Math 675: Analytic Theory of Numbers
Solutions to problem set # 2

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1. (a) Note that

$$\#\{(n,m) : n,m \le x \text{ and } \gcd(n,m)=1\} = \sum_{n,m\le x} \delta(\gcd(n,m)) = \sum_{n,m\le x} \sum_{d\mid\gcd(n,m)} \mu(d)$$

by Möbius inversion. Since $d \mid \gcd(n,m) \iff d \mid n$ and $d \mid m$, this equals

$$\sum_{d\le x} \mu(d) \sum_{n,m\le \frac{x}{d}} 1 = \sum_{d\le x} \mu(d) \left(\frac{x}{d} + O(1)\right)^2 = x^2 \sum_{d\le x} \frac{\mu(d)}{d^2} + O(x\log x).$$

To finish, note that $\sum_{d\le x} \frac{\mu(d)}{d^2} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} + O\left(\sum_{d>x} \frac{1}{d^2}\right) = \frac{1}{\zeta(2)} + O\left(\frac{1}{x}\right)$. Thus, since $\zeta(2) = \frac{\pi^2}{6}$, we conclude that

$$\#\{(n,m) : n,m \le x \text{ and } \gcd(n,m)=1\} = \frac{6}{\pi^2}x^2 + O(x\log x).$$

(b) Observe that $n \mapsto \mu(n)^2$ is the indicator function of the squarefrees, and that

$$\sum_{n=1}^{\infty} \frac{\mu(n)^2}{n^s} = \prod_p \left(1 + \frac{1}{p^s}\right) = \prod_p \left(\frac{1 - \frac{1}{p^{2s}}}{1 - \frac{1}{p^s}}\right) = \frac{\zeta(s)}{\zeta(2s)}$$

whenever $\sigma > 1$. Since $\frac{1}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{(n^2)^s}$, it follows that $\mu(n)^2 = 1 \star f$, where

$$f(n) = \begin{cases} \mu(m) & n = m^2 \text{ for some } m \in \mathbf{N} \\ 0 & \text{otherwise} \end{cases}.$$

Thus, $\#\{n \le x : n \text{ is squarefree}\}$ equals

$$\sum_{ab\le x} f(b) = \sum_{b\le x} f(b) \sum_{a\le \frac{x}{b}} 1 = \sum_{b\le x} f(b) \left(\frac{x}{b} + O(1)\right) = x \sum_{c\le\sqrt{x}} \frac{\mu(c)}{c^2} + O(\sqrt{x})$$

since f is bounded and supported on the squares. Analogously to the previous part, we have that $\sum_{c\le\sqrt{x}} \frac{\mu(c)}{c^2} = \frac{1}{\zeta(2)} + O\left(x^{-1/2}\right)$. We conclude that

$$\#\{n \le x : n \text{ is squarefree}\} = \frac{6}{\pi^2}x + O(\sqrt{x}).$$

(c) Observe that $n \mapsto \sum_{a^2b^3=n} \mu(b)^2$ equals the indicator function of the squarefulls, since n is squarefull if and only if $n = a^2b^3$ for some $a, b \in \mathbf{N}^2$, and each squarefull n has a unique factorization as $n = a^2b^3$ with b squarefree. Thus, with f as in the previous part, we have that $\#\{n \le x : n \text{ is squarefull}\}$ equals

$$\sum_{a^2b^3 \le x} \mu(b)^2 = \sum_{a^2b^3 \le x} \sum_{c|b} f(c) = \sum_{a^2b^3 \le x} \sum_{c^2|b} \mu(c) = \sum_{c \le x^{1/6}} \mu(c) \sum_{a^2b^3 \le \frac{x}{c^6}} 1.$$

We will now estimate $\sum_{a^2b^3 \le y} 1$ for a general $y > 0$ using the hyperbola method:

$$\sum_{a^2b^3 \le y} 1 = \sum_{a \le y^{1/5}} \sum_{b^3 \le \frac{y}{a^2}} 1 + \sum_{b \le y^{1/5}} \sum_{a^2 \le \frac{y}{b^3}} 1 - \sum_{a,b \le y^{1/5}} 1.$$

The first sum on the right-hand side is

$$\sum_{a \le y^{1/5}} \sum_{b^3 \le \frac{y}{a^2}} 1 = \sum_{a \le y^{1/5}} \left[\left(\frac{y}{a^2}\right)^{1/3} + O(1) \right] = 3y^{2/5} + \zeta(2/3)y^{1/3} + O(y^{1/5}),$$

where we have used partial summation to obtain $\sum_{a \le z} a^{-2/3} = 3z^{1/3} - 2 + \frac{2}{3}\int_1^\infty \frac{\{t\}}{t^{5/3}} \mathrm{d}t + O(z^{-2/3})$ the fact that $\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}} \mathrm{d}t$ whenever $\sigma > 0$ to obtain that $-2 + \frac{2}{3}\int_1^\infty \frac{\{t\}}{t^{5/3}} \mathrm{d}t = \zeta(2/3)$. The second sum is

$$\sum_{b \le y^{1/5}} \sum_{a^2 \le \frac{y}{b^3}} 1 = \sum_{b \le y^{1/5}} \left[\left(\frac{y}{b^3}\right)^{1/2} + O(1) \right] = \zeta(3/2)y^{1/2} - 2y^{2/5} + O(y^{1/5})$$

where we have used partial summation to obtain that $\sum_{b \le z} b^{-3/2} = 3 - \frac{3}{2}\int_1^\infty \frac{\{t\}}{t^{5/2}} \mathrm{d}t - 2z^{-1/2} + O(z^{-3/2})$ and the fact that $\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}} \mathrm{d}t$ whenever $\sigma > 0$ to obtain that $3 - \frac{3}{2}\int_1^\infty \frac{\{t\}}{t^{5/2}} \mathrm{d}t = \zeta(3/2)$. Finally, the third sum is $(y^{1/5} + O(1))^2 = y^{2/5} + O(y^{1/5})$, so that we have that $\sum_{a^2b^3 \le y} 1$ equals

$$3y^{2/5} + \zeta(2/3)y^{1/3} + \zeta(3/2)y^{1/2} - 2y^{2/5} - y^{2/5} + O(y^{1/5}) = \zeta(3/2)y^{1/2} + \zeta(2/3)y^{1/3} + O(y^{1/5})$$

Plugging this back into our estimate for the number of squarefulls below x yields

$$\sum_{c \le x^{1/6}} \mu(c) \left(\zeta(3/2) \left(\frac{x}{c^6}\right)^{1/2} + \zeta(2/3) \left(\frac{x}{c^6}\right)^{1/3} + O\left(\left(\frac{x}{c^6}\right)^{1/5} \right) \right),$$

which equals

$$\zeta(3/2)x^{1/2} \sum_{c \le x^{1/6}} \frac{\mu(c)}{c^3} + \zeta(2/3)x^{1/3} \sum_{c \le x^{1/6}} \frac{\mu(c)}{c^2} + O(x^{1/5}) = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + O(x^{1/5}),$$

where we have used that $\sum_{c \le x^{1/6}} \frac{\mu(c)}{c^3} = \frac{1}{\zeta(3)} + O(x^{-1/3})$ and $\sum_{c \le x^{1/6}} \frac{\mu(c)}{c^2} = \frac{1}{\zeta(2)} + O(x^{-1/6})$. We conclude that

$$\#\{n \le x : n \text{ is squarefull}\} = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + O(x^{1/5}).$$

2. (a) i. First, suppose that $n = p_1 \cdots p_k$ is squarefree. Then $-\mu(n)\log n = -(-1)^k \log n$, while on the other hand, since $\mu \star 1 = \delta$ and Λ is supported on prime powers,

$$(\mu \star (\Lambda - 1) + \delta)(n) = (\mu \star \Lambda)(n) = \sum_{i=1}^{k} (-1)^{k-1} \log p_i = -(-1)^k \log n.$$

If n is not squarefree, then $-\mu(n)\log n = 0$, and we split up into two subcases: 1) $n = p^a m$ where $a \geq 2$, m is squarefree, and $p \nmid m$ or 2) n is divisible by the square of two distinct primes p_1 and p_2. In the first subcase, we have $(\mu\star(\Lambda-1)+\delta)(n) = (\mu \star \Lambda)(n) = \mu(pm)\Lambda(p^{a-1}) + \mu(m)\Lambda(p^a) = -\mu(m)\log p + \mu(m)\log p = 0$. In the second subcase, we have that $(\mu \star (\Lambda - 1) + \delta)(n) = (\mu \star \Lambda)(n) = 0$ since if $m\ell = n$ with m squarefree, then $p_1 p_2 \mid \ell$, which would force $\Lambda(\ell) = 0$, and thus all terms of the sum $\sum_{d|n} \mu(d)\Lambda(n/d)$ must be zero.

ii. First, by the previous subpart, we have that $-\sum_{n\leq x} \mu(n)\log n$ equals

$$\sum_{ab\leq x} \mu(a)(\Lambda-1)(b)+1 = \sum_{a\leq x} \mu(a)\left(\psi\left(\frac{x}{a}\right) - \left\lfloor \frac{x}{a} \right\rfloor\right)+1 = \sum_{a\leq x} \mu(a)\left(\psi\left(\frac{x}{a}\right) - \frac{x}{a}\right)+O(x).$$

Since we are assuming the prime number theorem, we have that $\psi(x/a) = (1 + o(1))x/a$. Thus,

$$\sum_{a\leq x} \mu(a)\left(\psi\left(\frac{x}{a}\right) - \frac{x}{a}\right) = O(x) + o\left(x \sum_{a\leq x} \frac{1}{a}\right) = o(x \log x).$$

It follows that $m(x) := \sum_{n\leq x} \mu(n)\log n = o(x\log x)$. We will use partial summation, writing $\mu(n)$ as $\mu(n)\log n \frac{1}{\log n}$, to conclude:

$$\sum_{n\leq x} \mu(n) = \frac{m(x)}{\log x} + \int_2^x \frac{m(t)}{t(\log t)^2} \mathrm{d}t = o\left(x\right).$$

(b) i. Recall that we showed $1 \star \Lambda = \log$ in class, which implies, by Möbius inversion, that $\mu \star \log = \Lambda$. Thus, we have

$$\mu \star f - 2\gamma\delta = \mu \star \log - \mu \star \tau + 2\gamma\delta - 2\gamma\delta = \Lambda - \mu \star 1 \star 1 = \Lambda - 1,$$

since $\tau = 1 \star 1$, $\mu \star 1 = \delta$, and $\delta \star g = g$ for any arithmetic function g.

ii. Recalling from class that $\sum_{n\leq x} \log n = x\log x - x + O(\log x)$ and $\sum_{n\leq x} \tau(n) = x\log x + (2\gamma - 1)x + O(\sqrt{x})$, we have

$$\sum_{n\leq x} f(n) = x\log x - x - x\log x - (2\gamma-1)x + 2\gamma\lfloor x \rfloor + O(\sqrt{x}) \ll \sqrt{x}.$$

iii. Note that

$$\psi(x) - x = \sum_{n\leq x} (\Lambda - 1)(n) + O(1) = \sum_{n\leq x} (\mu \star f)(n) + O(1).$$

In addition to the estimate from the previous subpart, we will also need a bound on $\sum_{n\leq x} \frac{|f(n)|}{n}$. By the triangle inequality, $\sum_{n\leq x} \frac{|f(n)|}{n} \leq \sum_{n\leq x} \frac{\log n}{n} + \sum_{n\leq x} \frac{\tau(n)}{n} + O(1)$, and by partial summation,

$$\sum_{n\leq x} \frac{\log n}{n} = \frac{\lfloor x \rfloor \log x}{x} + \int_1^x \frac{\lfloor t \rfloor (\log t - 1)}{t^2} \mathrm{d}t = \int_1^x \frac{\log t}{t} \mathrm{d}t + O(\log x) \ll (\log x)^2$$

and

$$\sum_{n\le x}\frac{\tau(n)}{n} \ll \log x + \int_1^x \frac{t\log t}{t^2}\mathrm{d}t \ll (\log x)^2.$$

Thus, $\sum_{n\le x}\frac{|f(n)|}{n} \ll (\log x)^2$. Now, we will use the hyperbola method to estimate $\sum_{n\le x}(\mu\star f)(n) = \sum_{ab\le x}\mu(a)f(b)$. Let $yz = x$ be parameters to be chosen later. Then we have

$$\sum_{ab\le x}\mu(a)f(b) = \sum_{a\le y}\mu(a)\sum_{b\le x/a}f(b) + \sum_{b\le z}f(b)\sum_{a\le x/b}\mu(a) - \sum_{\substack{a\le y\\ b\le z}}\mu(a)f(b).$$

For the first sum, we have

$$\sum_{a\le y}\mu(a)\sum_{b\le x/a}f(b) \ll \sum_{a\le y}\sqrt{\frac{x}{a}} \ll \sqrt{xy}.$$

For the second sum, since we can assume that there exists a $g:[0,\infty)\to[0,\infty)$ such that $g(w)\to 0$ as $w\to\infty$ for which $\left|\sum_{n\le w}\mu(w)\right| \le wg(w)$, we have

$$\left|\sum_{b\le z}f(b)\sum_{a\le x/b}\mu(a)\right| \le \sum_{b\le z}|f(b)|g\left(\frac{x}{b}\right)\frac{x}{b} \ll x(\log z)^2 \sup_{w\in[x/z,x]} g(w).$$

For the third sum, we have

$$\sum_{\substack{a\le y\\ b\le z}}\mu(a)f(b) \ll \sqrt{z}yg(y).$$

Putting everything together, we have

$$\sum_{n\le x}(\mu\star f)(n) \ll \sqrt{xy} + x(\log z)^2\sup_{w\in[x/z,x]}g(w) + \sqrt{z}yg(y),$$

and just need to pick y and z suitably. Let $\varepsilon\in(0,1)$, and set $y=\varepsilon^2 x$. Then $z=\varepsilon^{-2}$, and our bound becomes

$$\sum_{n\le x}(\mu\star f)(n) \ll \varepsilon x + x(\log\varepsilon)^2\sup_{w\in[\varepsilon^2 x,x]}g(w) + \varepsilon x g(\epsilon^2 x).$$

Since $g(w)\to 0$ as $w\to\infty$, there exists a $K>0$ such that $|g(w)|\le\frac{\varepsilon}{(\log\varepsilon)^2}$ whenever $w\ge K$. Thus, for all $x\ge\frac{K}{\varepsilon^2}$, we must have $\sum_{n\le x}(\mu\star f)(n)\ll\varepsilon x$ for all $\varepsilon>0$. That is, $\sum_{n\le x}(\mu\star f)(n)=o(x)$. It follows that $\psi(x)-x=o(x)$, which is equivalent to the statement that $\psi(x)\sim x$.

3. (a) By the structure theorem for finitely generated abelian groups, $a(n)$ is multiplicative and $a(q^b)=p(b)$ for all prime powers q^b, where $p(b)$ denotes the number of partitions of b (i.e., tuples $(\lambda_1,\dots,\lambda_k)$ of positive integers such that $\lambda_1\ge\dots\ge\lambda_k$ and $\lambda_1+\dots+\lambda_k=b$). Observe that

$$\sum_{n=0}^{\infty}p(n)z^n = \prod_{j=1}^{\infty}\frac{1}{1-z^j}$$

whenever $|z| < 1$. Indeed, the product on the right-hand side converges absolutely by the test stated in class since $\left|(1-z^j)^{-1}-1\right| = |z^j|\left|(1-z^j)^{-1}\right| \ll_r |z|^j$ whenever $|z| \le r < 1$, and the partial sums on the left-hand side above satisfy $\sum_{n=1}^{N} p(n)|z|^n \le \prod_{j=1}^{N}(1-|z|^j)^{-1} \le \prod_{j=1}^{\infty}(1-|z|^j)^{-1}$ and so also $\sum_{n=1}^{\infty} p(n)z^n$ also converges absolutely since the sequence $\sum_{n=1}^{N} p(n)|z|^n$ of partial sums is increasing and bounded. It follows that $\sum_{j=0}^{\infty} \frac{a(q^j)}{q^{js}} = \prod_{j=1}^{\infty}(1-q^{-js})^{-1}$ whenever $\sigma > 1$. Now, it follows that

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_{q} \left(\prod_{j=1}^{\infty} \frac{1}{1-q^{-js}} \right)$$

whenever $\sigma > 1$. Indeed, the product on the right-hand side converges absolutely because its reciprocal converges absolutely, since $\sum_q \sum_{j=1}^{\infty} q^{-j\sigma} \ll \sum_q q^{-\sigma} \ll_\sigma 1$, and the left-hand side converges absolutely because the partial sums $\sum_{n=1}^{N} a(n)n^{-\sigma}$ are increasing and bounded (by $\prod_q \prod_{j=1}^{\infty}(1-q^{-j\sigma})^{-1}$). Finally, absolutely convergent infinite products are invariant under rearrangement, so we conclude that

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_{j=1}^{\infty} \prod_{q} \frac{1}{1-q^{-js}} = \prod_{j=1}^{\infty} \zeta(js)$$

whenever $\sigma > 1$.

(b) Let $b : \mathbf{N} \to \mathbf{R}$ denote the arithmetic function defined by

$$\sum_{n=1}^{\infty} \frac{c(n)}{n^s} = \prod_{j=3}^{\infty} \zeta(js)$$

for $\sigma > 1$. It follows from the previous part that the left-hand and right-hand sides above are an absolutely convergent Dirichlet series and infinite product, respectively, and also that $a = 1 \star 1_\square \star c$, where $1_\square$ denotes the indicator function of the squares. Also observe that c is supported on the cubefulls, and since $\sum_q \sum_{j=3}^{\infty} q^{-j\sigma} \ll \sum_q q^{-3\sigma} \ll_\sigma 1$ whenever $\sigma > \frac{1}{3}$, the Dirichlet series $\mathcal{D}c(s)$ is absolutely convergent when $\sigma > \frac{1}{3}$. By the hyperbola method,

$$\sum_{km^2 \le y} 1 = \sum_{k \le y^{1/3}} \sum_{m \le \sqrt{\frac{y}{k}}} 1 + \sum_{m \le y^{1/3}} \sum_{k \le \frac{y}{m^2}} 1 - \sum_{k,m \le y^{1/3}} 1 = \zeta(2)y + \zeta(1/2)\sqrt{y} + O(y^{1/3}),$$

since $\sum_{k \le y^{1/3}} \sum_{m \le \sqrt{\frac{y}{k}}} 1 = \sum_{k \le y^{1/3}} \left(\sqrt{\frac{y}{k}} + O(1) \right) = 2y^{2/3} + \zeta(1/2)y^{1/2} + O(y^{1/3})$, where we have used partial summation to obtain $\sum_{k \le z} k^{-1/2} = 2z^{1/2} + \zeta(1/2) + O(z^{-1/2})$, and $\sum_{m \le y^{1/3}} \sum_{k \le \frac{y}{m^2}} 1 = \sum_{m \le y^{1/3}} \left(\frac{y}{m^2} + O(1) \right) = \zeta(2)y - y^{2/3} + O(y^{1/3})$, where we have used partial summation to obtain $\sum_{m > z} \frac{1}{m^2} = \zeta(2) - \sum_{m \le z} \frac{1}{m^2} = -z^{-1} + O(z^{-2})$. It follows that $\sum_{n \le x} a(n) = \sum_{km^2\ell \le x} c(\ell)$ equals

$$\sum_{\ell \le x} c(\ell) \left(\zeta(2)\frac{x}{\ell} + \zeta(1/2)\frac{\sqrt{x}}{\sqrt{\ell}} + O\left(\left(\frac{x}{\ell} \right)^{1/3} \right) \right) = x\zeta(2) \sum_{\ell \le x} \frac{c(\ell)}{\ell} + O\left(\sqrt{x} + x^{1/3} \sum_{\ell \le x} \frac{c(\ell)}{\ell^{1/3}} \right).$$

Bounding $\sum_{\ell>x} \frac{c(\ell)}{\ell} \leq x^{-1/2} \sum_{\ell>x} \frac{c(\ell)}{\ell^{1/2}} \ll x^{-1/2}$ and $\sum_{\ell\leq x} \frac{c(\ell)}{\ell^{1/3}} \leq x^{1/6} \sum_{\ell\leq x} \frac{c(\ell)}{\ell^{1/2}} \ll x^{1/6}$ yields

$$\sum_{n\leq x} a(n) = x\zeta(2)\mathcal{D}c(1) + O(\sqrt{x}) = x \prod_{k=2}^{\infty} \zeta(k) + O(\sqrt{x}),$$

as desired.

4. (a) Since an absolutely convergent series is also convergent, it follows immediately from the definitions that $\sigma_c \leq \sigma_a$. If $F(s)$ converges, then $|f(n)| = o(n^\sigma)$. So, for all $\varepsilon > 0$, we have $|f(n)| \ll_\varepsilon n^{\sigma_c+\varepsilon/2}$, and thus $\frac{|f(n)|}{n^{\sigma_c+1+\varepsilon}} \ll_\varepsilon \frac{1}{n^{1+\varepsilon/2}}$. It follows that $F(\sigma_c + 1 + \varepsilon)$ is absolutely convergent. Since $\varepsilon > 0$ was arbitrary, this implies that $\sigma_a \leq \sigma_c + 1$.

   (b) First, suppose that $\sigma_c < +\infty$. Then $F(\sigma_c+1)$ certainly converges, and so $|f(n)| \ll n^{\sigma_c+1}$. Now suppose that there exists a $\theta \in \mathbf{R}$ such that $|f(n)| \ll n^\theta$. Then $\frac{|f(n)|}{n^{\theta+3/2}} \ll n^{-3/2}$, so $F(\theta + 3/2)$ converges absolutely, and thus certainly converges. It follows that $\sigma_c \leq \theta + 3/2 < +\infty$.

   (c) An example where $\sigma_a = \sigma_c$ is $\zeta(s)$, since $\zeta(s)$ is absolutely convegent when $\sigma > 1$, but the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (and if $\sigma_c < 1$, then the theorem we proved in class would force $\sum_{n=1}^{\infty} \frac{1}{n}$ to converge). An example where $\sigma_a = \sigma_c + 1$ is $F(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$, since $\sigma_a = 1$ in this case by the same reasoning as for $\zeta(s)$ and $\sigma_c = 0$ in this case since $F(\sigma) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^\sigma}$ converges for all $\sigma > 0$ by the alternating series test.

5. (a) Let $t \in \mathbf{R}$. By partial summation, we have

$$\sum_{n=1}^{N} \frac{(-1)^n n^{-it}}{(\log 2n)^2} = \frac{f(N)}{(\log 2N)^2} + 2\int_1^N \frac{f(y)}{y(\log 2y)^3} \mathrm{d}y$$

   for all $N \in \mathbf{N}$, where $f(y) = \sum_{n\leq y} (-1)^n n^{-it}$. Observe that $f(y)$ equals

$$\sum_{n\leq \frac{y}{2}} \left((2n)^{-it} - (2n+1)^{-it}\right) + O(1) \ll 1 + \sum_{n\leq \frac{y}{2}} \left|1 - e^{-it\log(1+1/2n)}\right| \ll_t 1 + \sum_{n\leq \frac{y}{2}} \frac{1}{n} \ll_t \log 2y.$$

   where we have used that $|e^{iu} - 1| \ll |u|$ for all $u \in \mathbf{R}$, which follows from considering the power series for e^z, and that $\log(1+\delta) \leq \delta$ for all $\delta > 0$. It follows that

$$\sum_{n=1}^{N} \frac{(-1)^n n^{-it}}{(\log 2n)^2} = 2\int_1^\infty \frac{f(y)}{y(\log 2y)^3} \mathrm{d}y + O_t\left(\frac{1}{\log 2N}\right),$$

   since the integral $\int_1^\infty \frac{f(y)}{y(\log 2y)^3} \mathrm{d}y$ is absolutely convergent and $\int_N^\infty \frac{f(y)}{y(\log 2y)^3} \mathrm{d}y \ll_t \frac{1}{\log 2N}$ by the estimate $|f(y)| \ll_t \log 2y$. Thus, $F(it)$ converges.

   (b) It follows from the previous part that $F(s)$ is absolutely convergent whenever $\sigma > 1$, and thus $H(s) = F(s)^2$ when $\sigma > 1$ and $h := f \star f$, where $f(n) = \frac{(-1)^n}{(\log 2n)^2}$. Consider the sequence of integers $a_n := (3 \cdot 5 \cdot 7 \cdot 11 \cdot 13)^n$. We will show that $h(a_n) \to \infty$ as $n \to \infty$, from which it follows that $H(it)$ diverges for all $t \in \mathbf{R}$ since then $|h(a_n)a_n^{-it}| = h(a_n)$ would be unbounded. Since all divisors of a_n are odd, we have

$$h(a_n) = \sum_{d|a_n} \frac{1}{(\log 2d)^2(\log 2n/d)^2} \geq \frac{\tau(a_n)}{(\log 2a_n)^4}.$$

Note also that $\tau(a_n) = (n+1)^5$ and, since $a_n = 15015^n$, $n = \frac{\log a_n}{\log 15015}$. Thus, $\tau(a_n) \geq (\frac{\log a_n}{\log 15015})^5$. It follows that $h(a_n) \gg \frac{(\log a_n)^5}{(\log 2a_n)^4}$ and hence, since $a_n \to \infty$ as $n \to \infty$ and $\frac{(\log x)^5}{(\log 2x)^4} \to \infty$ as $x \to \infty$, that $h(a_n) \to \infty$ as $n \to \infty$.