Math 675: Analytic Theory of Numbers
Problem set # 2

Due 2/7/2024

In all problems, n and m will always denote natural numbers.

1. (a) Prove an asymptotic for

$$\#\{(n,m) : n,m \le x \text{ and } \gcd(n,m)=1\}$$

with error term $\ll x \log x$. I.e., determine a main term $M(x)$ for which

$$\#\{(n,m) : n,m \le x \text{ and } \gcd(n,m)=1\} = M(x) + O(x\log x),$$

and prove it.

   (b) Prove an asymptotic for

$$\#\{n \le x : n \text{ is squarefree}\}$$

with error term $\ll \sqrt{x}$.

   (c) Prove an asymptotic for

$$\#\{n \le x : n \text{ is squarefull}\}$$

with error term $\ll x^{1/5}$.

2. Recall that δ denotes the arithmetic function defined by

$$\delta(n) = \begin{cases} 1 & n = 1 \\ 0 & \text{otherwise} \end{cases}.$$

In this problem, you will show that the prime number theorem is equivalent to the statement

$$\sum_{n\le x} \mu(n) = o(x). \tag{1}$$

   (a) i. Show that $-\mu \log = \mu \star (\Lambda - 1) + \delta$.

   ii. Prove (1) assuming the prime number theorem. (Hint: First prove that $\sum_{n\le x} \mu(n)\log n = o(x\log x)$.)

(b) Set $f(n) := \log n - \tau(n) + 2\gamma$

i. Show that $\Lambda - 1 = \mu \star f - 2\gamma\delta$.

ii. Show that $\sum_{n \le x} f(x) \ll \sqrt{x}$.

iii. Prove the prime number theorem assuming (1).

3. For each natural number n, let $a(n)$ denote the number of non-isomorphic abelian groups of order n.

(a) Show that

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_{k=1}^{\infty} \zeta(ks)$$

for $\sigma > 1$.

(b) Show that

$$\sum_{n \le x} a(n) = \left(\prod_{k=2}^{\infty} \zeta(k) \right) x + O\left(\sqrt{x}\right).$$

4. Let $f : \mathbf{N} \to \mathbf{C}$ be an arithmetic function and $F(s) = \mathcal{D}f(s)$ be its Dirichlet series. Denote the abscissas of convergence and of absolute convergence of $F(s)$ by σ_c and σ_a, respectively.

(a) Prove that $\sigma_c \le \sigma_a \le \sigma_c + 1$.

(b) Prove that $\sigma_c < +\infty$ if and only if there exists a $\theta \in \mathbf{R}$ such that $f(n) = O\left(n^{\theta}\right)$ for all $n \in \mathbf{N}$.

(c) Give an example of a Dirichlet series $F(s)$ for which $\sigma_a = \sigma_c$ and give an example of a Dirichlet series $F(s)$ for which $\sigma_a = \sigma_c + 1$.

5. (a) Show that the Dirichlet series $F(s) := \sum_{n=1}^{\infty} \frac{(-1)^n}{(\log 2n)^2 n^s}$ converges at every point on the line $\sigma = 0$.

(b) Define

$$H(s) := F(s)^2 = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}.$$

Show that $h(n)$ is unbounded, and deduce that $H(s)$ diverges everywhere on the line $\sigma = 0$.