1. (a) Note that 

\[
\# \{ (n, m) : n, m \leq x \text{ and } \gcd(n, m) = 1 \} = \sum_{n, m \leq x} \delta(\gcd(n, m)) = \sum_{n, m \leq x} \sum_{d \mid \gcd(n, m)} \mu(d)
\]

by M"obius inversion. Since \(d \mid \gcd(n, m) \iff d \mid n \text{ and } d \mid m\), this equals

\[
\sum_{d \leq x} \mu(d) \sum_{n, m \leq \frac{x}{d}} 1 = \sum_{d \leq x} \mu(d) \left( \frac{x}{d} + O(1) \right)^2 = x^2 \sum_{d \leq x} \frac{\mu(d)}{d^2} + O(x \log x).
\]

To finish, note that \(\sum_{d \leq x} \frac{\mu(d)}{d^2} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)} + O \left( \frac{1}{x} \right)\). Thus, since \(\zeta(2) = \frac{\pi^2}{6}\), we conclude that

\[
\# \{ (n, m) : n, m \leq x \text{ and } \gcd(n, m) = 1 \} = \frac{6}{\pi^2} x^2 + O(x \log x).
\]

(b) Observe that \(n \mapsto \mu(n)^2\) is the indicator function of the squarefrees, and that

\[
\sum_{n=1}^{\infty} \frac{\mu(n)^2}{n^s} = \prod_p \left( 1 + \frac{1}{p^s} \right) = \prod_p \left( 1 - \frac{1}{p^{2s}} \right) = \frac{\zeta(s)}{\zeta(2s)}
\]

whenever \(\sigma > 1\). Since \(\frac{1}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{(n^2)^s}\), it follows that \(\mu(n)^2 = 1 \ast f\), where

\[
f(n) = \begin{cases} 
\mu(m) & n = m^2 \text{ for some } m \in \mathbb{N} \\
0 & \text{otherwise}
\end{cases}
\]

Thus, \(\# \{ n \leq x : n \text{ is squarefree} \}\) equals

\[
\sum_{ab \leq x} f(b) = \sum_{b \leq x} f(b) \sum_{a \leq \frac{x}{b}} 1 = \sum_{b \leq x} f(b) \left( \frac{x}{b} + O(1) \right) = x \sum_{c \leq \sqrt{x}} \frac{\mu(c)}{c^2} + O(\sqrt{x})
\]

since \(f\) is bounded and supported on the squares. Analogously to the previous part, we have that \(\sum_{c \leq \sqrt{x}} \frac{\mu(c)}{c^2} = \frac{1}{\zeta(2)} + O \left( x^{-1/2} \right)\). We conclude that

\[
\# \{ n \leq x : n \text{ is squarefree} \} = \frac{6}{\pi^2} x + O(\sqrt{x}).
\]
(c) Observe that \( n \mapsto \sum_{a^2b^3=n} \mu(b)^2 \) equals the indicator function of the squarefulls, since \( n \) is squarefull if and only if \( n = a^2b^3 \) for some \( a, b \in \mathbb{N}^2 \), and each squarefull \( n \) has a unique factorization as \( n = a^2b^3 \) with \( b \) squarefree. Thus, with \( f \) as in the previous part, we have that \( \#\{n \leq x : n \text{ is squarefull}\} \) equals

\[
\sum_{a^2b^3 \leq x} \mu(b)^2 = \sum_{a^2b^3 \leq x} \sum_{c \mid b} f(c) = \sum_{a^2b^3 \leq x} \sum_{c \mid b} \mu(c) = \sum_{c \leq x^{1/6}} \mu(c) \sum_{a^2b^3 \leq x} 1.
\]

We will now estimate \( \sum_{a^2b^3 \leq y} \) for a general \( y > 0 \) using the hyperbola method:

\[
\sum_{a^2b^3 \leq y} 1 = \sum_{a \leq y^{1/5}} \sum_{b \leq y^{2/5}} 1 + \sum_{b \leq y^{1/5}} \sum_{a^2 \leq y} 1 - \sum_{a, b \leq y^{1/5}} 1.
\]

The first sum on the right-hand side is

\[
\sum_{a \leq y^{1/5}} \sum_{b \leq y^{2/5}} 1 = \sum_{a \leq y^{1/5}} \left[ \left( \frac{y}{a^2} \right)^{1/3} + O(1) \right] = 3y^{2/5} + \zeta(2/3)y^{1/3} + O(y^{1/5}),
\]

where we have used partial summation to obtain \( \sum_{a \leq x} a^{-2/3} = 3x^{1/3} - 2 + \frac{2}{3} \int_1^x \frac{t}{t^2} dt + O(x^{-2/3}) \) the fact that \( \zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{t}{t^2} dt \) whenever \( \sigma > 0 \) to obtain that

\[
-2 + \frac{2}{3} \int_1^\infty \frac{t}{t^2} dt = \zeta(2/3).
\]

The second sum is

\[
\sum_{b \leq y^{1/5}} \sum_{a^2 \leq y} 1 = \sum_{b \leq y^{1/5}} \left[ \left( \frac{y}{b^3} \right)^{1/2} + O(1) \right] = \zeta(3/2)y^{1/2} - 2y^{2/5} + O(y^{1/5})
\]

where we have used partial summation to obtain that \( \sum_{b \leq x} b^{-3/2} = 3 - \left( \frac{3}{2} \right) \int_1^x \frac{t}{t^2} dt - 2x^{-1/2} + O(x^{-3/2}) \) and the fact that \( \zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{t}{t^2} dt \) whenever \( \sigma > 0 \) to obtain that

\[
3 - \left( \frac{3}{2} \right) \int_1^\infty \frac{t}{t^2} dt = \zeta(3/2).
\]

Finally, the third sum is \( (y^{1/5} + O(1))^2 = y^{2/5} + O(y^{1/5}) \), so that we have that \( \sum_{a^2b^3 \leq y} 1 \) equals

\[
3y^{2/5} + \zeta(2/3)y^{1/3} + \zeta(3/2)y^{1/2} - 2y^{2/5} - y^{2/5} + O(y^{1/5}) = \zeta(3/2)y^{1/2} + \zeta(2/3)y^{1/3} + O(y^{1/5}).
\]

Plugging this back into our estimate for the number of squarefulls below \( x \) yields

\[
\sum_{c \leq x^{1/6}} \mu(c) \left( \zeta(3/2) \left( \frac{x}{c^2} \right)^{1/2} + \zeta(2/3) \left( \frac{x}{c^{1/6}} \right)^{1/3} + O \left( \left( \frac{x}{c^{1/5}} \right)^{1/5} \right) \right),
\]

which equals

\[
\zeta(3/2)x^{1/2} \sum_{c \leq x^{1/6}} \frac{\mu(c)}{c^2} + \zeta(2/3)x^{1/3} \sum_{c \leq x^{1/6}} \frac{\mu(c)}{c^{1/6}} + O(x^{1/5}) = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + O(x^{1/5}),
\]

where we have used that \( \sum_{c \leq x^{1/6}} \frac{\mu(c)}{c^2} = \frac{1}{\zeta(3)} + O(x^{-1/3}) \) and \( \sum_{c \leq x^{1/6}} \frac{\mu(c)}{c^{1/6}} = \frac{1}{\zeta(2)} + O(x^{-1/6}) \). We conclude that

\[
\#\{n \leq x : n \text{ is squarefull}\} = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + O(x^{1/5}).
\]
2. (a) i. First, suppose that $n = p_1 \cdots p_k$ is squarefree. Then $-\mu(n) \log n = -(1)^k \log n$, while on the other hand, since $\mu \ast 1 = \delta$ and $\Lambda$ is supported on prime powers,

$$(\mu \ast (\Lambda - 1) + \delta)(n) = (\mu \ast \Lambda)(n) = \sum_{i=1}^{k} (-1)^{k-1} \log p_i = -(1)^k \log n.$$  

If $n$ is not squarefree, then $-\mu(n) \log n = 0$, and we split up into two subcases: 1) $n = p^m \ell$ where $a \geq 2$, $m$ is squarefree, and $p \nmid m$ or 2) $n$ is divisible by the square of two distinct primes $p_1$ and $p_2$. In the first subcase, we have that $(\mu \ast (\Lambda - 1) + \delta)(n) = (\mu \ast \Lambda)(m) = \mu(p^m)\Lambda(p^m) = -\mu(m) \log p + \mu(m) \log p = 0$. In the second subcase, we have that $(\mu \ast (\Lambda - 1) + \delta)(n) = (\mu \ast \Lambda)(n) = 0$ since $m \ell = n$ with $m$ squarefree, then $p_1 p_2 \mid \ell$, which would force $\Lambda(\ell) = 0$, and thus all terms of the sum $\sum_{d \mid n} \mu(d) \Lambda(n/d)$ must be zero.

ii. First, by the previous subpart, we have that $-\sum_{n \leq x} \mu(n) \log n$ equals

$$\sum_{n \leq x} \mu(n)(\Lambda(n) - 1)(b) + 1 = \sum_{n \leq x} \mu(n) \left( \psi \left( \frac{x}{n} \right) - \left\lfloor \frac{x}{n} \right\rfloor \right) + 1 = \sum_{n \leq x} \mu(n) \left( \psi \left( \frac{x}{n} \right) - \frac{x}{n} \right) + O(x).$$

Since we are assuming the prime number theorem, we have that $\psi(x/a) = (1 + o(1))x/a$. Thus,

$$\sum_{n \leq x} \mu(n) \left( \psi \left( \frac{x}{n} \right) - \frac{x}{n} \right) = O(x) + o \left( x \sum_{n \leq x} \frac{1}{n} \right) = o(x \log x).$$

It follows that $m(x) := \sum_{n \leq x} \mu(n) \log n = o(x \log x)$. We will use partial summation, writing $\mu(n)$ as $\mu(n) \log n \frac{1}{\log n}$, to conclude:

$$\sum_{n \leq x} \mu(n) = \frac{m(x)}{\log x} + \int_{2}^{x} \frac{m(t)}{t \log t^2} dt = o(x).$$

(b) i. Recall that we showed $1 \ast \Lambda = \log$ in class, which implies, by M"obius inversion, that $\mu \ast \log = \Lambda$. Thus, we have

$$\mu \ast f - 2\gamma \delta = \mu \ast \log - \mu \ast \tau + 2\gamma \delta - 2\gamma \delta = \Lambda - \mu \ast 1 \ast 1 = \Lambda - 1,$$

since $\tau = 1 \ast 1$, $\mu \ast 1 = \delta$, and $\delta \ast g = g$ for any arithmetic function $g$.

ii. Recalling from class that $\sum_{n \leq x} \log n = x \log x - x + O(\log x)$ and $\sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x})$, we have

$$\sum_{n \leq x} f(n) = x \log x - x - x \log x - (2\gamma - 1)x + 2\gamma x + O(\sqrt{x}) \ll \sqrt{x}.$$  

iii. Note that

$$\psi(x) - x = \sum_{n \leq x} (\Lambda(n) - 1(n) + O(1) = \sum_{n \leq x} (\mu \ast f)(n) + O(1).$$

In addition to the estimate from the previous subpart, we will also need a bound on $\sum_{n \leq x} \frac{f(n)}{n}$. By the triangle inequality, $\sum_{n \leq x} \frac{f(n)}{n} \leq \sum_{n \leq x} \frac{\log n}{n} + \sum_{n \leq x} \frac{\tau(n)}{n} + O(1)$, and by partial summation,

$$\sum_{n \leq x} \frac{\log n}{n} = \frac{x \log x}{x} + \int_{1}^{x} \frac{\log t - 1}{t} \, dt = \int_{1}^{x} \frac{\log t}{t} \, dt + O(\log x) \ll (\log x)^2.$$
and
\[ \sum_{n \leq x} \frac{\tau(n)}{n} \ll \log x + \int_{1}^{x} \frac{t \log t}{t^2} dt \ll (\log x)^2. \]

Thus, \( \sum_{n \leq x} \frac{|f(n)|}{n} \ll (\log x)^2. \) Now, we will use the hyperbola method to estimate \( \sum_{n \leq x} (\mu \ast f)(n) = \sum_{ab \leq x} \mu(a)f(b) \). Let \( yz = x \) be parameters to be chosen later. Then we have
\[
\sum_{ab \leq x} \mu(a)f(b) = \sum_{a \leq y} \mu(a) \sum_{b \leq x/a} f(b) + \sum_{b \leq z} f(b) \sum_{a \leq x/b} \mu(a) - \sum_{a \leq y} \mu(a)f(b) - \sum_{b \leq z} f(b) \sum_{b \leq z} \mu(a).
\]

For the first sum, we have
\[
\sum_{a \leq y} \mu(a) \sum_{b \leq x/a} f(b) \ll \sum_{a \leq y} \sqrt{\frac{x}{a}} \ll \sqrt{xy}.
\]

For the second sum, since we can assume that there exists a \( g : [0, \infty) \to [0, \infty) \) such that \( g(w) \to 0 \) as \( w \to \infty \) for which \( |\sum_{n \leq w} \mu(n)| \leq wg(w) \), we have
\[
\left| \sum_{b \leq z} f(b) \sum_{a \leq x/b} \mu(a) \right| \leq \sum_{b \leq z} |f(b)|g\left(\frac{x}{b}\right) \frac{x}{b} \ll x(\log z)^2 \sup_{w \in [x/z, x]} g(w).
\]

For the third sum, we have
\[
\sum_{a \leq y} \mu(a)f(b) \ll \sqrt{yz}(y).
\]

Putting everything together, we have
\[
\sum_{n \leq x} (\mu \ast f)(n) \ll \sqrt{xy} + x(\log z)^2 \sup_{w \in [x/z, x]} g(w) + \sqrt{yz}(y),
\]

and just need to pick \( y \) and \( z \) suitably. Let \( \varepsilon \in (0, 1) \), and set \( y = \varepsilon^2 x \). Then \( z = \varepsilon^{-2} \), and our bound becomes
\[
\sum_{n \leq x} (\mu \ast f)(n) \ll \varepsilon x + x(\log z)^2 \sup_{w \in [\varepsilon^2 x, x]} g(w) + \varepsilon x \varepsilon x = o(x).
\]

Since \( g(w) \to 0 \) as \( w \to \infty \), there exists a \( K > 0 \) such that \( |g(w)| \leq \frac{\varepsilon}{(\log \varepsilon)^2} \) whenever \( w \geq K \). Thus, for all \( x \geq K \varepsilon^2 \), we must have \( \sum_{n \leq x} (\mu \ast f)(n) \ll \varepsilon x \) for all \( \varepsilon > 0 \). That is, \( \sum_{n \leq x} (\mu \ast f)(n) = o(x) \). It follows that \( \psi(x) = o(x) \), which is equivalent to the statement that \( \psi(x) \sim x \).

3. (a) By the structure theorem for finitely generated abelian groups, \( a(n) \) is multiplicative and \( a(q^n) = p(b) \) for all prime powers \( q^n \), where \( p(b) \) denotes the number of partitions of \( b \) (i.e., tuples \( (\lambda_1, \ldots, \lambda_k) \) of positive integers such that \( \lambda_1 \geq \cdots \geq \lambda_k \) and \( \lambda_1 + \cdots + \lambda_k = b \)). Observe that
\[
\sum_{n=0}^{\infty} p(n)z^n = \prod_{j=1}^{\infty} \frac{1}{1 - z^i}.
\]
whenever $|z| < 1$. Indeed, the product on the right-hand side converges absolutely by the test stated in class since $|(1 - z^j)^{-1} - 1| = |z^j||(1 - z^j)^{-1}| \leq |z|^{j+1}$ whenever $|z| \leq r < 1$, and the partial sums on the left-hand side above satisfy $\sum_{n=1}^{N} p(n) |z|^n \leq \prod_{n=1}^{N} (1 - |z|^n)^{-1}$ and so also $\sum_{n=1}^{\infty} p(n) z^n$ also converges absolutely. Indeed, the product on the right-hand side converges absolutely because the partial sums $\sum_{n=1}^{\infty} a(n) n^{-\sigma}$ are increasing and bounded (by $\prod_{j=1}^{\infty} (1 - q^{-j\sigma})^{-1}$). Finally, absolutely convergent infinite products are invariant under rearrangement, so we conclude that

$$\sum_{n=1}^{\infty} a(n) n^{-\sigma} = \prod_{q} \prod_{j=1}^{\infty} \frac{1}{1 - q^{-j\sigma}} = \prod_{j=1}^{\infty} \zeta(j\sigma)$$

whenever $\sigma > 1$.

(b) Let $b : \mathbb{N} \to \mathbb{R}$ denote the arithmetic function defined by

$$\sum_{n=1}^{\infty} \frac{c(n)}{n^s} = \prod_{j=3}^{\infty} \zeta(js)$$

for $\sigma > 1$. It follows from the previous part that the left-hand and right-hand sides above are an absolutely convergent Dirichlet series and infinite product, respectively, and also that $a = 1 \ast \mathbb{1}_{\square} \ast c$, where $\mathbb{1}_{\square}$ denotes the indicator function of the squares. Also observe that $c$ is supported on the cubefulls, and since $\sum_{q} \sum_{j=1}^{\infty} q^{-j\sigma} \ll \sum_{q} q^{-3\sigma} \ll \sigma 1$ whenever $\sigma > \frac{1}{3}$, the Dirichlet series $Dc(s)$ is absolutely convergent when $\sigma > \frac{1}{3}$. By the hyperbola method,

$$\sum_{km^2 \leq y} 1 = \sum_{k \leq y^{1/3}} \sum_{m \leq \sqrt{\frac{y}{k}}} 1 + \sum_{m \leq y^{1/3}} \sum_{k \leq \frac{y}{m^2}} 1 - \sum_{k,m \leq y^{1/3}} 1 = \zeta(2)y + \zeta(1/2)\sqrt{y} + O(y^{1/3}),$$

since $\sum_{k \leq y^{1/3}} \sum_{m \leq \sqrt{\frac{y}{k}}} 1 = \sum_{k \leq y^{1/3}} \left(\sqrt{\frac{y}{k}} + O(1)\right) = 2y^{2/3} + \zeta(1/2)y^{1/2} + O(y^{1/3})$, where we have used partial summa tion to obtain $\sum_{k \leq y^{1/3}} k^{-1/2} = 2z^{1/2} + \zeta(1/2) + O(z^{-1/2})$, and $\sum_{m \leq y^{1/3}} \sum_{k \leq \frac{y}{m^2}} 1 = \sum_{m \leq y^{1/3}} \left(\frac{y}{m^2} + O(1)\right) = \zeta(2)y - y^{2/3} + O(y^{1/3})$, where we have used partial summa tion to obtain $\sum_{m \leq z} \frac{1}{m^2} = \zeta(2) - \zeta(2) - \sum_{m \leq \frac{1}{m}} = -z^{-1} + O(z^{-2})$. It follows that $\sum_{n \leq x} a(n) = \sum_{km^2 \leq \ell\leq x} c(\ell)$ equals

$$\sum_{\ell \leq x} c(\ell) \left(\zeta(2)\frac{x}{\ell} + \zeta(1/2)\sqrt{\frac{x}{\ell}} + O\left(\frac{x}{\ell}^{1/3}\right)\right) = x\zeta(2) \sum_{\ell \leq x} \frac{c(\ell)}{\ell} + O\left(\sqrt{x} + \frac{x^{1/3}}{\ell^{1/3}} \sum_{\ell \leq x} c(\ell)\right).$$
Bounding $\sum_{t>x} \frac{c(t)}{t^2} \leq x^{-1/2} \sum_{t>x} \frac{c(t)}{t^{3/2}} \ll x^{-1/2}$ and $\sum_{t\leq x} \frac{c(t)}{t^{3/2}} \leq x^{1/6} \sum_{t\leq x} \frac{c(t)}{t^{1/2}} \ll x^{1/6}$ yields

$$\sum_{n \leq x} a(n) = x\zeta(2)\mathcal{D}c(1) + O(\sqrt{x}) = x \prod_{k=2}^{\infty} \zeta(k) + O(\sqrt{x}),$$

as desired.

4. (a) Since an absolutely convergent series is also convergent, it follows immediately from the definitions that $\sigma_c \leq \sigma_a$. If $F(s)$ converges, then $|f(n)| = o(n^\sigma)$. So, for all $\varepsilon > 0$, we have $|f(n)| \ll_n n^{\sigma_c + \varepsilon/2}$, and thus $\frac{|f(n)|}{n^{\sigma_c +\varepsilon/2}} \ll_{\varepsilon}$. It follows that $F(\sigma_c + 1 + \varepsilon)$ is absolutely convergent. Since $\varepsilon > 0$ was arbitrary, this implies that $\sigma_a \leq \sigma_c + 1$.

(b) First, suppose that $\sigma_c < +\infty$. Then $F(\sigma_c + 1)$ certainly converges, and so $|f(n)| \ll n^{\sigma_c + 1}$. Now suppose that there exists a $\theta \in \mathbb{R}$ such that $|f(n)| \ll n^\theta$. Then $\frac{|f(n)|}{n^{\sigma_c + \varepsilon/2}} \ll n^{-3/2}$, so $F(\theta + 3/2)$ converges absolutely, and thus certainly converges. It follows that $\sigma_c \leq \theta + 3/2 < +\infty$.

(c) An example where $\sigma_a = \sigma_c$ is $\zeta(s)$, since $\zeta(s)$ is absolutely convergent when $\sigma > 1$, but the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (and if $\sigma_c < 1$, then the theorem we proved in class would force $\sum_{n=1}^{\infty} \frac{1}{n}$ to converge). An example where $\sigma_a = \sigma_c + 1$ is $F(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$, since $\sigma_a = 1$ in this case by the same reasoning as for $\zeta(s)$ and $\sigma_c = 0$ in this case since $F(\sigma) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^\sigma}$ converges for all $\sigma > 0$ by the alternating series test.

5. (a) Let $t \in \mathbb{R}$. By partial summation, we have

$$\sum_{n=1}^{N} \frac{(-1)^n n^{-it}}{(\log 2n)^2} = \frac{f(N)}{(\log 2N)^2} + 2 \int_{1}^{N} \frac{f(y)}{y(\log 2y)^3} dy$$

for all $N \in \mathbb{N}$, where $f(y) = \sum_{n\leq y} (-1)^n n^{-it}$. Observe that $f(y)$ equals

$$\sum_{n \leq \frac{y}{2}} ((2n)^{-it} - (2n+1)^{-it}) + O(1) \ll 1 + \sum_{n \leq \frac{y}{2}} |1 - e^{-it \log(1 + 1/2n)}| \ll t + \sum_{n \leq \frac{y}{2}} \frac{1}{n} \ll t \log 2y,$$

where we have used that $|e^{iu} - 1| \ll |u|$ for all $u \in \mathbb{R}$, which follows from considering the power series for $e^z$, and that $\log(1 + \delta) \leq \delta$ for all $\delta > 0$. It follows that

$$\sum_{n=1}^{N} \frac{(-1)^n n^{-it}}{(\log 2n)^2} = 2 \int_{1}^{\infty} \frac{f(y)}{y(\log 2y)^3} dy + O_t \left( \frac{1}{\log 2N} \right),$$

since the integral $\int_{1}^{\infty} \frac{f(y)}{y(\log 2y)^3} dy$ is absolutely convergent and $\int_{N}^{\infty} \frac{f(y)}{y(\log 2y)^3} dy \ll t \frac{1}{\log 2N}$ by the estimate $|f(y)| \ll t \log 2y$. Thus, $F(it)$ converges.

(b) It follows from the previous part that $F(s)$ is absolutely convergent whenever $\sigma > 1$, and thus $H(s) = F(s)^2$ when $\sigma > 1$ and $h := f * f$, where $f(n) = \frac{(-1)^n}{(\log 2n)^2}$. Consider the sequence of integers $a_n := (3 \cdot 5 \cdot 7 \cdot 11 \cdot 13)^n$. We will show that $h(a_n) \to \infty$ as $n \to \infty$, from which it follows that $H(it)$ diverges for all $t \in \mathbb{R}$ since then $|h(a_n) a_n^{-it}| = h(a_n)$ would be unbounded. Since all divisors of $a_n$ are odd, we have

$$h(a_n) = \sum_{d|a_n} \frac{1}{(\log 2d)^2(\log 2n/d)^2} \geq \frac{\tau(a_n)}{(\log 2a_n)^4},$$
Note also that \( \tau(a_n) = (n + 1)^5 \) and, since \( a_n = 15015^n, n = \frac{\log a_n}{\log 15015} \). Thus, \( \tau(a_n) \geq \left( \frac{\log a_n}{\log 15015} \right)^5 \). It follows that \( h(a_n) \gg \frac{(\log a_n)^5}{(\log 2a_n)^4} \) and hence, since \( a_n \to \infty \) as \( n \to \infty \) and \( \frac{(\log x)^5}{(\log 2x)^4} \to \infty \) as \( x \to \infty \), that \( h(a_n) \to \infty \) as \( n \to \infty \).