Math 675: Analytic Theory of Numbers
Problem set # 4

Due 3/13/2024

1. (a) Prove that if $N \in \mathbf{N}$ is not a prime power, then

$$\psi(N) = N - \sum_{|\gamma| \leq N^2} \frac{N^\rho}{\rho} - \log 2\pi + o(1).$$

   (b) Prove that $\zeta(s)$ must have infinitely many nontrivial zeros.

   (c) Prove that, for each $\varepsilon > 0$, there is at least one nontrivial zero $\rho = \beta + i\gamma$ with $\beta \geq \frac{1}{2} - \varepsilon$. Conclude that we cannot have $\psi(x) = x + O(x^{1/2-\varepsilon})$.

2. In class, we proved that there exists a constant $c_0 > 0$ such that $\zeta(s)$ does not vanish in the region $\sigma \geq 1 - \frac{c_0}{\log(|t|+2)}$. Define, for each $t \in \mathbf{R}$, $\delta_t := \frac{c_0}{\log(|t|+2)}$.

   (a) Show that, when $\sigma \geq 1 - \frac{\delta_t}{2}$ and $|t| \geq 3$, $\frac{\zeta'}{\zeta}(s) \ll \log^2 |t|$.

   (b) In the same range of s, improve the above bound to $\frac{\zeta'}{\zeta}(s) \ll \log |t|$. (Hint: Prove that $\mathrm{Re}\Big\{-\frac{\zeta'}{\zeta}(s)\Big\} \ll \log|t|$ for $\sigma > 1 - \delta_t$ and that $\frac{\zeta'}{\zeta}(1 + \delta_t + it) \ll \log |t|$.)

   (c) Show that, when $\sigma \geq 1 - \frac{\delta_t}{2}$ and $|t| \geq 3$, $|\log \zeta(s)| \leq \log\log|t| + O(1)$. (Hint: Show that $\log \zeta(s) = \log \zeta(1 + \delta_t + it) + O(1)$ when $\sigma \leq 1 + \delta_t$.)

   (d) Prove that there exists a constant $c > 0$ such that

$$\sum_{n \leq x} \mu(n) \ll \frac{x}{e^{c\sqrt{\log x}}}.$$

3. Fix $\varepsilon \in (0, 1/2)$ and assume the Riemann Hypothesis.

   (a) Prove that $\frac{\zeta'}{\zeta}(s) \ll_\varepsilon \log|t|$ and $\log \zeta(s) \ll_\varepsilon \log |t|$ whenever $\sigma \geq \frac{1}{2} + \varepsilon$ and $|t| \geq 2$.

   (b) Prove that $\frac{\zeta'}{\zeta}(s) \ll_\varepsilon (\log|t|)^{2\max\{1-\sigma,0\}+\varepsilon}$ and $\log \zeta(s) \ll_\varepsilon (\log|t|)^{2\max\{1-\sigma,0\}+\varepsilon}$ whenever $\sigma \geq \frac{1}{2} + \varepsilon$ and $|t| \geq 2$. Infer the Lindelöf Hypothesis.

4. Prove that the Riemann Hypothesis is equivalent to both of the following statements:

   (a) For each $\varepsilon > 0$, we have $\psi(x) = x + O_\varepsilon(x^{1/2+\varepsilon})$.

   (b) For each $\varepsilon > 0$, we have $\sum_{n \leq x} \mu(n) \ll_\varepsilon x^{1/2+\varepsilon}$.