Math 675: Analytic Theory of Numbers
Problem set # 3

Due 2/21/2024

1. Show that
$$\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|)$$
in a neighborhood of $s = 1$, where $\gamma := 1 - \int_1^\infty \frac{\{t\}}{t^2} \mathrm{d}t$ denotes Euler's constant.

2. Define the sequence of *Bernoulli polynomials* $(b_n(x))_{n=1}^\infty$ on $[0,1]$ by setting $b_0(x) = 1$ and, for $n \geq 1$, setting $b_n(x)$ to be the unique polynomial satisfying $b_n'(x) = nb_{n-1}(x)$ and $\int_0^1 b_n(x)\mathrm{d}x = 0$. We then define the *Bernoulli functions* $B_n(x)$ by $B_n(x) := b_n(\{x\})$ and the *Bernoulli numbers* B_n by $B_n = B_n(0)$.

   (a) For $n \neq 1$, show that $b_n(1) = b_n(0) = B_n$. Conclude that the 1-periodic function $B_n(x)$ is continuous. In addition, show that $\int_0^x B_n(t)\mathrm{d}t = \frac{B_{n+1}(x) - B_{n+1}}{n+1}$ for all $n \geq 1$ and $x \in \mathbf{R}$.

   (b) Given integers $a < b$ and $k \geq 1$ and $f \in C^\infty([a,b])$, prove that
$$\sum_{a<n\leq b} f(n) = \int_a^b f(t)\mathrm{d}t + \sum_{r=1}^k \frac{(-1)^r B_r}{r!}(f^{(r-1)}(b) - f^{(r-1)}(a)) + (-1)^{k+1}\int_a^b \frac{B_k(t) f^{(k)}(t)}{k!}\mathrm{d}t.$$
   This is the *Euler–Maclurin summation formula.*

   (c) Let $m \in \mathbf{Z}$ and $k \in \mathbf{N}$. Show that
$$\int_0^1 B_k(x) e^{-2\pi i m x}\mathrm{d}x = -1_{m\neq 0}\frac{k!}{(2\pi i m)^k}.$$
   Conclude that
$$B_k(x) = -\frac{k!}{(2\pi i)^k}\sum_{m\neq 0}\frac{e^{2\pi i m x}}{m^k}$$
   for $k \geq 2$.

   (d) For $k \geq 1$, show that $B_{2k+1} = 0$ and that
$$B_{2k} = \frac{(-1)^{k-1}(2k)!}{2^{2k-1}\pi^{2k}}\sum_{m\geq 1}\frac{1}{m^{2k}}.$$
   Deduce a formula for $\zeta(2k)$.

   (e) Prove the recursion formula $B_n = -(n+1)^{-1}\sum_{k=2}^{n+1}\binom{n+1}{k}B_{n+1-k}$ for $n \geq 1$, and derive from it the exact values of $\zeta(2), \zeta(4), \zeta(6)$, and $\zeta(8)$.

3. (a) By applying the Euler–Maclaurin summation formula to $f(t) := t^{-s}$, show that, in the notation of the previous problem,

$$\zeta(s) = \frac{s}{s-1} + \sum_{0 \leq r \leq k} \frac{B_{r+1}}{r+1} \binom{s+r-1}{r} - \binom{s+k}{k+1} \int_1^\infty B_{k+1}(t) t^{-s-k-1} \mathrm{d}t \qquad (1)$$

whenever $s > 1$.

(b) Use (1) to provide an alternative proof that $\zeta(s)$ can be meromorphically continued to all of $\mathbf{C}$ with its only singularity being a simple pole at $s = 1$ with residue 1.

4. Let $\tau^*(n)$ denote the number of odd divisors of an integer n and set $T^*(x) := \sum_{n \leq x} \tau^*(n)$.

(a) Determine the Dirichlet series associated to $\tau^*(n)$ and use Perron's formula to deduce the asymptotic estimate

$$T^*(x) = \frac{x}{2}(\log 2x + 2\gamma - 1) + O_\varepsilon(x^{1/2+\varepsilon}). \qquad (2)$$

(b) Set $T(x) := \sum_{n \leq x} \tau(n)$. Show that $T^*(x) = T(x) - T(x/2)$ and deduce an improved error term in (2).

5. (a) An integer is called *cube-free* if there is no prime p such that $p^3 \mid n$. Obtain an asymptotic for the number of cube-free integers in $[1, x]$.

(b) An integer is called *cube-full* if $p^3 \mid n$ whenever $p \mid n$. Obtain an asymptotic for the number of cube-full integers in $[1, x]$.