with $0 < r_1 < r_2$, $f(z)$ be holmorphic on the annulus $|z| \in (r_1, r_2)$, and $r \in (r_1, r_2)$. Denote the maxima of $f$ on the three circles $|z| = r_1$, $|z| = r_2$, and $|z| = r$, respectively, by $M_1, M_2$, and $M$. Then,*

$$M^{\log \frac{r_2}{r_1}} \leq M_1^{\log \frac{r_2}{r}} M_2^{\log \frac{r}{r_1}}.$$

Note that the desired bounds follow trivially when $\sigma \geq 1 + \varepsilon$, so we may as well assume that $\sigma < 1 + \varepsilon$. Since we are assuming RH, $\log \zeta(s)$ is holomorphic in $\sigma > 1/2$. Let $s \in \mathbf{C}$ with $1 + \varepsilon > \sigma \geq \frac{1}{2} + \varepsilon$ and $|t| \geq 2$, and let $\sigma_0 \in (1, t)$ be a parameter to be chosen shortly. We apply the three circles theorem to $f(s) = \log \zeta(s)$ with the circles centered at $\sigma_0 + it$ with radii $r_1 = \sigma_0 - \left(1 + \frac{\varepsilon}{2}\right)$, $r = \sigma_0 - \sigma$, and $r_2 = \sigma_0 - \frac{1+\varepsilon}{2}$. The maximum $M_1$ of $\log \zeta$ on the circle $|z - (\sigma_0 + it)| = r_1$ is $\ll_\varepsilon 1$, and, by the previous part, the maximum $M_2$ of $\log \zeta$ on the circle $|z - (\sigma_0 + it)| = r_2$ is $\ll_\varepsilon \log |t|$. The desired bound $\log \zeta(s) \ll_\varepsilon (\log |t|)^{2 \max\{1-\sigma, 0\} + \varepsilon}$ thus follows immediately from the three circles theorem by taking $\sigma_0 = \log \log t$, since

$$\frac{\log(r/r_1)}{\log(r_2/r_1)} = \frac{\log \left(1 + \frac{1+\varepsilon/2-\sigma}{\sigma_0 - 1 - \varepsilon/2}\right)}{\log \left(1 + \frac{1/2}{\sigma_0 - 1 - \varepsilon/2}\right)} = 2(1-\sigma) + \varepsilon + O_\varepsilon\left(\frac{1}{\sigma_0}\right).$$

The proof that $\frac{\zeta'}{\zeta}(s) \ll_\varepsilon (\log |t|)^{2 \max\{1-\sigma, 0\} + \varepsilon}$ is identical, except that, since $\frac{\zeta'}{\zeta}$ has a pole at $s = 1$, one also has to notice that the assumption $|t| \geq 2$ ensures that the annulus $r_1 < |z - (\sigma_0 + it)| < r_2$ does not contain $z = 1$.

Finally, to deduce the Lindelöf Hypothesis, the above tells us that

$$\zeta\left(\frac{1}{2} + \varepsilon + it\right) \ll_\varepsilon 1$$

for all $\varepsilon > 0$, so that $\zeta(1/2 + it) \ll_\varepsilon t^\varepsilon$ follows from the Phragmén–Lindelöf principle.

4. (a) First, assume RH. By the explicit formula, we have

$$\psi(x) = x - \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} + O\left(\frac{x}{T}(\log xT)^2\right).$$

for all $x, T \geq 2$ with $x \in \frac{1}{2} + \mathbf{Z}$. By RH, $\left|\sum_{|\gamma| \leq T} \frac{x^\rho}{\rho}\right| \leq x^{1/2}(\log T)^2$. Taking $T = x^{1/2}$ in the above and using that $|\psi(x) - \psi(x+y)| \ll \log x$ for all $x \geq 2$ and $y \in [0, 1]$, we thus obtain that $\psi(x) = x + O\left(x^{1/2}(\log x)^2\right) = x + O_\varepsilon(x^{1/2+\varepsilon})$.

For the other direction, we argue as in the first problem. Let $\varepsilon > 0$ and $F(s)$ be the Dirichlet series from the third part of the first problem. Suppose that $\psi(x) = x + O\left(x^{1/2+\varepsilon}\right)$. Then, $F(s)$ converges, and is thus analytic, in the half-plane $\sigma > \frac{1}{2} + \varepsilon$. Since $-\frac{\zeta'}{\zeta}(s) = F(s) + \zeta(s)$ for all $\sigma > \frac{1}{2} + \varepsilon$ as well by the principle of analytic continuation, and $F(s) + \zeta(s)$ is analytic except for a simple pole at $s = 1$, it follows that $-\frac{\zeta'}{\zeta}(s)$ is analytic except for a simple pole at $s = 1$ in the half-plane $\sigma > \frac{1}{2} + \varepsilon$. Since $-\frac{\zeta'}{\zeta}(s)$ has a pole wherever $\zeta(s)$ has a zero, it follows that $\zeta(s)$ has no nontrivial zeros $\rho$ with $\beta > \frac{1}{2} + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude that any nontrivial zero $\rho$ of $\zeta(s)$ must satisfy $\beta \leq \frac{1}{2}$. By the functional equation, if $\rho$ is a nontrivial zero of $\zeta(s)$, then $1 - \rho$ is as well. It follows that any nontrivial zero of $\zeta(s)$ must have $\beta = \frac{1}{2}$, i.e., that RH must hold.

(b) First, assume RH, and let $\varepsilon > 0$. We argue as in the fourth part of the second problem, except that we can take a contour that goes further into the critical strip. As before, it suffices to prove the result for $x \in \frac{1}{2} + \mathbf{Z}$. We have, by Perron's formula, that, whenever $x \geq T \geq 2$,

$$\sum_{n \leq x} \mu(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{1}{\zeta(s)} x^s \frac{\mathrm{d}s}{s} + O\left(\frac{x \log x}{T}\right)$$

where $c = 1 + \frac{1}{\log x}$, so that $x^c \asymp x$. Set $\alpha = \frac{1}{2} + \frac{\varepsilon}{2}$ and let $R_T$ denote the rectangular contour (traversed counterclockwise) with vertices at $c \pm iT$ and $\alpha \pm iT$. Note that $R_T$ is completely contained within the region $\sigma \geq \frac{1}{2} + \varepsilon$. Since $\frac{x^s}{s\zeta(s)}$ is analytic in an open neighborhood of $R_T$ (by the assumption of RH), we have $\frac{1}{2\pi i} \int_{R_T} \frac{x^s}{s\zeta(s)} \mathrm{d}s = 0$. Thus,

$$\int_{c-iT}^{c+iT} \frac{x^s}{s\zeta(s)} \mathrm{d}s \ll \left| \int_{\alpha-iT}^{c-iT} \frac{x^s}{s\zeta(s)} \mathrm{d}s \right| + \left| \int_{\alpha+iT}^{c+iT} \frac{x^s}{s\zeta(s)} \mathrm{d}s \right| + \left| \int_{\alpha-iT}^{\alpha+iT} \frac{x^s}{s\zeta(s)} \mathrm{d}s \right|.$$

We can now bound the three integrals on the right-hand side using the estimates from the previous problem, which tells us that $\frac{1}{\zeta(s)} \ll_\varepsilon |t|^{\varepsilon/2}$, say. For the integrals over the horizontal lines, we thus have

$$\left| \int_{\alpha \pm iT}^{c \pm iT} \frac{x^s}{s\zeta(s)} \mathrm{d}s \right| \ll_\varepsilon \frac{T^{\varepsilon/2}}{T} \int_\alpha^c x^\sigma \mathrm{d}\sigma \ll \frac{x}{T^{1-\varepsilon/2}},$$

and for the integral along the vertical strip, we have

$$\left| \int_{\alpha-iT}^{\alpha+iT} \frac{x^s}{s\zeta(s)} \mathrm{d}s \right| \ll_\varepsilon x^\alpha \int_{-T}^{T} \frac{|t|^\varepsilon}{|t|+1} \mathrm{d}t \ll x^{1/2+\varepsilon/2} T^{\varepsilon/2} (\log T)^2.$$

Hence,

$$\sum_{n \leq x} \mu(n) \ll_\varepsilon \frac{x \log x}{T} + \frac{x}{T^{1-\varepsilon/2}} + x^{1/2+\varepsilon/2} T^{\varepsilon/2} (\log T)^2.$$

Taking $T = \sqrt{x}$ then yields

$$\sum_{n \leq x} \mu(n) \ll_\varepsilon x^{1/2+\varepsilon}.$$

For the other direction, we again argue analogously to the first problem. Set $M(x) := \sum_{n \leq x} \mu(n)$. Let $\varepsilon > 0$, so that $M(x) \ll x^{1/2+\varepsilon}$. By partial summation, we have

$$\sum_{n=1}^{N} \frac{\mu(n)}{n^s} = \frac{M(N)}{N^s} + s \int_1^N \frac{M(t)}{t^{s+1}} \mathrm{d}t$$

for all $N \in \mathbf{N}$ and $s \in \mathbf{C}$. Since $M(x) \ll x^{1/2+\varepsilon}$, it follows that $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ converges whenever $\sigma > \frac{1}{2} + \varepsilon$, and thus is analytic in this half-plane. But, $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$ when $\sigma > 1$, and thus this relation holds in the wider half-plane $\sigma > \frac{1}{2} + \varepsilon$ by the principle of analytic continuation. Since $\frac{1}{\zeta(s)}$ has a pole wherever $\zeta(s)$ has a zero, it follows from the analyticity of $\frac{1}{\zeta(s)}$ in $\sigma > \frac{1}{2} + \varepsilon$ that $\zeta(s)$ has no zeros with $\sigma > \frac{1}{2} + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we thus have that any nontrivial zero of $\zeta(s)$ satisfies $\beta \leq \frac{1}{2}$. Again, since $1 - \rho$ is a nontrivial zero of $\zeta(s)$ whenever $\rho$ is a nontrivial zero, we conclude that any nontrivial zero of $\zeta(s)$ satisfies $\beta = \frac{1}{2}$, i.e., RH holds.