Math 675: Analytic Theory of Numbers
Solutions to problem set # 3

March 4, 2024

1. Recall that
$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}\mathrm{d}t$$
whenever $\sigma > 0$. Thus, whenever $\sigma > 1/2$ (say), we certainly have
$$\begin{aligned}\zeta(s) &= \frac{1}{s-1} + 1 - \int_1^\infty \frac{\{t\}}{t^{s+1}}\mathrm{d}t - (s-1)\int_1^\infty \frac{\{t\}}{t^{s+1}}\mathrm{d}t \\ &= \frac{1}{s-1} + 1 - \int_1^\infty \frac{\{t\}}{t^{s+1}}\mathrm{d}t + O\left(|s-1|\right).\end{aligned}$$
Now, since $t^{-2} - t^{-(s+1)} = t^{-2}(1 - t^{-(s-1)}) = t^{-2}(1 - e^{-(s-1)\log t})$ for all $t \in [1, \infty)$, it follows using the power series for $z \mapsto e^z$ centered at 0 that $t^{-2} - t^{-(s+1)} = t^{-2}(1 - (1 + O(|s-1|\log t))) = O(|s-1|t^{-2}\log t)$ in a neighborhood of $s = 1$. Thus,
$$\int_1^\infty \frac{\{t\}}{t^{s+1}}\mathrm{d}t = \int_1^\infty \frac{\{t\}}{t^2}\mathrm{d}t + O\left(|s-1|\int_1^\infty \frac{\{t\}\log t}{t^2}\mathrm{d}t\right) = \int_1^\infty \frac{\{t\}}{t^2}\mathrm{d}t + O(|s-1|)$$
in a neighborhood of $s = 1$. We can thus deduce that
$$\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|)$$
in a neighborhood of $s = 1$.

2. (a) We have, by definition, that $b_0(0) = 1 = b_0(1)$. Now suppose that $n \geq 1$. Then, we have that
$$0 = (n+1)\int_0^1 b_n(x)\mathrm{d}x = \int_0^1 (n+1)b_n(x)\mathrm{d}x = \int_0^1 b'_{n+1}(x)\mathrm{d}x = b_{n+1}(1) - b_{n+1}(0).$$
Thus, $b_n(0) = b_n(1)$ for all $n \geq 2$ as well. It follows that $b_n(0) = b_n(1)$ whenever $n \neq 1$. This implies that $B_n(x)$ is continuous and periodic when $n \neq 1$. Similarly, when $n \geq 1$, since $\int_0^m B_n(t)\mathrm{d}t = 0$ for any $m \in \mathbf{N}$, we have
$$\int_0^x B_n(t)\mathrm{d}t = \frac{1}{n+1}\int_0^{\{x\}} (n+1)b_n(t)\mathrm{d}t = \frac{b_{n+1}(\{x\}) - b_{n+1}(0)}{n+1} = \frac{B_{n+1}(x) - B_{n+1}}{n+1}.$$
(b) By partial summation and writing $\lfloor t \rfloor = t - \{t\}$, we have
$$\sum_{a<n\leq b} f(n) = tf(t)\Big|_a^b - \int_a^b \lfloor t \rfloor f'(t)\mathrm{d}t = tf(t)\Big|_a^b - \int_a^b tf'(t)\mathrm{d}t + \int_a^b \{t\}f'(t)\mathrm{d}t.$$

Integrating by parts gives $\int_a^b tf'(t)\mathrm{d}t = tf(t)\big|_a^b - \int_a^b f(t)\mathrm{d}t$. Thus,

$$\sum_{a<n\le b} f(n) = \int_a^b f(t)\mathrm{d}t + \int_a^b \{t\}f'(t)\mathrm{d}t.$$

Since $B_1(t) = \{t\} - 1/2$, we get $\int_a^b \{t\}f'(t)\mathrm{d}t = \int_a^b B_1(t)f'(t)\mathrm{d}t + \frac{f(b)-f(a)}{2}$, so that

$$\sum_{a<n\le b} f(n) = \int_a^b f(t)\mathrm{d}t + \frac{f(b)-f(a)}{2} + \int_a^b B_1(t)f'(t)\mathrm{d}t,$$

which is the desired identity for $k=1$. We will proceed by induction, having done the base case $k=1$. Suppose that the identity holds for a general $k \ge 1$, so that

$$\sum_{a<n\le b} f(n) = \int_a^b f(t)\mathrm{d}t + \sum_{r=1}^{k} \frac{(-1)^r B_r}{r!}(f^{(r-1)}(b) - f^{(r-1)}(a)) + (-1)^{k+1}\int_a^b \frac{B_k(t)f^{(k)}(t)}{k!}\mathrm{d}t.$$

We have that $\int_a^b \frac{B_k(t)f^{(k)}(t)}{k!}\mathrm{d}t$ equals

$$\int_a^b \frac{(k+1)B_k(t)f^{(k)}(t)}{(k+1)!}\mathrm{d}t = \int_a^b \frac{B'_{k+1}(t)f^{(k)}(t)}{(k+1)!}\mathrm{d}t = \frac{B_{k+1}(t)f^{(k)}(t)\big|_a^b}{(k+1)!} - \int_a^b \frac{B_{k+1}(t)f^{(k+1)}(t)}{(k+1)!}\mathrm{d}t$$

by integrating by parts. Using that $a, b \in \mathbf{Z}$, so that $B_{k+1}(a) = B_{k+1}(b) = B_{k+1}$, we obtain $\frac{B_{k+1}(t)f^{(k)}(t)\big|_a^b}{(k+1)!} = \frac{B_{k+1}(f^{(k)}(b)-f^{(k)}(a))}{(k+1)!}$, so that

$$\sum_{a<n\le b} f(n) = \int_a^b f(t)\mathrm{d}t + \sum_{r=1}^{k+1} \frac{(-1)^r B_r}{r!}(f^{(r-1)}(b) - f^{(r-1)}(a)) + (-1)^{k+2}\int_a^b \frac{B_{k+1}(t)f^{(k+1)}(t)}{(k+1)!}\mathrm{d}t,$$

as desired.

(c) Note that, when $k=0$, we have

$$\int_0^1 B_k(x)e^{-2\pi imx}\mathrm{d}x = \int_0^1 e^{-2\pi imx}\mathrm{d}x = 1_{m=0}.$$

When $k=1$, we have that $\int_0^1 B_k(x)e^{-2\pi imx}\mathrm{d}x$ equals

$$\int_0^1 xe^{-2\pi imx}\mathrm{d}x - \frac{1}{2}\int_0^1 e^{-2\pi imx}\mathrm{d}x = \begin{cases} \frac{1}{2} & m=0 \\ -\frac{1}{2\pi im} & m \ne 0 \end{cases} - \frac{1}{2}1_{m=0} = 1_{m\ne 0}\frac{-1}{2\pi im}$$

by integrating by parts. This proves the desired identity in the $k=1$ case. Note that, when $m=0$ and $k \ge 1$, we have $\int_0^1 B_k(x)e^{-2\pi imx}\mathrm{d}x = \int_0^1 B_k(x)\mathrm{d}x = 0$, proving the desired identity in this case as well. We now proceed by induction, supposing that

$$\int_0^1 B_k(x)e^{-2\pi imx}\mathrm{d}x = -1_{m\ne 0}\frac{k!}{(2\pi im)^k}$$

for a general $k \ge 1$ and $m \ne 0$. Then, $-1_{m\ne 0}\frac{(k+1)!}{(2\pi im)^k}$ equals

$$\int_0^1 (k+1)B_k(x)e^{-2\pi imx}\mathrm{d}x = \int_0^1 B'_{k+1}(x)e^{-2\pi imx}\mathrm{d}x = 2\pi im\int_0^1 B_{k+1}(x)e^{-2\pi imx}\mathrm{d}x$$

by integrating by parts. Dividing through by $2\pi i m$ yields

$$\int_0^1 B_{k+1}(x)e^{-2\pi i m x}\mathrm{d}x = -1_{m\neq 0}\frac{(k+1)!}{(2\pi i m)^{k+1}},$$

as desired.

Note that, when $k \geq 2$, the Fourier series $\sum_{m\neq 0}\frac{-k!}{(2\pi i m)^k}e^{2\pi i m x}$ of $B_k(x)$ is absolutely convergent. Since each such B_k is continuous, it follows that the Fourier series converges to $B_k(x)$ pointwise, i.e., that

$$B_k(x) = -\frac{k!}{(2\pi i)^k}\sum_{m\neq 0}\frac{e^{2\pi i m x}}{m^k}$$

when $k \geq 2$.

(d) Let $k \geq 1$. Then, by the previous part, $B_{2k+1} = B_{2k+1}(0) = -\frac{(2k+1)!}{(2\pi i)^{2k+1}}\sum_{m\neq 0}\frac{1}{m^{2k+1}}$, which equals

$$-\frac{(2k+1)!}{(2\pi i)^{2k+1}}\left(\sum_{m=1}^{\infty}\frac{1}{m^{2k+1}} + \sum_{m=1}^{\infty}\frac{1}{(-m)^{2k+1}}\right) = -\frac{(2k+1)!}{(2\pi i)^{2k+1}}\left(\sum_{m=1}^{\infty}\frac{1}{m^{2k+1}} - \sum_{m=1}^{\infty}\frac{1}{m^{2k+1}}\right) = 0.$$

We also have that

$$B_{2k} = B_{2k}(0) = -\frac{(2k)!}{(2\pi i)^{2k}}\sum_{m\neq 0}\frac{1}{m^{2k}} = \frac{(-1)^{k-1}(2k)!}{(2\pi)^{2k}}2\sum_{m\geq 1}\frac{1}{m^{2k}} = \frac{(-1)^{k-1}(2k)!}{2^{2k-1}\pi^{2k}}\sum_{m\geq 1}\frac{1}{m^{2k}}.$$

Rearranging yields

$$\zeta(2k) = (-1)^{k-1}\frac{2^{2k-1}\pi^{2k}B_{2k}}{(2k)!}.$$

(e) Let $n \geq 1$, so that $b_n^{(k)}(x) = n(n-1)\cdots(n+1-k)b_{n-k}(x)$ for all $k = 1,\ldots,n$. Thus, $b_n^{(k)}(0) = n(n-1)\cdots(n+1-k)b_{n-k}(0) = n(n-1)\cdots(n+1-k)B_{n-k}$. It follows that the k^{th} coefficient of $b_n(x)$ is $\binom{n}{k}B_{n-k}$, so that

$$b_n(x) = B_n + \sum_{k=1}^{n}\binom{n}{k}B_{n-k}x^k.$$

Integrating both sides over $[0,1]$ yields

$$0 = B_n + \sum_{k=1}^{n}\binom{n}{k}\frac{B_{n-k}}{k+1} = B_n + \frac{1}{n+1}\sum_{k=1}^{n}\binom{n+1}{k+1}B_{n-k} = B_n + \frac{1}{n+1}\sum_{k=2}^{n+1}\binom{n+1}{k}B_{n+1-k}.$$

Rearranging gives $B_n = -(n+1)^{-1}\sum_{k=2}^{n+1}\binom{n+1}{k}B_{n+1-k}$, as desired. To compute $\zeta(2), \zeta(4), \zeta(6)$, and $\zeta(8)$, we now just need to compute B_2, B_4, B_6, and B_8 using the fact that $B_0 = 1$, $B_{2k+1} = 0$ for all $k \geq 1$, and this recurrence. We have $B_1 = -\frac{1}{2}$,

$$B_2 = -\frac{1}{3}\left(\binom{3}{2}B_1 + \binom{3}{3}B_0\right) = \frac{1}{6},$$

$$B_4 = -\frac{1}{5}\left(\binom{5}{3}B_2 + \binom{5}{4}B_1 + \binom{5}{5}B_0\right) = -\frac{1}{30},$$

and
$$B_6 = -\frac{1}{7}\left(\binom{7}{3}B_4 + \binom{7}{5}B_2 + \binom{7}{6}B_1 + \binom{7}{7}B_0\right) = \frac{1}{42}.$$
Thus, $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$, $\zeta(6) = \frac{\pi^6}{945}$, and $\zeta(8) = \frac{\pi^8}{9450}$.

3. (a) By Euler–Maclaurin summation, we have
$$\sum_{n=2}^{N} \frac{1}{n^s} = \int_1^N t^{-s}\mathrm{d}s - \sum_{r=1}^{k+1} \frac{B_r}{r}\binom{s+r-2}{r-1}\left(N^{-s+1-r}-1\right) - \int_1^N B_{k+1}(t)\binom{s+k}{k+1}t^{-s-k-1}\mathrm{d}t$$
for all $N \in \mathbf{N}$, since $\frac{\mathrm{d}^r}{\mathrm{d}t^r}t^{-s} = (-1)^r s(s+1)\cdots(s+r-1)t^{-s-r} = (-1)^r r!\binom{s+r-1}{s-1}t^{-s-r} = (-1)^r r!\binom{s+r-1}{r}t^{-s-r}$ for all $r \in \mathbf{N}$. Letting $s > 1$, shifting the index on the sum on the right-hand side, and taking $N \to \infty$ then yields
$$\begin{aligned}\zeta(s) &= 1 + \int_1^\infty t^{-s}\mathrm{d}s + \sum_{0\le r\le k} \frac{B_{r+1}}{r+1}\binom{s+r-1}{r} - \binom{s+k}{k+1}\int_1^\infty B_{k+1}(t)t^{-s-k-1}\mathrm{d}t \\ &= \frac{s}{s-1} + \sum_{0\le r\le k} \frac{B_{r+1}}{r+1}\binom{s+r-1}{r} - \binom{s+k}{k+1}\int_1^\infty B_{k+1}(t)t^{-s-k-1}\mathrm{d}t,\end{aligned}$$
since $\int_1^\infty t^{-s}\mathrm{d}s = \frac{1}{s-1}$.

(b) Note that $\sum_{0\le r\le k} \frac{B_{r+1}}{r+1}\binom{s+r-1}{r}$, being a polynomial in s for every integer $k \ge 0$, is entire, as is $\binom{s+k}{k+1}$. The integral $\int_1^\infty B_{k+1}(t)t^{-s-k-1}\mathrm{d}t$ converges absolutely whenever $\sigma > -k$, and thus defines an analytic function in this half-plane. It follows that, for every integer $k \ge 0$,
$$\frac{s}{s-1} + \sum_{0\le r\le k} \frac{B_{r+1}}{r+1}\binom{s+r-1}{r} - \binom{s+k}{k+1}\int_1^\infty B_{k+1}(t)t^{-s-k-1}\mathrm{d}t.$$
is a meromorphic function on the half-plane $\sigma > -k$ that is analytic except for a simple pole at $s = 1$ with residue 1. Since the above equals $\zeta(s)$ for $s \in (1,\infty)$, by the principle of analytic continuation, we can meromorphically continue $\zeta(s)$ to the whole complex plane with its only singularity being a simple pole at $s = 1$ with residue 1.

4. (a) Note that τ^* is multiplicative and thus, by looking at the values of τ^* on prime powers, that
$$F(s) := \mathcal{D}\tau^*(s) = \frac{1}{1-2^{-s}}\prod_{p>2}\left(\frac{1}{1-p^{-s}}\right)^2 = \frac{(1-2^{-s})^2}{1-2^{-s}}\zeta(s)^2 = (1-2^{-s})\zeta(s)^2$$
for $\sigma > 1$. Let $\delta > 0$ and $S \ge 1$ be parameters to be chosen later, and assume that $x > 0$ satisfies $x - \frac{1}{2} \in \mathbf{Z}$. We apply Perron's formula with $c = 1 + \frac{1}{\log x}$ to obtain
$$\begin{aligned}T(x) &= \frac{1}{2\pi i}\int_{c-iS}^{c+iS} F(s)x^s\frac{\mathrm{d}s}{s} + O\left(\frac{x^{1+c}}{S}\sum_{x/2<n\le 2x}\frac{1}{n^c|x-n|} + \frac{x^c}{S}\sum_{n=1}^\infty \frac{1}{n^c}\right) \\ &= \frac{1}{2\pi i}\int_{c-iS}^{c+iS} F(s)x^s\frac{\mathrm{d}s}{s} + O\left(\frac{x\log x}{S}\right).\end{aligned}$$

Let $R_{\delta,S}$ denote the rectangular contour with vertices at $c \pm iS$, $\delta \pm iS$ traversed counterclockwise. Inside of this contour, $\frac{F(s)x^s}{s}$ has a pole of order 2 at $s = 1$ with residue $\frac{x}{2}\log 2x + \frac{x}{2}(2\gamma - 1)$, where we have used the first problem and the power series expansions $\frac{1}{s} = \sum_{j=0}^{\infty}(-1)^j(s-1)^j$, $x^s = x\sum_{j=0}^{\infty}\frac{(\log x)^j}{j!}(s-1)^j$, and $(1-2^{-s}) = \frac{1}{2} + 2\log 2(s-1) + \ldots$ to help compute the residue. Thus,

$$\frac{1}{2\pi i}\int_{c-iS}^{c+iS} F(s)x^s \frac{\mathrm{d}s}{s} = \frac{x}{2}\left(\log 2x + 2\gamma - 1\right) - \frac{1}{2\pi i}\left(\int_{\delta - iS}^{\delta + iS} + \int_{\delta - iS}^{c - iS} - \int_{\delta + iS}^{c+iS}\right) F(s)x^s\frac{\mathrm{d}s}{s}.$$

Using that $\zeta(s) \ll_\varepsilon (1+|t|)^{(1-\sigma+\varepsilon/2)/2}$ whenever $|s-1| \gg 1$ and $\sigma \in [0, 1+c]$, that $|\delta + it| \asymp_\delta 1 + |t|$, and that $|1 - 2^{-\delta - it}| \leq 1 + 2^{-\delta} \ll 1$ for all real t, we can bound

$$\int_{\delta - iS}^{\delta + iS} F(s)x^s\frac{\mathrm{d}s}{s} \ll_{\varepsilon,\delta} x^\delta \int_{-S}^{S}(1+|t|)^{-\delta+\varepsilon/2}\mathrm{d}t \ll x^\delta S^{1-\delta+\varepsilon/2}$$

and, assuming $x > S$,

$$\int_{\delta \pm iS}^{c \pm iS} F(s)x^s\frac{\mathrm{d}s}{s} \ll_{\varepsilon,\delta} S^{\varepsilon/2}\int_\delta^c S^{-\sigma}x^\sigma \mathrm{d}\sigma \ll \frac{S^{\varepsilon/2}}{\log(x/S)}\left(\frac{x}{S}\right)^c.$$

Thus,

$$\left|T(x) - \frac{x}{2}\left(\log 2x + 2\gamma - 1\right)\right| \ll_{\varepsilon,\delta} \frac{x\log x}{S} + x^\delta S^{1-\delta+\varepsilon/2} + \frac{S^{\varepsilon/2}}{\log(x/S)}\left(\frac{x}{S}\right)^c.$$

Choosing $S = \sqrt{x}$ bounds the above by $\ll_{\varepsilon,\delta} x^{1/2+\varepsilon/2} + x^{1/2+\delta/2+\varepsilon/2} + x^{1/2+\varepsilon/4}$. Taking $\delta = \varepsilon$ yields the desired asymptotic.

(b) By $\mathcal{D}\tau^*(s) = (1-2^{-s})\zeta(s)^2$, we have

$$\sum_{n=1}^{\infty}\frac{\tau^*(n)}{n^s} = \sum_{n=1}^{\infty}\frac{\tau(n)}{n^s} - \sum_{n=1}^{\infty}\frac{\tau(n)}{(2n)^s} = \sum_{n=1}^{\infty}\frac{\tau(n)}{n^s} - \sum_{\substack{n\in\mathbf{N} \\ n \text{ even}}}\frac{\tau(n/2)}{n^s}$$

for $\sigma > 1$. Thus,

$$\tau^*(n) = \begin{cases}\tau(n) & n \text{ odd} \\ \tau(n) - \tau(n/2) & n \text{ even}\end{cases}$$

for all $n \in \mathbf{N}$. It follows that $T^*(x) = T(x) - \sum_{\substack{n \leq x \\ n \text{ even}}}\tau(n/2) = T(x) - \sum_{n \leq x/2}\tau(n) = T(x) - T(x/2)$. Using the asymptotic for $T(x)$ from class, we conclude that $T^*(x)$ equals

$$x\log x + (2\gamma - 1)x - \frac{x}{2}\log(x/2) - (2\gamma-1)\frac{x}{2} + O(\sqrt{x}) = \frac{x}{2}\left(\log 2x + (2\gamma - 1)\right) + O(\sqrt{x}).$$

5. Both of the desired asymptotics can be proven using either elementary methods or Perron's formula. We will use Perron's formula in our solutions.

(a) Let $f : \mathbf{N} \to \{0, 1\}$ denote the indicator function of the cube-frees. Then f is multiplicative, and

$$\mathcal{D}f(s) = \prod_p\left(1 + p^{-s} + p^{-2s}\right) = \prod_p \frac{1 - p^{-3s}}{1 - p^{-s}} = \frac{\zeta(s)}{\zeta(3s)}$$

for $\sigma > 1$. Let $\delta > 0$ and $T \geq 1$ be a parameter to be chosen later, and assume that $x > 0$ satisfies $x \in \frac{1}{2} + \mathbf{Z}$. We apply Perron's formula with $c = 1 + \frac{1}{\log x}$ to obtain that the number of cube-free integers below x equals

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{\zeta(s)}{\zeta(3s)}x^s\frac{\mathrm{d}s}{s} + O\left(\frac{x\log x}{T}\right).$$

Let R_T denote the rectangular contour with vertices at $c \pm iT$, $\frac{1}{2} \pm iT$ traversed counterclockwise. Inside of this contour, $\frac{\zeta(s)x^s}{\zeta(3s)s}$ has a simple pole at $s = 1$ with residue $\frac{x}{\zeta(3)}$. Thus,

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{\zeta(s)}{\zeta(3s)}x^s\frac{\mathrm{d}s}{s} = \frac{x}{\zeta(3)} - \frac{1}{2\pi i}\left(\int_{1/2-iT}^{1/2+iT} + \int_{1/2-iT}^{c-iT} + \int_{1/2+iT}^{c+iT}\right)\frac{\zeta(s)}{\zeta(3s)}x^s\frac{\mathrm{d}s}{s}.$$

Note that, whenever $s \geq \frac{1}{2}$, we have

$$\frac{1}{\zeta(3s)} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^{3s}} \ll \sum_{n=1}^{\infty}\frac{1}{n^{3/2}} \ll 1.$$

Using this esimate, along with that $\zeta(s) \ll_\delta (1+|t|)^{(1-\sigma+\delta)/2}$ whenever $|s-1| \gg 1$ and $\sigma \in [0, 1+\delta]$ and that $|\sigma + it| \asymp 1 + |t|$ for $\sigma \in [1/2, 1+\delta]$, we can bound

$$\int_{1/2-iT}^{1/2+iT}\frac{\zeta(s)}{\zeta(3s)}x^s\frac{\mathrm{d}s}{s} \ll_\delta \sqrt{x}\int_{-T}^{T}(1+|t|)^{(1/2+\delta)/2-1}\mathrm{d}t \ll \sqrt{x}T^{1/4+\delta/2}.$$

and

$$\int_{1/2\pm iT}^{c\pm iT}\frac{\zeta(s)}{\zeta(3s)}x^s\frac{\mathrm{d}s}{s} \ll_\delta T^{(\delta-1)/2}\int_{1/2}^{c}T^{-\sigma/2}x^\sigma\mathrm{d}\sigma \ll_\delta T^{(\delta-1)/2-1/4}x.$$

Thus, picking $\delta = 1/8$, we have that the number of cube-free integers below x equals

$$\frac{x}{\zeta(3)} + O\left(\frac{x\log x}{T} + x^{1/2}T^{5/16} + xT^{-11/16}\right).$$

Choosing $T = \sqrt{x}$ yields that the number of cube-free integers below x equals

$$\frac{x}{\zeta(3)} + O(x^{21/32})$$

whenever $x \in \frac{1}{2} + \mathbf{Z}$ is positive. This estimate can be extended to all $x > 0$ by noting that $\left|\frac{\lfloor x\rfloor+1/2}{\zeta(3)} - \frac{\lfloor x\rfloor-1/2}{\zeta(3)}\right| \ll 1$, which can be absorbed into the error term $O(x^{21/32})$. We conclude that the number of cube-frees below $x > 0$ is $\sim \frac{x}{\zeta(3)}$.

(b) First, observe that $n \in \mathbf{N}$ is cube-full if and only if $n = a^3b^4c^5$ for some $a, b, c \in \mathbf{N}$, and that this representation is unique when bc is squarefree. Indeed, if $n = p_1^{m_1}\cdots p_k^{m_k}$ is the prime factorization of n, then we have

$$b = \prod_{i=1}^{k}p_i^{1_{3|(m_i-1)}} \qquad \text{and} \qquad c = \prod_{i=1}^{k}p_i^{1_{3|(m_i-2)}},$$

which determines a as well. Thus,

$$\mathcal{D}f(s) = \sum_{a,b,c\in\mathbf{N}} \frac{\mu(bc)^2}{(a^3b^4c^5)^s} = \zeta(3s)F(s),$$

where $F(s) := \sum_{b,c\in\mathbf{N}} \frac{\mu(bc)^2}{(b^4c^5)^s}$, when $\sigma > 1$. Observe that $F(s)$ converges absolutely for $\sigma > \frac{1}{4}$, and thus in analytic in this half-plane, and, further, satisfies $F(s) \ll 1$ when $\sigma \geq \frac{5}{18}$, say. We, thus, have $\mathcal{D}f(s) = \zeta(3s)F(s)$ for all $s \in \mathbf{C}$ by the principle of analytic continuation. Let $\delta > 0$ and $T \geq 1$ be parameters to be chosen later, and assume that $x > 0$ satisfies $x \in \frac{1}{2} + \mathbf{Z}$. We apply Perron's formula with $c = \frac{1}{3} + \frac{1}{\log x}$ to obtain that the number of cube-full integers below x equals

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \zeta(s)F(s)x^s\frac{\mathrm{d}s}{s} + O\left(\frac{x^{1/3}\log x}{T}\right).$$

Let R_T denote the rectangular contour with vertices at $c \pm iT$ and $\frac{5}{18} \pm iT$ traversed counterclockwise. Inside of this contour, $\frac{\zeta(3s)F(s)x^s}{s}$ has a simple pole at $s = \frac{1}{3}$ with residue $3F(1/3)x^{1/3}$. Thus,

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \zeta(3s)F(s)x^s\frac{\mathrm{d}s}{s} = 3F(1/3)x^{1/3} - \frac{1}{2\pi i}\left(\int_{5/18-iT}^{5/18+iT} + \int_{5/18-iT}^{c-iT} + \int_{5/18+iT}^{c+iT}\right)\zeta(3s)F(s)x^s\frac{\mathrm{d}s}{s}.$$

Using, yet again, that $\zeta(s) \ll_\delta (1+|t|)^{(1-\sigma+\delta)/2}$ whenever $|s-1| \gg 1$ and $\sigma \in [0, 1+\delta]$, and that $|\sigma + it| \asymp 1 + |t|$ for $\sigma \in [5/18, 1]$, we can bound

$$\int_{5/18-iT}^{5/18+iT} \zeta(3s)F(s)x^s\frac{\mathrm{d}s}{s} \ll_\delta x^{5/18}\int_{-T}^{T}(1+|3t|)^{(1/6+\delta)/2-1}\mathrm{d}t \ll_\delta x^{5/18}T^{(1/6+\delta)/2}$$

and

$$\int_{5/18\pm iT}^{c\pm iT} \zeta(3s)F(s)x^s\frac{\mathrm{d}s}{s} \ll_\delta T^{(\delta-1)/2}\int_{5/18}^{c} T^{-3\sigma/2}x^\sigma\mathrm{d}\sigma \ll_\delta T^{(\delta-1)/2-5/12}x^{1/3}.$$

Thus, picking $\delta = 1/6$, we have that the number of cube-full integers below x equals

$$3F(1/3)x^{1/3} + O\left(\frac{x^{1/3}\log x}{T} + x^{5/18}T^{1/6} + T^{-5/6}x^{1/3}\right).$$

Choosing $T = x^{1/18}$ yields that the number of cube-full integers below x equals

$$3F(1/3)x^{1/3} + O\left(x^{31/108}\right)$$

whenever $x \in \frac{1}{2} + \mathbf{Z}$. This estimate can be extended to all $x > 0$ by noting that $3F(1/3)(\lfloor x\rfloor + 1/2)^{1/3} - 3F(1/3)(\lfloor x\rfloor - 1/2)^{1/3} \ll 1$, which can be absorbed into the error term. We therefore conclude that the number of cube-fulls below $x > 0$ is $\sim 3\left(\sum_{b,c\in\mathbf{N}} \frac{\mu(bc)^2}{b^{4/3}c^{5/3}}\right)x^{1/3}$.