Math 675: Analytic Theory of Numbers
Problem set # 6

Due 4/10/2024

In all problems, n and q will always denote natural numbers, m will always denote a nonnegative integer, p will always denote a prime, and χ will always denote a Dirichlet character.

1. (a) Let $\chi \neq \chi_0$ be a primitive Dirichlet character modulo q. Prove that if χ is odd and x is not a prime power, then

$$\psi(x,\chi) = -\sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{L'}{L}(0,\chi) + \sum_{m=1}^{\infty} \frac{x^{1-2m}}{2m-1}.$$

   (b) Let $\chi \neq \chi_0$ be a primitive Dirichlet character modulo q. Prove that if χ is even and x is not a prime power, then

$$\psi(x,\chi) = -\sum_{\rho} \frac{x^{\rho}}{\rho} - \log x - b(\chi) - \frac{1}{2}\log\left(1 - x^{-2}\right),$$

   where $b(\chi) = \lim_{s\to 0}\left(\frac{L'}{L}(s,\chi) - \frac{1}{s}\right)$.

2. Let χ_1 and χ_2 be primitive quadratic characters modulo q_1 and q_2, respectively, with $q_1 \neq q_2$, and set

$$F(s) := \zeta(s)L(s,\chi_1)L(s,\chi_2)L(s,\chi_1\chi_2),$$

   so that $F(s)$ is analytic in all of $\mathbf{C}$ except for a simple pole at $s = 1$ with residue $\lambda := L(1,\chi_1)L(1,\chi_2)L(1,\chi_1\chi_2)$.

   (a) Prove that, when $\sigma > 1$,

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

   with $a_1 = 1$ and $a_n \geq 0$ for all n.

   (b) Prove that, when $|s-2| < 1$,

$$F(s) = \sum_{m=0}^{\infty} b_m (2-s)^m$$

with $b_0 \geq 1$ and $b_m \geq 0$ for all m. Deduce that

$$F(s) - \frac{\lambda}{s-1} = \sum_{m=0}^{\infty} (b-\lambda)(2-s)^m$$

whenever $|s-2| < 2$.

(c) Prove that

$$F(s) \ll q_1^2 q_2^2 \qquad \text{and} \qquad \frac{\lambda}{s-1} \ll q_1^2 q_2^2$$

for $|s-2| = \frac{3}{2}$. Deduce that

$$|b_m - \lambda| \ll q_1^2 q_2^2 \left(\frac{2}{3}\right)^m$$

for all m.

(d) Prove that there exists an absolute constant $c_0 > 0$ such that, for real $s \in [7/8, 1]$,

$$F(s) - \frac{\lambda}{s-1} \geq 1 - \lambda \frac{(2-s)^M - 1}{1-s} - c_0 q_1^2 q_2^2 e^{-M/4}$$

for any $M \in \mathbf{N}$.

(e) By picking M for which $e^{-1/4}/2 \leq c_0 q_1^2 q_2^2 e^{-M/4} < 1/2$, deduce that

$$F(s) > \frac{1}{2} - \frac{c_1 \lambda}{1-s} (q_1 q_2)^{8(1-s)}$$

for $s \in [7/8, 1]$ for some absolute constant $c_1 > 0$.

(f) Fix $\varepsilon > 0$ for the remainder of the problem. Supposing first that there is a primitive quadratic character χ_1 for which $L(s, \chi_1)$ has a real zero β_1 in $[1 - \varepsilon/16, 1]$, prove that

$$c_1 \lambda > \frac{1}{2}(1-\beta_1)(q_1 q_2)^{-8(1-\beta_1)}.$$

(g) Supposing now that there is no primitive quadratic character χ for which $L(s, \chi)$ has a real zero in $(1 - \varepsilon/16, 1)$, let $\beta_1 \in (1 - \varepsilon/16, 1)$ be arbitrary, and deduce that

$$c_1 \lambda > \frac{1}{2}(1-\beta_1)(q_1 q_2)^{-8(1-\beta_1)}.$$

(h) Prove Siegel's theorem.

3. Fix $a \in \mathbf{Z}$ nonzero. Show that

$$\#\{p \leq x : p + a \text{ is squarefree}\} = c(a)\,\mathrm{li}(x) + O_A\left(\frac{x}{(\log x)^A}\right),$$

where

$$c(a) = \prod_{p \nmid a} \left(1 - \frac{1}{p(p-1)}\right).$$