Math 675: Analytic Theory of Numbers
Solutions to problem set # 1

January 28, 2024

1. (a) Suppose first that n is squarefull. Note that 1 is squarefull since the definition $p \mid n \implies p^2 \mid n$ is vacuous in this case, and $1 = 1^2 \cdot 1^3$. If $n > 1$ is squarefull, then n has prime factorization $n = p_1^{e_1} \cdots p_k^{e_k}$ with $e_i \geq 2$ for each $i = 1, \ldots, k$, and so we can write $n = a^2 b^3$ for $a = p_1^{\lfloor e_1/2 \rfloor - \epsilon(e_1)} \cdots p_k^{\lfloor e_k/2 \rfloor - \epsilon(e_k)}$ and $b = p_1^{\epsilon(e_1)} \cdots p_k^{\epsilon(e_k)}$, where we define $\epsilon : \mathbf{N} \to \{0, 1\}$ by $\epsilon(e) = 0$ if e is even and $\epsilon(e) = 1$ if e is odd. Indeed, if e is even, then $2\left(\lfloor \frac{e}{2} \rfloor - \epsilon(e)\right) + 3\epsilon(e) = 2\frac{e}{2} = e$ and, if e is odd, then $2\left(\lfloor \frac{e}{2} \rfloor - \epsilon(e)\right) + 3\epsilon(e) = 2\left(\frac{e-1}{2} - 1\right) + 3 = e$.
   Now suppose that $n = a^2 b^3$ for some $a, b \in \mathbf{N}$. If p is a prime for which $p \mid n$, then $p \mid a^2$ or $p \mid b^3$, and thus $p \mid a$ or $p \mid b$. In either case, $p^2 \mid n$. We conclude that n is squarefull.

   (b) First of all, the squares are squarefull, so

$$\#\{n \leq x : n \text{ is squarefull}\} \geq \lfloor \sqrt{x} \rfloor \gg \sqrt{x}.$$

   For the upper bound, we have

$$\#\{n \leq x : n \text{ is squarefull}\} \leq \sum_{a \leq \sqrt{x}} \sum_{b \leq \sqrt[3]{x/a^2}} 1 \leq \sum_{a \leq \sqrt{x}} \sqrt[3]{\frac{x}{a^2}} \ll x^{1/3} (\sqrt{x})^{1/3} \ll \sqrt{x}.$$

2. (a) Every integer $n \leq N$ that is not squarefree is divisible by the square of at least one prime, so we indeed have

$$Q(N) \geq N - \sum_p \sum_{n \leq N} 1_{p^2 \mid n} = N - \sum_p \left\lfloor \frac{N}{p^2} \right\rfloor.$$

   (b) The first inequality follows from the fact that all primes greater than 2 are odd and that not all odd integers are primes. The second inequality follows from the general inequality $2k(2k+2) < (2k+1)^2$ using

$$\frac{1}{2}\left(\frac{1}{2k} - \frac{1}{2k+2}\right) = \frac{1}{2k(2k+2)}.$$

   The equality follows by telescoping.

   (c) By the previous two parts, we have

$$Q(N) \geq N - \sum_p \left\lfloor \frac{N}{p^2} \right\rfloor \geq N - \sum_p \frac{N}{p^2} > N - \frac{N}{2} = \frac{N}{2}.$$

(d) Let $n \geq 2$, and consider the set $S = \{n - m : 1 \leq m < n \text{ and } m \text{ is squarefree}\} \subset \{1, \dots, n-1\}$. By the previous part, we have $|S| = Q(n-1) > \frac{n-1}{2}$. On the other hand, the set of squarefree integers in $\{1, \dots, n-1\}$ also has size greater than $\frac{n-1}{2}$. Thus, S and the squarefrees in $\{1, \dots, n-1\}$ have nontrivial intersection by the pigeonhole principle. It follows that there exist squarefree $m, m' \in \{1, \dots, n-1\}$ such that $n = m + m'$.

3. (a) Observe that $|\psi(x) - \theta(x)| = \psi(x) - \theta(x)$, so we have

$$|\psi(x) - \theta(x)| = \sum_{\substack{p^a \leq x \\ a \geq 2}} \log p \leq \sum_{p \leq \sqrt{x}} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p \leq \log x \sum_{p \leq \sqrt{x}} 1 \leq \sqrt{x} \log x.$$

(b) That $\psi(x) \sim x \iff \theta(x) \sim x$ follows immediately from the first part, so it remains to show that $\pi(x) \sim \frac{x}{\log x} \iff \theta(x) \sim x$. By writing $\pi(x) = \sum_{p \leq x} \frac{\log p}{\log p}$ and applying partial summation, we have

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t(\log t)^2} \mathrm{d}t = \frac{\theta(x)}{\log x} + O\left(\int_2^x \frac{\mathrm{d}t}{(\log t)^2} \right)$$

where we have used Chebyshev's upper bound $\theta(t) \ll \psi(t) \ll t$. Since

$$\int_2^x \frac{\mathrm{d}t}{(\log t)^2} = \int_2^{\sqrt{x}} \frac{\mathrm{d}t}{(\log t)^2} + \int_{\sqrt{x}}^x \frac{\mathrm{d}t}{(\log t)^2} \ll \sqrt{x} + \frac{x}{(\log x)^2} \ll \frac{x}{(\log x)^2},$$

it follows that $\left| \pi(x) - \frac{\theta(x)}{\log x} \right| \ll \frac{x}{(\log x)^2}$. This immediately yields that $\pi(x) \sim \frac{x}{\log x} \iff \theta(x) \sim x$.

4. (a) Writing $\omega(n) = \sum_p 1_{p|n}$, we have

$$\sum_{n \leq x} \omega(n) = \sum_{n \leq x} \sum_p 1_{p|n} = \sum_{p \leq x} \sum_{n \leq x} 1_{p|n} = \sum_{p \leq x} \left(\frac{x}{p} + O(1) \right) = x \log\log x + O(x),$$

where we have used Mertens' second theorem.

(b) We have

$$\sum_{n \leq x} \omega(n)^2 = \sum_{n \leq x} \left(\sum_p 1_{p|n} \right)^2 = \sum_{n \leq x} \sum_{p,q} 1_{p|n} 1_{q|n} = \sum_{n \leq x} \sum_p 1_{p|n} + \sum_{n \leq x} \sum_{p \neq q} 1_{pq|n}.$$

The first sum on the right-hand side equals $x \log\log x + O(x)$ by the previous part. For the second sum on the right-hand side, we have

$$\sum_{n \leq x} \sum_{p \neq q} 1_{pq|n} = \sum_{\substack{pq \leq x \\ p \neq q}} \sum_{n \leq x} 1_{pq|n} = x \sum_{pq \leq x} \frac{1}{pq} + O(x),$$

where we have used that $\sum_{p \leq x} \sum_{n \leq x} 1_{p^2|n} \ll \sum_{p \leq x} \frac{x}{p^2} \ll x$ and $\sum_{pq \leq x} 1 \ll x$ (by unique factorization). Using the hyperbola method,

$$\sum_{pq \leq x} \frac{1}{pq} = 2 \sum_{p \leq \sqrt{x}} \frac{1}{p} \sum_{q \leq x/p} \frac{1}{q} - \sum_{p,q \leq \sqrt{x}} \frac{1}{pq} = 2 \sum_{p \leq \sqrt{x}} \frac{\log\log \frac{x}{p} + O(1)}{p} - (\log\log \sqrt{x} + O(1))^2,$$

which equals $2\left((\log\log x)^2+O(\log\log x)\right)-(\log\log x+O(1))^2=(\log\log x)^2+O(\log\log x)$ since $\log\log y=\log\log x+O(1)$ whenever $\sqrt{x}\le y\le x$. Putting everything together, we conclude that $\sum_{n\le x}\omega(n)^2$ equals

$$x\left((\log\log x)^2+O(\log\log x)\right)+O\left(x\log\log x\right)=x(\log\log x)^2+O(x\log\log x).$$

(c) Note that, by the previous two parts,

$$\sum_{n\le x}|\omega(n)-\log\log x|^2=\sum_{n\le x}\omega(n)^2-2\log\log x\sum_{n\le x}\omega(n)+x(\log\log x)^2\ll x\log\log x.$$

Set $E=\left\{n\le x:|\omega(n)-\log\log x|\ge(\log\log x)^{3/4}\right\}$, say. Then

$$\sum_{n\le x}|\omega(n)-\log\log x|^2\ge|E|(\log\log x)^{3/2}$$

Comparing with the upper bound above, it follows that $|E|\ll\frac{x}{\sqrt{\log\log x}}$. Thus, for all but $\ll\frac{x}{\sqrt{\log\log x}}$ natural numbers $n\le x$, $|\omega(n)-\log\log x|<(\log\log x)^{3/4}$. Since $|\log\log n-\log\log x|<1$ for all $\sqrt{x}\le n\le x$, it follows that $|\omega(n)-\log\log n|<(\log\log x)^{3/4}$ for all but $\ll\frac{x}{\log\log x}$ natural numbers $n\le x$. This implies the weaker qualitative statement that $\omega(n)\sim\log\log n$ for all but $o(x)$ natural numbers $n\le x$.

5. (a) Let $n\in\mathbf{N}$, and note that there is an obvious bijection f from

$$\left\{(d_1,\dots,d_k)\in\mathbf{N}^k:d_1\cdots d_k=n\right\}\to\left\{(d,(d_1,\dots,d_{k-1}))\in\mathbf{N}\times\mathbf{N}^{k-1}:d\mid n,\ d_1\cdots d_{k-1}=\frac{n}{d}\right\}$$

given by $f(d_1,\dots,d_k)=(d_1,(d_2,\dots,d_{k-1}))$.

(b) Observe, first, that both $\mathcal{D}\tau(s)$ and $\zeta(s)=\mathcal{D}1(s)$ (and thus $\zeta(s)^k$ as well) are absolutely convergent in the half-plane $\sigma>1$. The previous part says that $\tau_k=\tau_{k-1}\star 1$ for all $k\ge 2$. Thus, $\tau_k=\overbrace{1\star\cdots\star 1}^{k\text{ times}}$, and so $\mathcal{D}\tau(s)=\mathcal{D}(1\star\cdots\star 1)(s)=\zeta(s)^k$.

(c) We will use the hyperbola method. In class, we showed that $\sum_{n\le x}\tau(n)=x\log x+(2\gamma-1)x+O(\sqrt{x})$, and using the first part, we have

$$\sum_{n\le x}\tau_3(n)=\sum_{n\le x}\sum_{d|n}\tau(d)=\sum_{nm\le x}\tau(m)=\sum_{n\le x^{1/3}}\sum_{m\le x/n}\tau(m)+\sum_{m\le x^{2/3}}\sum_{n\le x/m}\tau(m)-\sum_{\substack{n\le x^{1/3}\\ m\le x^{2/3}}}\tau(m).$$

For the first sum, we have

$$\begin{aligned}\sum_{n\le x^{1/3}}\sum_{m\le x/n}\tau(m)&=\sum_{n\le x^{1/3}}\left(\frac{x}{n}\log\frac{x}{n}+(2\gamma-1)\frac{x}{n}+O\left(\sqrt{\frac{x}{n}}\right)\right)\\&=x(\log x)(\log x^{1/3})-x\frac{(\log x^{1/3})^2}{2}+c_1x\log x+c_2x+O(x^{2/3}\log x),\end{aligned}$$

for some constants $c_1,c_2\in\mathbf{R}$, where we have used that

$$\begin{aligned}\sum_{n\le y}\frac{\log n}{n}&=\frac{(y+f(y))\log y}{y}-\int_1^y\frac{(t+f(t))(1-\log t)}{t^2}\mathrm{d}t\\&=\log y-\int_1^y\frac{(1-\log t)}{t}\mathrm{d}t-\int_1^\infty\frac{f(t)(1-\log t)}{t^2}\mathrm{d}t+O\left(\int_y^\infty\frac{(1-\log t)}{t^2}\mathrm{d}t+\frac{\log y}{y}\right)\\&=\frac{(\log y)^2}{2}+c+O\left(\frac{\log y}{y}\right)\end{aligned}$$

for some function $f(z) \ll 1$ and constant $c \in \mathbf{R}$ by partial summation. For the second sum on the right-hand side,

$$\sum_{m \le x^{2/3}} \sum_{n \le x/m} \tau(m) = \sum_{m \le x^{2/3}} \tau(m) \left(\frac{x}{m} + O(1) \right)$$
$$= x \sum_{m \le x^{2/3}} \frac{\tau(m)}{m} + O\left(x^{2/3} \log x \right)$$
$$= x \frac{(\log x^{2/3})^2}{2} + c_3 x \log x^{2/3} + c_4 + O\left(x^{2/3} \right)$$

for some constants $c_3, c_4 \in \mathbf{R}$, where we have used that

$$\sum_{m \le y} \frac{\tau(m)}{m} = \frac{y \log y + (2\gamma - 1)y + g(y)}{y} + \int_1^y \frac{t \log t + (2\gamma - 1)t + g(t)}{t^2} \mathrm{d}t$$
$$= \frac{(\log y)^2}{2} + 2\gamma \log y + c' + O\left(\frac{1}{\sqrt{y}} \right)$$

for some function $g(z) \ll \sqrt{z}$ and constant $c' \in \mathbf{R}$ by partial summation. Finally, for the third sum, we have

$$\sum_{\substack{n \le x^{1/3} \\ m \le x^{2/3}}} \tau(m) = \lfloor x^{1/3} \rfloor \left(x^{2/3} \log x^{2/3} + (2\gamma - 1)x^{2/3} + O\left(x^{1/3} \right) \right) = c_5 x \log x + c_6 x + O\left(x^{2/3} \log x \right)$$

for some constants $c_5, c_6 \in \mathbf{R}$. Putting everything together yields

$$\sum_{n \le x} \tau(n) = \left(\frac{1}{3} - \frac{(1/3)^2}{2} + \frac{(2/3)^2}{2} \right) x (\log x)^2 + c_7 x \log x + c_8 x + O\left(x^{2/3} \log x \right)$$

for some constants $c_7, c_8 \in \mathbf{R}$. Since $\frac{1}{3} - \frac{(1/3)^2}{2} + \frac{(2/3)^2}{2} = \frac{1}{2}$, this is the desired asymptotic.

(d) We proceed via induction, arguing analogously to the previous part. So assume that, for a general $k \ge 2$, there exists a polynomial $P_k \in \mathbf{R}[z]$ of degree $k-1$ and with leading coefficient $\frac{1}{(k-1)!}$ such that

$$\sum_{n \le y} \tau_k(n) = y P_k(\log y) + O\left(y^{1-1/k} (\log y)^{k-2} \right)$$

for all $y \ge 2$. Using the first part, we have that $\sum_{n \le x} \tau_{k+1}(n)$ equals

$$\sum_{n \le x} \sum_{d \mid n} \tau_k(d) = \sum_{nm \le x} \tau_k(m) = \sum_{n \le x^{1/(k+1)}} \sum_{m \le x/n} \tau_k(m) + \sum_{m \le x^{k/(k+1)}} \sum_{n \le x/m} \tau_k(m) - \sum_{\substack{n \le x^{1/(k+1)} \\ m \le x^{k/(k+1)}}} \tau_k(m).$$

For the first sum on the right-hand side, we have

$$\sum_{n \le x^{1/(k+1)}} \sum_{m \le x/n} \tau_k(m) = \sum_{n \le x^{1/(k+1)}} \left(\frac{x}{n} P_k \left(\log \frac{x}{n} \right) + O\left(\left(\frac{x}{n} \right)^{1-1/k} (\log x)^{k-2} \right) \right)$$
$$= x \sum_{n \le x^{1/(k+1)}} \frac{1}{n} P_k \left(\log \frac{x}{n} \right) + O\left(x^{1-1/(k+1)} (\log x)^{k-2} \right).$$

Observe that $P_k\left(\log \frac{x}{n}\right)$ equals

$$P_k(\log x - \log n) = \frac{1}{(k-1)!}\sum_{i=0}^{k-1}\binom{k-1}{i}(\log x)^i(-\log n)^{k-1-i} + \sum_{i+i'<k-1} a'_{i,i'}(\log x)^i(\log n)^{i'}$$

for some $a_{i,i'} \in \mathbf{R}$. Thus, it suffices to obtain asymptotics for $\sum_{n\le y}\frac{(\log n)^j}{n}$ for each $j = 0, \dots, k-1$, and we have already taken care of the cases $j = 0, 1$ previously. By partial summation, for $j \ge 1$, we have

$$\begin{aligned}\sum_{n\le y}\frac{(\log n)^j}{n} &= \frac{(y+f(y))(\log y)^j}{y} + \int_1^y \frac{(t+f(t))(\log t - j)(\log t)^{j-1}}{t^2}\mathrm{d}t \\ &= (\log y)^j + \int_1^\infty \frac{f(t)(\log t - j)(\log t)^{j-1}}{t^2}\mathrm{d}t + \int_1^y \frac{(\log t - j)(\log t)^{j-1}}{t}\mathrm{d}t + O\left(\frac{(\log y)^j}{y}\right) \\ &= \frac{(\log y)^{j+1}}{j+1} + Q_j(\log y) + O\left(\frac{(\log y)^j}{y}\right)\end{aligned}$$

for some function $f(z) \ll 1$ and polynomial $Q_j \in \mathbf{R}[z]$ of degree at most j, since $\int_1^y \frac{(\log t)^i}{t}\mathrm{d}t = \frac{(\log y)^{i+1}}{i+1}$ for all integers $i \in \mathbf{N}$. It follows that

$$\begin{aligned}\sum_{n\le y}\frac{1}{n}P_k\left(\log\frac{x}{n}\right) &= \frac{-1}{(k-1)!}\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{(\log x)^i}{k-i}(-\log y)^{k-i} + R_k(\log y) + O\left(\frac{(\log y)^{k-1}}{y}\right) \\ &= \frac{(\log x)^k - \left(\log\frac{x}{y}\right)^k}{k!} + R_k(\log y) + O\left(\frac{(\log y)^{k-1}}{y}\right)\end{aligned}$$

for some polynomial $R_k \in \mathbf{R}[z]$ of degree at most $k-1$, since

$$\frac{1}{(k-1)!}\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{z^i(-w)^{k-i}}{k-i} = \sum_{i=0}^{k-1}\frac{z^i(-w)^{k-i}}{i!(k-i)!} = \frac{1}{k!}\left(z^k - \sum_{i=0}^{k}\binom{k}{i}z^i(-w)^{k-i}\right) = \frac{z^k-(z-w)^k}{k!}.$$

for all $z, w \in \mathbf{R}$. Thus,

$$\sum_{n\le x^{1/(k+1)}}\sum_{m\le x/n}\tau_k(m) = x\frac{(\log x)^k - \left(\log x^{k/(k+1)}\right)^k}{k!} + xR_k\left(\log x^{1/(k+1)}\right) + O\left(x^{1-1/(k+1)}(\log x)^{k-1}\right).$$

For the second sum on the right-hand side, we have that $\sum_{m\le x^{k/(k+1)}}\sum_{n\le x/m}\tau_k(m)$ equals

$$\sum_{m\le x^{k/(k+1)}}\tau_k(m)\left(\frac{x}{m}+O(1)\right) = x\sum_{m\le x^{k/(k+1)}}\frac{\tau_k(m)}{m} + O\left(x^{1-1/(k+1)}(\log x)^{k-1}\right).$$

By partial summation,

$$\begin{aligned}\sum_{m\le y}\frac{\tau_k(m)}{m} &= \frac{yP_k(\log y)+g(y)}{y} + \int_1^y \frac{tP_k(\log t)+g(t)}{t^2}\mathrm{d}t \\ &= \frac{yP_k(\log y)+g(y)}{y} + \int_1^\infty \frac{g(t)}{t^2}\mathrm{d}t + \int_1^y \frac{P_k(\log t)}{t}\mathrm{d}t + O\left(\int_y^\infty \frac{(\log t)^{k-2}}{t^{1+1/k}}\mathrm{d}t\right) \\ &= \frac{(\log y)^k}{k!} + S_k(\log y) + O\left(\frac{(\log y)^{k-2}}{y^{1/k}}\right)\end{aligned}$$

for some $g(z) \ll z^{1-1/k}(\log z)^{k-2}$ and polynomial $S_k \in \mathbf{R}[z]$ of degree at most $k-1$. Thus,

$$\sum_{m \le x^{k/(k+1)}} \sum_{n \le x/m} \tau_k(m) = x\frac{\left(\log x^{k/(k+1)}\right)^k}{k!} + xS_k\left(\log x^{k/(k+1)}\right) + O\left(x^{1-1/(k+1)}(\log x)^{k-1}\right).$$

Finally, we have

$$\begin{aligned}\sum_{\substack{n \le x^{1/(k+1)} \\ m \le x^{k/(k+1)}}} \tau_k(m) &= \left\lfloor x^{1/(k+1)} \right\rfloor \left(x^{k/(k+1)} P_k\left(\log x^{k/(k+1)}\right) + O\left(\left(x^{k/(k+1)}\right)^{1-1/k}(\log x)^{k-2}\right)\right) \\ &= xT_k(\log x) + O\left(x^{1-1/(k+1)}(\log x)^{k-1}\right)\end{aligned}$$

for some polynomial $T_k \in \mathbf{R}[z]$ of degree at most $k-1$. Putting everything together, we have that $\sum_{n \le x} \tau_{k+1}(n)$ equals

$$\begin{aligned}& x\left(\frac{(\log x)^k - \left(\log x^{k/(k+1)}\right)^k}{k!} + \frac{\left(\log x^{k/(k+1)}\right)^k}{k!}\right) + xU_k(\log x) + O\left(x^{1-1/(k+1)}(\log x)^{k-1}\right) \\ &= \frac{x(\log x)^k}{k!} + xU_k(\log x) + O\left(x^{1-1/(k+1)}(\log x)^{k-1}\right)\end{aligned}$$

for some polynomial $U_k \in \mathbf{R}[z]$ of degree at most $k-1$, giving the desired asymptotic.

6. (a) Since 1 is multiplicative, this follows from the fact that $\tau_k = \overbrace{1 \star \cdots \star 1}^{k \text{ times}}$ (which we showed in the previous problem) and that the Dirichlet convolution of two multiplicative functions is multiplicative.

   (b) The lower bound $\tau_k(p^a) \ge k$ follows from noting that $\prod_{i=1}^k d_i = p^a$ for all $(d_1, \ldots, d_k) \in \{(p^a, 1, \ldots, 1), \ldots, (1, \ldots, 1, p^a)\}$. For the upper bound $\tau_k(p^a) \le \min\left\{k^a, (a+1)^{k-1}\right\}$, observe that if $\prod_{i=1}^k d_i = p^a$, then $d_i = p^{a_i}$ for some nonnegative integers a_i that satisfy $\sum_{i=1}^k a_i = a$. Since the a_i are nonnegative, we must have $0 \le a_i \le a$ for each $i = 1, \ldots, k$. This gives at most $(a+1)^{k-1}$ choices of $a_1, \ldots, a_{k-1}$, and since any choice of $a_1, \ldots, a_{k-1}$ uniquely determines a_k, it follows that $\tau_k(p^a) \le (a+1)^{k-1}$. Note also that any k-tuple of nonnegative integers $(a_1, \ldots, a_k)$ for which $\sum_{i=1}^k a_i = a$ corresponds to a unique a-tuple of integers $(b_1, \ldots, b_a)$ with each $b_j \in \{1, \ldots, k\}$ by taking b_j to be the smallest index ℓ for which $\sum_{i=1}^{\ell} a_i \ge j$. It follows that $\tau_k(p^a) \le k^a$ as well. To obtain $k^{\omega(n)} \le \tau_k(n) \le \min\left\{k^{\Omega(n)}, \tau(n)^{k-1}\right\}$ for $n \in \mathbf{N}$ with prime factorization $n = p_1^{a_1} \cdots p_\ell^{a_\ell}$, we simply take the product of the inequalities just obtained over all $i = 1, \ldots, \ell$, and use that $\omega(n) = \ell$, $\Omega(\ell) = \sum_{i=1}^{\ell} a_i$, and $\tau(n) = \prod_{i=1}^{\ell}(a_i + 1)$.

   (c) Since, as observed in the previous part, $\tau_k(p^a)$ equals the number of k-tuples of nonnegative integers $(a_1, \ldots, a_k)$ that satisfy $\sum_{i=1}^k a_i = a$, we immediately have $\tau_k(p^a) = \binom{a+k-1}{k-1}$ by stars and bars.

   (d) By the definition of $\Omega(n; y)$, we have $y^{\Omega(n;y)} \le n$. Taking log of both sides and rearranging yields $\Omega(n; y) \le \frac{\log n}{\log y}$.

(e) We factor n as the product of $\prod_{\substack{p^a \| n \\ p \le y}} p^a$ and $\prod_{\substack{p^a \| n \\ p > y}} p^a$ and use the multiplicativity of τ and the bounds from the second part to obtain

$$\tau_k(n) \le \prod_{\substack{p^a \| n \\ p \le y}} (a+1)^{k-1} \prod_{\substack{p^a \| n \\ p > y}} k^a.$$

For the second inequality, we use that if $p^a \| n$, then $a = \left\lfloor \frac{\log n}{\log p} \right\rfloor \le \frac{\log n}{\log 2} \le 2 \log n$ and the trivial bound $\pi(y) \le y$ and the bound on $\Omega(n; y)$ from the previous part to obtain

$$\prod_{\substack{p^a \| n \\ p \le y}} (a+1)^{k-1} \prod_{\substack{p^a \| n \\ p > y}} k^a \le (2\log n + 1)^{(k-1)y}\, k^{\log n/\log y}.$$

(f) We pick $y = \frac{\log n}{(\log\log n)^3}$. With this choice of y, we have

$$k^{\log n/\log y} = n^{\log k/(\log\log n - 3\log\log\log n)} = n^{(1+o(1))\log k/\log\log n}$$

and

$$(2\log n + 1)^{(k-1)y} = (2\log n + 1)^{(k-1)\log n/(\log\log n)^3} \ll n^{O((k-1)/(\log\log n)^2)}.$$

It now follows from the previous part that $\tau_k(n) \le n^{(1+o(1))\log k/\log\log n}$.

(g) By the previous part, there exists some function $f : [1,\infty) \to \mathbb{R}$ such that $f(x) \to 0$ as $x \to \infty$ for which $\tau(n) \le n^{(1+f(n))\log 2/\log\log n}$ for all $n \ge e^e$, say. Let $\varepsilon > 0$, and let x be large enough so that $\frac{(1+f(x))\log 2}{\log\log x} \le \varepsilon$, which must exist since $\log\log x \to \infty$ and $f(x) \to 0$ as $x \to \infty$. Then, for all $n \ge x$, we have $\tau(n) \le n^\varepsilon$. By taking $C_\varepsilon = \max_{n<x} \frac{\tau(n)}{n^\varepsilon}$, we obtain that $\tau(n) \le (C_\varepsilon + 1)n^\varepsilon$ for all $n \in \mathbf{N}$. This certainly implies that $\tau(n) \ll_\varepsilon n^\varepsilon$.