Math 675: Analytic Theory of Numbers
Problem set # 7

Due 4/22/2024

In all problems, m, n, and d will always denote natural numbers, p will always denote a prime and $\mathcal{P}$ a set of primes, and (λ_i) will always denote a sequence of real numbers satisfying $\lambda_1 = 1$ and $\lambda_d = 0$ for all $d > z$. We define $\pi(x, z) = \#\{n \leq x : p \mid n \implies p > z\}$ and $P_z := \prod_{p \leq z} p$, and use (n, m) and $[n, m]$ to denote the greatest common divisor and least common multiple of n and m, respectively. Recall from class that $\mathcal{P}(z) = \prod_{\substack{p \leq z \\ p \in \mathcal{P}}} p$.

1. (a) Prove that
$$\pi(x, z) \leq \sum_{n \leq x} \left(\sum_{d \mid (n, P_z)} \lambda_d \right)^2 .$$
   (b) Deduce that if $|\lambda_d| \leq 1$ for all $d \in \mathbf{N}$, then
$$\pi(x, z) \leq \sum_{d_1, d_2 \leq z} \frac{\lambda_{d_1} \lambda_{d_2}}{[d_1, d_2]} x + O\left(z^2\right).$$
   (c) Show that
$$\sum_{d_1, d_2 \leq z} \frac{\lambda_{d_1} \lambda_{d_2}}{[d_1, d_2]} = \sum_{m \leq z} \phi(m) \left(\sum_{\substack{m \mid d \\ d \leq z}} \frac{\lambda_d}{d} \right)^2 .$$

2. (a) Show that if
$$u_m = \sum_{\substack{m \mid d \\ d \leq z}} \frac{\lambda_d}{d},$$
   then
$$\frac{\lambda_m}{m} = \sum_{m \mid d} \mu\left(\frac{d}{m}\right) u_d.$$
   (b) Show that
$$\sum_{d_1, d_2 \leq z} \frac{\lambda_{d_1} \lambda_{d_2}}{[d_1, d_2]}$$

has minimum value $\frac{1}{V(z)}$, where

$$V(z) := \sum_{d \le z} \frac{\mu(d)^2}{\phi(d)}.$$

(Hint: Use Lagrange multipliers.)

(c) Show that with the choice of $u_m = \frac{\mu(m)}{\phi(m)V(z)}$ in the first part of the problem, we have $|\lambda_d| \le 1$.

(d) Prove that

$$\pi(x, z) \le \frac{x}{V(z)} + O\left(z^2\right).$$

Deduce that $\pi(x) \ll \frac{x}{\log x}$ by setting $z = x^{1/2-\varepsilon}$ above.

3. (a) Prove that if f is a multiplicative function and $d_1, d_2 \in \mathbf{N}$, then

$$f([d_1, d_2]) \cdot f((d_1, d_2)) = f(d_1)f(d_2).$$

(b) Let $\mathcal{P}$ be a set of primes and (a_n) a sequence of integers. Set $N(x, z) := \#\{n \le x : (a_n, \mathcal{P}(z)) = 1\}$ and $N(d) := \#\{n \le x : d \mid a_n\}$, and assume that $N(d) = X/f(d) + R_d$ for some multiplicative function f and some real number X. Prove the *Selberg sieve inequality*

$$N(x, z) \le \frac{X}{U(z)} + O\left(\sum_{d_1, d_2 \le z} |R_{[d_1,d_2]}|\right)$$

where

$$U(z) = \sum_{\substack{d \le z \\ d \mid \mathcal{P}(z)}} \frac{\mu(d)^2}{f_1(d)} \qquad \text{and} \qquad f_1 = f \star \mu.$$

(Hint: Start from

$$N(x, z) \le \sum_{n \le x} \left(\sum_{d \mid (a_n, \mathcal{P}(z))} \lambda_d\right)^2$$

and then mimic the arguments in the previous two problems.)

(c) In the notation above, show that

$$U(z) \ge \sum_{m \le z} \frac{1}{\tilde{f}(m)},$$

where $\tilde{f}$ is the completely multiplicative function defined by $\tilde{f}(p) = f(p)$ for all primes p.

4. Let $\pi_2(x)$ denote the number of pairs of twin primes $(p, p+2)$ with $p \le x$. Using Selberg's sieve, prove that

$$\pi_2(x) \ll \frac{x}{\log^2 x}.$$