Math 675: Analytic Theory of Numbers
Problem set # 7

Due 4/22/2024

In all problems, \( m, n, \) and \( d \) will always denote natural numbers, \( p \) will always denote a prime and \( \mathcal{P} \) a set of primes, and \( (\lambda_i) \) will always denote a sequence of real numbers satisfying \( \lambda_1 = 1 \) and \( \lambda_d = 0 \) for all \( d > z \). We define \( \pi(x, z) = \#\{n \leq x \mid n \implies p > z\} \) and \( P_z := \prod_{p \leq z} p \), and use \( (n, m) \) and \( [n, m] \) to denote the greatest common divisor and least common multiple of \( n \) and \( m \), respectively. Recall from class that \( \mathcal{P}(z) = \prod_{p \leq z} p \).

1. (a) Prove that
   \[
   \pi(x, z) \leq \sum_{n \leq x} \left( \sum_{d|(n, P_z)} \lambda_d \right)^2.
   \]

(b) Deduce that if \( |\lambda_d| \leq 1 \) for all \( d \in \mathbb{N} \), then
   \[
   \pi(x, z) \leq \sum_{d_1, d_2 \leq z} \frac{\lambda_{d_1} \lambda_{d_2}}{[d_1, d_2]} x + O(z^2).
   \]

(c) Show that
   \[
   \sum_{d_1, d_2 \leq z} \frac{\lambda_{d_1} \lambda_{d_2}}{[d_1, d_2]} = \sum_{m \leq z} \phi(m) \left( \sum_{m|d} \frac{\lambda_d}{d} \right)^2.
   \]

2. (a) Show that if
   \[
   u_m = \sum_{\substack{m|d \\leq z}} \frac{\lambda_d}{d},
   \]
   then
   \[
   \frac{\lambda_m}{m} = \sum_{m|d} \mu\left(\frac{d}{m}\right) u_d.
   \]

(b) Show that
   \[
   \sum_{d_1, d_2 \leq z} \frac{\lambda_{d_1} \lambda_{d_2}}{[d_1, d_2]} \]
has minimum value $\frac{1}{V(z)}$, where

$$V(z) := \sum_{d \leq z} \frac{\mu(d)^2}{\phi(d)}.$$  

(Hint: Use Lagrange multipliers.)

(c) Show that with the choice of $u_m = \frac{\mu(m)}{\phi(m)V(z)}$ in the first part of the problem, we have $|\lambda_d| \leq 1$.

(d) Prove that

$$\pi(x, z) \leq \frac{x}{V(z)} + O\left( z^2 \right).$$

Deduce that $\pi(x) \ll \frac{x}{\log x}$ by setting $z = x^{1/2 - \varepsilon}$ above.

3. (a) Prove that if $f$ is a multiplicative function and $d_1, d_2 \in \mathbb{N}$, then

$$f([d_1, d_2]) \cdot f((d_1, d_2)) = f(d_1)f(d_2).$$

(b) Let $\mathcal{P}$ be a set of primes and $(a_n)$ a sequence of integers. Set $N(x, z) := \#\{n \leq x : (a_n, \mathcal{P}(z)) = 1\}$ and $N(d) := \#\{n \leq x : d | a_n\}$, and assume that $N(d) = X/f(d) + R_d$ for some multiplicative function $f$ and some real number $X$. Prove the Selberg sieve inequality

$$N(x, z) \leq \frac{X}{U(z)} + O\left( \sum_{d_1, d_2 \leq z} |R_{[d_1, d_2]}| \right)$$

where

$$U(z) = \sum_{d \leq z} \frac{\mu(d)^2}{f_1(d)}$$

and $f_1 = f \ast \mu$.

(Hint: Start from

$$N(x, z) \leq \sum_{n \leq x} \left( \sum_{d | (a_n, \mathcal{P}(z))} \lambda_d \right)^2$$

and then mimic the arguments in the previous two problems.)

(c) In the notation above, show that

$$U(z) \geq \sum_{m \leq z} \frac{1}{f(m)},$$

where $\tilde{f}$ is the completely multiplicative function defined by $\tilde{f}(p) = f(p)$ for all primes $p$.

4. Let $\pi_2(x)$ denote the number of pairs of twin primes $(p, p + 2)$ with $p \leq x$. Using Selberg’s sieve, prove that

$$\pi_2(x) \ll \frac{x}{\log^2 x}.$$