Math 675: Analytic Theory of Numbers
Problem set # 5

Due 3/27/2024

In all problems, n, k, and q will always denote a natural number, p will always denote a prime, and χ will always denote a Dirichlet character.

1. For all $n, q \in \mathbf{N}$, we define the *Ramanujan sum*

$$c_q(n) = \sum_{a \in (\mathbf{Z}/q\mathbf{Z})^\times} e_q(an).$$

   (a) Prove that, for any fixed $n \in \mathbf{N}$, $c_q(n)$ is a multiplicative function of q.

   (b) Prove that

$$\sum_{d|q} c_d(n) = \begin{cases} q & q \mid n \\ 0 & \text{otherwise} \end{cases}.$$

   (c) Prove that

$$c_q(n) = \frac{\mu(q/(q,n))}{\phi(q/(q,n))}\phi(q),$$

   where ϕ denotes Euler's totient function.

2. Let χ be a real non-principal Dirichlet character modulo p^k.

   (a) Prove that, when p is odd, $\chi(n) = \left(\frac{n}{p}\right)$ and χ has conductor p.

   (b) Prove that, when $p^k = 8$, there are three possibilities for χ: two of conductor 8 and one of conductor 4.

   (c) Prove that, when $p = 2$ and $k > 3$, $\chi(n) = \psi(n \pmod 8)$, where ψ is one of the three characters from the previous part.

3. Let $R(n)$ denote the number of ordered pairs of integers a, b such that $a^2 + b^2 = n$ with $a \geq 0$ and $b > 0$ and let $r(n)$ denote the number of such pairs with a and b relatively prime. Let

$$\chi_{-4} = \left(\frac{-4}{n}\right) = \begin{cases} 1 & n \equiv 1 \bmod 4 \\ -1 & n \equiv 3 \bmod 4 \\ 0 & 2 \mid n \end{cases}$$

   denote the non-principal character modulo 4.

(a) Prove that

$$R(n) = \sum_{d^2|n} r(n/d^2) = \sum_{d|n} \chi_{-4}(d).$$

(b) Show that

$$\sum_{n=1}^{\infty} \frac{R(n)}{n^s} = \zeta(s)L(s, \chi_{-4})$$

when $\sigma > 1$.

(c) Show that

$$\sum_{n=1}^{\infty} \frac{r(n)}{n^s} = \frac{\zeta(s)L(s, \chi_{-4})}{\zeta(2s)}$$

when $\sigma > 1$.

4. (a) Let $f_1, f_2 : \mathbf{N} \to \mathbf{C}$ be totally multiplicative functions satisfying $|f_1(n)|, |f_2(n)| \leq 1$ for all $n \in \mathbf{N}$. Prove that

$$\sum_{n=1}^{\infty} \frac{(1 \star f_1)(n)(1 \star f_2)(n)}{n^s}$$

equals

$$\frac{\zeta(s)\mathcal{D}f_1(s)\mathcal{D}f_2(s)\mathcal{D}(f_1 f_2)(s)}{\mathcal{D}(f_1 f_2)(2s)}$$

and

$$\frac{\prod_p \left(1 - f_1(p)f_2(p)p^{-2s}\right)}{\prod_p \left(1 - p^{-s}\right)\left(1 - f_1(p)p^{-s}\right)\left(1 - f_2(p)p^{-s}\right)\left(1 - f_1(p)f_2(p)p^{-s}\right)}$$

when $\sigma > 1$.

(b) By considering the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{\left|\sum_{d|n} \chi(d) d^{-it}\right|^2}{n^s},$$

prove that $L(1 + it, \chi) \neq 0$.

5. Recall the definitions

$$\psi(x, \chi) = \sum_{n \leq x} \chi(n)\Lambda(n) \qquad \text{and} \qquad \psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a \bmod q}} \Lambda(n).$$

Assuming the Generalized Riemann Hypothesis, prove the following statements.

(a) $\psi(x, \chi) = 1_{\chi = \chi_0} x + O\left(x^{1/2} \log^2(qx)\right)$ for all $x \geq 1$ and Dirichlet characters χ modulo q.

(b) $\psi(x; q, a) = \frac{x}{\phi(q)} + O\left(x^{1/2} \log^2(qx)\right)$ for all $x \geq 1$ and $(a, q) = 1$.