Math 675: Analytic Theory of Numbers
Solutions to problem set # 3

March 4, 2024

1. Recall that
\[
\zeta(s) = \frac{s}{s - 1} - s \int_1^\infty \frac{\{t\}}{ts+1}dt
\]
whenever \(\sigma > 0\). Thus, whenever \(\sigma > 1/2\) (say), we certainly have
\[
\zeta(s) = \frac{1}{s - 1} + 1 - \int_1^\infty \frac{\{t\}}{ts+1}dt - (s - 1) \int_1^\infty \frac{\{t\}}{ts+1}dt
= \frac{1}{s - 1} + 1 - \int_1^\infty \frac{\{t\}}{ts+1}dt + O(|s - 1|).
\]
Now, since \(t^{-2} - t^{-(s+1)} = t^{-2}(1 - t^{-(s-1)}) = t^{-2}(1 - e^{-(s-1)\log t})\) for all \(t \in [1, \infty)\), it follows using the power series for \(z \mapsto e^z\) centered at 0 that \(t^{-2} - t^{-(s+1)} = t^{-2}(1 - (1 + O(|s - 1|\log t))) = O(|s - 1|t^{-2}\log t)\) in a neighborhood of \(s = 1\). Thus,
\[
\int_1^\infty \frac{\{t\}}{ts+1}dt = \int_1^\infty \frac{\{t\}}{t^2}dt + O\left(|s - 1|\int_1^\infty \frac{\{t\}\log t}{t^2}dt\right) = \int_1^\infty \frac{\{t\}}{t^2}dt + O(|s - 1|)
\]
in a neighborhood of \(s = 1\). We can thus deduce that
\[
\zeta(s) = \frac{1}{s - 1} + \gamma + O(|s - 1|)
\]
in a neighborhood of \(s = 1\).

2. \(\text{(a)}\) We have, by definition, that \(b_0(0) = 1 = b_0(1)\). Now suppose that \(n \geq 1\). Then, we have that
\[
0 = (n + 1) \int_0^1 b_n(x)dx = \int_0^1 (n + 1)b_n(x)dx = \int_0^1 b'_n(x)dx = b_{n+1}(1) - b_{n+1}(0).
\]
Thus, \(b_n(0) = b_n(1)\) for all \(n \geq 2\) as well. It follows that \(b_n(0) = b_n(1)\) whenever \(n \neq 1\). This implies that \(B_n(x)\) is continuous and periodic when \(n \neq 1\). Similarly, when \(n \geq 1\), since \(\int_0^m B_n(t)dt = 0\) for any \(m \in \mathbb{N}\), we have
\[
\int_0^x B_n(t)dt = \frac{1}{n + 1} \int_0^{\{x\}} (n + 1)b_n(t)dt = \frac{b_{n+1}(\{x\}) - b_{n+1}(0)}{n + 1} = \frac{B_{n+1}(x) - B_{n+1}}{n + 1}.
\]
\(\text{(b)}\) By partial summation and writing \([t] = t - \{t\}\), we have
\[
\sum_{a < n \leq b} f(n) = \int_a^b \{t\} f(t)dt = \int_a^b [t] f'(t)dt = \int_a^b t f(t)dt + \int_a^b \{t\} f'(t)dt.
\]
Integrating by parts gives \( f_a^b tf'(t)dt = tf(t)|_a^b - f_a^b f(t)dt \). Thus,

\[
\sum_{a<n\leq b} f(n) = \int_a^b f(t)dt + \int_a^b \{t\} f'(t)dt.
\]

Since \( B_1(t) = \{t\} - 1/2 \), we get \( \int_a^b \{t\} f'(t)dt = \int_a^b B_1(t)f'(t)dt + \frac{f(b) - f(a)}{2} \), so that

\[
\sum_{a<n\leq b} f(n) = \int_a^b f(t)dt + \frac{f(b) - f(a)}{2} + \int_a^b B_1(t)f'(t)dt,
\]

which is the desired identity for \( k = 1 \). We will proceed by induction, having done the base case \( k = 1 \). Suppose that the identity holds for a general \( k \geq 1 \), so that

\[
\sum_{a<n\leq b} f(n) = \int_a^b f(t)dt + \sum_{r=1}^{k} \frac{(-1)^r B_r}{r!} (f^{(r-1)}(b) - f^{(r-1)}(a)) + (-1)^{k+1} \int_a^b \frac{B_{k+1}(t)f^{(k)}(t)}{k!} dt.
\]

We have that \( \int_a^b \frac{B_k(t)f^{(k)}(t)}{k!} dt \) equals

\[
\int_a^b \frac{(k+1)B_k(t)f^{(k)}(t)}{(k+1)!} dt = \int_a^b \frac{B_{k+1}(t)f^{(k)}(t)}{(k+1)!} dt = \int_a^b \frac{B_{k+1}(t)f^{(k+1)}(t)}{(k+1)!} dt
\]

by integrating by parts. Using that \( a, b \in \mathbb{Z} \), so that \( B_{k+1}(a) = B_{k+1}(b) = B_{k+1} \), we obtain \( \frac{B_{k+1}(t)f^{(k)}(t)}{(k+1)!} |_a^b = B_{k+1}(f^{(k)}(b) - f^{(k)}(a)) \), so that

\[
\sum_{a<n\leq b} f(n) = \int_a^b f(t)dt + \sum_{r=1}^{k+1} \frac{(-1)^r B_r}{r!} (f^{(r-1)}(b) - f^{(r-1)}(a)) + (-1)^{k+2} \int_a^b \frac{B_{k+1}(t)f^{(k+1)}(t)}{(k+1)!} dt,
\]

as desired.

(c) Note that, when \( k = 0 \), we have

\[
\int_0^1 B_k(x)e^{-2\pi imx}dx = \int_0^1 e^{-2\pi imx}dx = 1_{m=0}.
\]

When \( k = 1 \), we have that \( \int_0^1 B_k(x)e^{-2\pi imx}dx \) equals

\[
\int_0^1 xe^{-2\pi imx}dx - \frac{1}{2} \int_0^1 e^{-2\pi imx}dx = \begin{cases} \frac{1}{2} & m = 0 \\ -\frac{1}{2\pi im} & m \neq 0 \end{cases}
\]

by integrating by parts. This proves the desired identity in the \( k = 1 \) case. Note that, when \( m = 0 \) and \( k \geq 1 \), we have \( \int_0^1 B_k(x)e^{-2\pi imx}dx = \int_0^1 B_k(x)dx = 0 \), proving the desired identity in this case as well. We now proceed by induction, supposing that

\[
\int_0^1 B_k(x)e^{-2\pi imx}dx = -1_{m \neq 0} \frac{k!}{(2\pi im)^k}
\]

for a general \( k \geq 1 \) and \( m \neq 0 \). Then, \( -1_{m \neq 0} \frac{(k+1)!}{(2\pi im)^{k+1}} \) equals

\[
\int_0^1 (k+1)B_k(x)e^{-2\pi imx}dx = \int_0^1 B_{k+1}(x)e^{-2\pi imx}dx = 2\pi im \int_0^1 B_{k+1}(x)e^{-2\pi imx}dx
\]
by integrating by parts. Dividing through by $2\pi im$ yields

$$
\int_0^1 B_{k+1}(x)e^{-2\pi imx}dx = -1_{m\neq 0} \frac{(k+1)!}{(2\pi im)^{k+1}},
$$
as desired.

Note that, when $k \geq 2$, the Fourier series $\sum_{m\neq 0} \frac{-k!}{(2\pi im)^2} e^{2\pi imx}$ of $B_k(x)$ is absolutely convergent. Since each such $B_k$ is continuous, it follows that the Fourier series converges to $B_k(x)$ pointwise, i.e., that

$$
B_k(x) = -\frac{k!}{(2\pi i)^k} \sum_{m\neq 0} \frac{e^{2\pi imx}}{m^k}
$$

when $k \geq 2$.

(d) Let $k \geq 1$. Then, by the previous part, $B_{2k+1} = B_{2k+1}(0) = -\frac{(2k+1)!}{(2\pi i)^{2k+1}} \sum_{m\neq 0} \frac{1}{m^{2k+1}}$, which equals

$$
-\frac{(2k+1)!}{(2\pi i)^{2k+1}} \left( \sum_{m=1}^{\infty} \frac{1}{m^{2k+1}} + \sum_{m=1}^{\infty} \frac{1}{(-m)^{2k+1}} \right) = -\frac{(2k+1)!}{(2\pi i)^{2k+1}} \left( \sum_{m=1}^{\infty} \frac{1}{m^{2k+1}} - \sum_{m=1}^{\infty} \frac{1}{m^{2k+1}} \right) = 0.
$$

We also have that

$$
B_{2k} = B_{2k}(0) = -\frac{(2k)!}{(2\pi i)^{2k}} \sum_{m\neq 0} \frac{1}{m^{2k}} = -\frac{(-1)^k}{(2\pi i)^{2k}} \sum_{m=1}^{\infty} \frac{1}{m^{2k}} = -\frac{(-1)^k(2k)!}{2^{2k-1} \pi^{2k}} \sum_{m=1}^{\infty} \frac{1}{m^{2k}}.
$$

Rearranging yields

$$
\zeta(2k) = (-1)^k \frac{2^{2k-1} \pi^{2k} B_{2k}}{(2k)!}.
$$

(e) Let $n \geq 1$, so that $b_n^{(k)}(x) = n(n-1) \cdots (n+1-k)b_{n-k}(x)$ for all $k = 1, \ldots, n$. Thus, $b_n^{(k)}(0) = n(n-1) \cdots (n+1-k)b_{n-k}(0) = n(n-1) \cdots (n+1-k)B_{n-k}$. It follows that the $k^{th}$ coefficient of $b_n(x)$ is $\binom{n}{k} B_{n-k}$, so that

$$
b_n(x) = B_n + \sum_{k=1}^{n} \binom{n}{k} B_{n-k} x^k.
$$

Integrating both sides over $[0,1]$ yields

$$
0 = B_n + \sum_{k=1}^{n} \binom{n}{k} B_{n-k} = B_n + \frac{1}{n+1} \sum_{k=1}^{n} \binom{n+1}{k+1} B_{n-k} = B_n + \frac{1}{n+1} \sum_{k=2}^{n+1} \binom{n+1}{k} B_{n+1-k}.
$$

Rearranging gives $B_n = -(n+1)^{-1} \sum_{k=2}^{n+1} \binom{n+1}{k} B_{n+1-k}$, as desired. To compute $\zeta(2), \zeta(4), \zeta(6)$, and $\zeta(8)$, we now just need to compute $B_2, B_4, B_6$, and $B_8$ using the fact that $B_0 = 1$, $B_{2k+1} = 0$ for all $k \geq 1$, and this recurrence. We have $B_1 = -\frac{1}{2}$,

$$
B_2 = -\frac{1}{3} \left( \binom{3}{2} B_1 + \binom{3}{3} B_0 \right) = \frac{1}{6},
$$

$$
B_4 = -\frac{1}{5} \left( \binom{5}{3} B_2 + \binom{5}{4} B_1 + \binom{5}{5} B_0 \right) = -\frac{1}{30}.
$$

3
3. (a) By Euler–Maclaurin summation, we have
\[
\sum_{n=2}^{N} \frac{1}{n^s} = \int_{1}^{N} t^{-s} \, dt - \sum_{r=1}^{k+1} \frac{B_r}{r} \left( \frac{s + r - 2}{r - 1} \right) (N^{-s+1-r} - 1) - \int_{1}^{N} B_{k+1}(t) \left( \frac{s + k}{k + 1} \right) t^{-s-k-1} \, dt
\]
for all \( N \in \mathbb{N} \), since \( \frac{d^r}{dt^r} t^{-s} = (-1)^r s(s+1) \cdots (s+r-1) t^{-s-r} = (-1)^r r! \left( \frac{s + r - 1}{s - 1} \right) t^{-s-r} \) for all \( r \in \mathbb{N} \). Letting \( s > 1 \), shifting the index on the sum on the right-hand side, and taking \( N \to \infty \) then yields
\[
\zeta(s) = 1 + \int_{1}^{\infty} t^{-s} \, dt + \sum_{0 \leq r \leq k} \frac{B_{r+1}}{r+1} \left( \frac{s + r - 1}{r} \right) - \left( \frac{s + k}{k + 1} \right) \int_{1}^{\infty} B_{k+1}(t) t^{-s-k-1} \, dt
\]
\[
= \frac{s}{s-1} + \sum_{0 \leq r \leq k} \frac{B_{r+1}}{r+1} \left( \frac{s + r - 1}{r} \right) - \left( \frac{s + k}{k + 1} \right) \int_{1}^{\infty} B_{k+1}(t) t^{-s-k-1} \, dt,
\]
since \( \int_{1}^{\infty} t^{-s} \, dt = \frac{1}{s-1} \).

(b) Note that \( \sum_{0 \leq r \leq k} \frac{B_{r+1}}{r+1} \left( \frac{s + r - 1}{r} \right) \), being a polynomial in \( s \) for every integer \( k \geq 0 \), is entire, as is \( \left( \frac{s + k}{k + 1} \right) \). The integral \( \int_{1}^{\infty} B_{k+1}(t) t^{-s-k-1} \, dt \) converges absolutely whenever \( \sigma > -k \), and thus defines an analytic function in this half-plane. It follows that, for every integer \( k \geq 0 \),
\[
\frac{s}{s-1} + \sum_{0 \leq r \leq k} \frac{B_{r+1}}{r+1} \left( \frac{s + r - 1}{r} \right) - \left( \frac{s + k}{k + 1} \right) \int_{1}^{\infty} B_{k+1}(t) t^{-s-k-1} \, dt.
\]
is a meromorphic function on the half-plane \( \sigma > -k \) that is analytic except for a simple pole at \( s = 1 \) with residue 1. Since the above equals \( \zeta(s) \) for \( s \in (1, \infty) \), by the principle of analytic continuation, we can meromorphically continue \( \zeta(s) \) to the whole complex plane with its only singularity being a simple pole at \( s = 1 \) with residue 1.

4. (a) Note that \( \tau^* \) is multiplicative and thus, by looking at the values of \( \tau^* \) on prime powers, that
\[
F(s) := D\tau^*(s) = \frac{1}{1 - 2^{-s}} \prod_{p>2} \left( \frac{1}{1 - p^{-s}} \right)^{2} = \frac{(1 - 2^{-s})^2}{1 - 2^{-s}} \zeta(s)^2 = (1 - 2^{-s}) \zeta(s)^2
\]
for \( \sigma > 1 \). Let \( \delta > 0 \) and \( S \geq 1 \) be parameters to be chosen later, and assume that \( x > 0 \) satisfies \( x - \frac{1}{2} \in \mathbb{Z} \). We apply Perron’s formula with \( c = 1 + \frac{1}{\log x} \) to obtain
\[
T(x) = \frac{1}{2\pi i} \int_{c-iS}^{c+iS} F(s)x^s \frac{ds}{s} + O \left( \frac{x^{1+c}}{S} \sum_{x/2<n\leq 2x} \frac{1}{n^c} + \frac{x^c}{S} \sum_{n=1}^{\infty} \frac{1}{n^c} \right)
\]
\[
= \frac{1}{2\pi i} \int_{c-iS}^{c+iS} F(s)x^s \frac{ds}{s} + O \left( \frac{x \log x}{S} \right).
\]
Let $R_{\delta,S}$ denote the rectangular contour with vertices at $c \pm iS$, $\delta \pm iS$ traversed counterclockwise. Inside of this contour, \( \frac{F(s)x^s}{s} \) has a pole of order 2 at $s = 1$ with residue $\frac{\pi}{2} \log 2 + \frac{\pi}{2}(2\gamma - 1)$, where we have used the first problem and the power series expansions $\frac{1}{s} = \sum_{j=0}^{\infty} \frac{(-1)^j(s-1)^j}{j!}$, $x^s = x \sum_{j=0}^{\infty} \frac{(\log x)^j}{j!}(s-1)^j$, and $(1 - 2^{-s}) = \frac{1}{2} + \log 2 + (s-1) + \ldots$ to help compute the residue. Thus,

\[
\frac{1}{2\pi i} \int_{c-iS}^{c+iS} F(s)x^s \frac{ds}{s} = \frac{x}{2} (\log 2x + 2\gamma - 1) - \frac{1}{2\pi i} \left( \int_{\delta-iS}^{\delta+iS} + \int_{c-iS}^{c+iS} - \int_{\delta+iS}^{c+iS} \right) F(s)x^s \frac{ds}{s}.
\]

Using that $\zeta(s) \ll \epsilon (1 + |t|)^{(1 - \sigma + \epsilon/2)/2}$ whenever $|s - 1| \gg 1$ and $\sigma \in [0, 1 + \epsilon]$, that $|\delta + it| \gg 1 + |t|$, and that $|1 - 2^{-\delta - it}| \ll 1 + 2^{-\delta} \ll 1$ for all real $t$, we can bound

\[
\int_{\delta-iS}^{\delta+iS} F(s)x^s \frac{ds}{s} \ll_{\epsilon, \delta} x^\delta \int_{-S}^{S} (1 + |t|)^{-\delta + \epsilon/2} dt \ll x^\delta S^{1-\delta + \epsilon/2}
\]

and, assuming $x > S$,

\[
\int_{\delta+iS}^{c+iS} F(s)x^s \frac{ds}{s} \ll_{\epsilon, \delta} S^{\epsilon/2} \int_{\delta}^{c} S^{-\sigma} x^\sigma d\sigma \ll \frac{S^{\epsilon/2}}{\log(x/S)} (x/S)^{c}.
\]

Thus,

\[
|T(x) - \frac{x}{2} (\log 2x + 2\gamma - 1)| \ll \epsilon, \delta \quad \frac{x \log x}{S} + x^\delta S^{1-\delta + \epsilon/2} + \frac{S^{\epsilon/2}}{\log(x/S)} (x/S)^{c}.
\]

Choosing $S = \sqrt{x}$ bounds the above by $\ll \epsilon, \delta \quad x^{1/2+\epsilon/2} + x^{1/2+\delta/2+\epsilon/2} + x^{1/2+\epsilon/4}$. Taking $\delta = \epsilon$ yields the desired asymptotic.

(b) By $D\tau^s (s) = (1 - 2^{-s}) \zeta(s)^2$, we have

\[
\sum_{n=1}^{\infty} \frac{\tau^s (n)}{n^s} = \sum_{n=1}^{\infty} \frac{\tau (n)}{n^s} - \sum_{n=1}^{\infty} \frac{\tau (n)}{(2n)^s} = \sum_{n=1}^{\infty} \frac{\tau (n)}{n^s} - \sum_{n \in \mathbb{N} \text{ even}} \frac{\tau(n/2)}{n^s}
\]

for $\sigma > 1$. Thus,

\[
\tau^s (n) = \begin{cases} 
\tau (n) & \text{n odd} \\
\tau (n) - \tau (n/2) & \text{n even}
\end{cases}
\]

for all $n \in \mathbb{N}$. It follows that $T^* (x) = T(x) - \sum_{n \text{ even}}^{} \tau(n/2) = T(x) - \sum_{n \leq x/2}^{} \tau(n) = T(x) - T(x/2)$. Using the asymptotic for $T(x)$ from class, we conclude that $T^* (x)$ equals

\[
x \log x + (2\gamma - 1)x - \frac{x}{2} \log(x/2) - (2\gamma - 1)\frac{x}{2} + O(\sqrt{x}) = \frac{x}{2} (\log 2x + (2\gamma - 1)) + O(\sqrt{x}).
\]

5. Both of the desired asymptotics can be proven using either elementary methods or Perron’s formula. We will use Perron’s formula in our solutions.

(a) Let $f : \mathbb{N} \rightarrow \{0, 1\}$ denote the indicator function of the cube-frees. Then $f$ is multiplicative, and

\[
D \tau^s (s) = \prod_{p}^{} (1 + p^{-s} + p^{-2s}) = \prod_{p}^{} \frac{1 - p^{-3s}}{1 - p^{-s}} = \frac{\zeta (s)}{\zeta(3s)}
\]
for $\sigma > 1$. Let $\delta > 0$ and $T \geq 1$ be a parameter to be chosen later, and assume that $x > 0$ satisfies $x \in \frac{1}{2} + \mathbb{Z}$. We apply Perron’s formula with $c = 1 + \frac{1}{\log x}$ to obtain that the number of cube-free integers below $x$ equals

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta(s)}{\zeta(3s)} \frac{x^s}{s} ds + O\left(\frac{x \log x}{T}\right).$$

Let $R_T$ denote the rectangular contour with vertices at $c \pm iT$, $\frac{1}{2} \pm iT$ traversed counterclockwise. Inside of this contour, $\frac{\zeta(s)x^s}{\zeta(3s)s}$ has a simple pole at $s = 1$ with residue $\frac{x}{\zeta(3)}$. Thus,

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta(s)}{\zeta(3s)} \frac{x^s}{s} ds = \frac{1}{\zeta(3)} \int_{1/2-iT}^{1/2+iT} \frac{\zeta(s)}{\zeta(3s)} \frac{x^s}{s} ds + O\left(\frac{x \log x}{T}\right).$$

Note that, whenever $s \geq 1$, we have

$$\frac{1}{\zeta(3s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{3s}} \ll \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \ll 1.$$

Using this estimate, along with that $\zeta(s) \ll_{\delta} (1 + |t|)^{(1-\sigma+\delta)/2}$ whenever $|s - 1| \gg 1$ and $0 \leq \sigma \leq 1 + \delta$, we can bound

$$\int_{1/2-iT}^{1/2+iT} \frac{\zeta(s)}{\zeta(3s)} \frac{x^s}{s} ds \ll_{\delta} \sqrt{x} \int_{-T}^{T} (1 + |t|)^{(1/2+\delta)/2-1/2} dt \ll \sqrt{x} T^{1/4+\delta/2}.$$

and

$$\int_{1/2\pm iT}^{c \pm iT} \frac{\zeta(s)}{\zeta(3s)} \frac{x^s}{s} ds \ll_{\delta} T^{(\delta-1)/2} \int_{1/2}^{c} T^{-\sigma/2} x^\sigma d\sigma \ll_{\delta} T^{(\delta-1)/2-1/4} x.$$

Thus, picking $\delta = 1/8$, we have that the number of cube-free integers below $x$ equals

$$\frac{x}{\zeta(3)} + O\left(x \log x + x^{1/2} T^{5/16} + x T^{-11/16}\right).$$

Choosing $T = \sqrt{x}$ yields that the number of cube-free integers below $x$ equals

$$\frac{x}{\zeta(3)} + O(x^{21/32})$$

whenever $x \in \frac{1}{2} + \mathbb{Z}$ is positive. This estimate can be extended to all $x > 0$ by noting that $\left|\frac{x^{1/2}}{\zeta(3)} - \frac{x^{-1/2}}{\zeta(3)}\right| \ll 1$, which can be absorbed into the error term $O(x^{21/32})$. We conclude that the number of cube-frees below $x > 0$ is $\sim \frac{x}{\zeta(3)}$.

(b) First, observe that $n \in \mathbb{N}$ is cube-full if and only if $n = a^3 b^4 c^5$ for some $a, b, c \in \mathbb{N}$, and that this representation is unique when $bc$ is squarefree. Indeed, if $n = p_1^{m_1} \cdots p_k^{m_k}$ is the prime factorization of $n$, then we have

$$b = \prod_{i=1}^{k} p_i^{13(m_i-1)} \quad \text{and} \quad c = \prod_{i=1}^{k} p_i^{13(m_i-2)},$$

where $m_i$ are the exponents in the prime factorization of $n$, and $b$ and $c$ are the unique representations of $n$ as $a^3 b^4 c^5$. We can express $n$ as $n = \prod_{i=1}^{k} p_i^{13(m_i-1)} \cdot \prod_{i=1}^{k} p_i^{13(m_i-2)}$, and by the uniqueness of the prime factorization, this expression is unique. This completes the proof that $n$ is cube-full if and only if $n = a^3 b^4 c^5$. 


which determines \( a \) as well. Thus,

\[
\mathcal{D}f(s) = \sum_{a,b,c \in \mathbb{N}} \frac{\mu(bc)^2}{(a^3b^4c^5)^s} = \zeta(3s)F(s),
\]

where \( F(s) := \sum_{b,c \in \mathbb{N}} \frac{\mu(bc)^2}{(b^2c^5)^s} \), when \( \sigma > 1 \). Observe that \( F(s) \) converges absolutely for \( \sigma > \frac{1}{3} \), and thus in analytic in this half-plane, and, further, satisfies \( F(s) \ll 1 \) when \( \sigma \geq \frac{5}{18} \), say. We, thus, have \( \mathcal{D}f(s) = \zeta(3s)F(s) \) for all \( s \in \mathbb{C} \) by the principle of analytic continuation. Let \( \delta > 0 \) and \( T \geq 1 \) be parameters to be chosen later, and assume that \( x > 0 \) satisfies \( x \in \frac{1}{2} + \mathbb{Z} \). We apply Perron’s formula with \( c = \frac{1}{3} + \frac{1}{\log x} \) to obtain that the number of cube-full integers below \( x \) equals

\[
\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \zeta(s)F(s)x^s \frac{ds}{s} + O\left(\frac{x^{1/3} \log x}{T}\right).
\]

Let \( R_T \) denote the rectangular contour with vertices at \( c \pm iT \) and \( \frac{5}{18} \pm iT \) traversed counterclockwise. Inside of this contour, \( \frac{\zeta(3s)F(s)x^s}{s} \) has a simple pole at \( s = \frac{1}{3} \) with residue \( 3F(1/3)x^{1/3} \). Thus,

\[
\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \zeta(3s)F(s)x^s \frac{ds}{s} = 3F(1/3)x^{1/3} - \frac{1}{2\pi i} \left( \int_{5/18-iT}^{5/18+iT} + \int_{5/18-iT}^{c-iT} + \int_{c+iT}^{5/18+iT} \right) \zeta(3s)F(s)x^s \frac{ds}{s}.
\]

Using, yet again, that \( \zeta(s) \ll \delta (1 + |t|)^{(1-\sigma+\delta)/2} \) whenever \( |s-1| \gg 1 \) and \( \sigma \in [0, 1+\delta] \), and that \( |\sigma + it| \asymp 1 + |t| \) for \( \sigma \in [5/18, 1] \), we can bound

\[
\int_{5/18-iT}^{5/18+iT} \zeta(3s)F(s)x^s \frac{ds}{s} \ll \delta x^{5/18} \int_{-T}^{T} (1+|3t|)^{(1/6+\delta)/2-1} dt \ll \delta x^{5/18} T^{(1/6+\delta)/2}
\]

and

\[
\int_{5/18-iT}^{c+iT} \zeta(3s)F(s)x^s \frac{ds}{s} \ll \delta T^{(\delta-1)/2} \int_{5/18}^{c} T^{-3\delta/2} x^\sigma d\sigma \ll \delta T^{(\delta-1)/2-5/12} x^{1/3}.
\]

Thus, picking \( \delta = 1/6 \), we have that the number of cube-full integers below \( x \) equals

\[
3F(1/3)x^{1/3} + O\left(\frac{x^{1/3} \log x}{T} + x^{5/18} T^{1/6} + T^{-5/6} x^{1/3}\right).
\]

Choosing \( T = x^{1/18} \) yields that the number of cube-full integers below \( x \) equals

\[
3F(1/3)x^{1/3} + O\left(x^{31/108}\right)
\]

whenever \( x \in \frac{1}{2} + \mathbb{Z} \). This estimate can be extended to all \( x > 0 \) by noting that \( 3F(1/3)(|x| + 1/2)^{1/3} - 3F(1/3)(|x| - 1/2)^{1/3} \ll 1 \), which can be absorbed into the error term. We therefore conclude that the number of cube-fulls below \( x > 0 \) is \(~ 3 \left( \sum_{b,c \in \mathbb{N}} \frac{\mu(bc)^2}{b^3/3^2/3} \right) x^{1/3} \).