Math 675: Analytic Theory of Numbers
Problem set # 1

Due 1/24/2024

In all problems, n will always denote a natural number and p will always denote a prime.

1. We say that $n \in \mathbf{N}$ is *squarefull* if $p^2 \mid n$ for all primes $p \mid n$.
   (a) Show that n is squarefull if and only if there exist $a, b \in \mathbf{N}$ such that $n = a^2 b^3$.
   (b) Show that
   $$\#\{n \leq x : n \text{ is squarefull}\} \asymp \sqrt{x}.$$
2. We say that $n \in \mathbf{N}$ is *squarefree* if it is not divisible by any square other than 1. Let
   $$Q(x) := \#\{n \leq x : n \text{ is squarefree}\}$$
   denote the number of squarefree integers not exceeding x.
   (a) Show that
   $$Q(N) \geq N - \sum_{p \text{ prime}} \left\lfloor \frac{N}{p^2} \right\rfloor$$
   for every $N \in \mathbf{N}$.
   (b) Justify each of the relations in the following string of in/equalities:
   $$\sum_{p \text{ prime}} \frac{1}{p^2} < \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} < \frac{1}{4} + \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{1}{2k} - \frac{1}{2k+2} \right) = \frac{1}{2}.$$
   (c) Deduce that $Q(N) > \frac{N}{2}$ for every $N \in \mathbf{N}$.
   (d) Show that every integer $n \geq 2$ can be written as the sum of two squarefree numbers.
3. Recall the definitions
   $$\pi(x) = \#\{p \leq x : p \text{ prime}\}, \qquad \psi(x) = \sum_{n \leq x} \Lambda(n), \qquad \text{and} \qquad \theta(x) = \sum_{\substack{p \leq x \\ p \text{ prime}}} \log p.$$
   (a) Show that $|\psi(x) - \theta(x)| \leq \sqrt{x} \log x$ for all $x \geq 1$.

(b) Show that the asymptotic relations $\pi(x) \sim \frac{x}{\log x}$, $\psi(x) \sim x$, and $\theta(x) \sim x$ are equivalent as $x \to \infty$.

4. Recall that $\omega(n)$ denotes the number of distinct prime factors of n, so that if $n = p_1^{a_1} \cdots p_k^{a_k}$ is the prime factorization of n, then $\omega(n) = k$.

   (a) Show that
$$\sum_{n \le x} \omega(n) = x \log\log x + O(x).$$

   (b) Show that
$$\sum_{n \le x} \omega(n)^2 = x(\log\log x)^2 + O(x \log\log x).$$

   (c) Deduce that all but at most $o(x)$ integers $n \le x$ have $\sim \log\log n$ distinct prime factors.

5. Recall that $\tau(n) = \#\{d \in \mathbf{N} : d \mid n\}$ denotes the divisor function of n. More generally, we can define the *k-fold divisor function* by
$$\tau_k(n) = \#\left\{(d_1, d_2, \ldots, d_k) \in \mathbf{N}^k : d_1 d_2 \cdots d_k = n\right\}$$
for all $k \in \mathbf{N}$. Note that $\tau_2 = \tau$ and $\tau_1 \equiv 1$.

   (a) Show that
$$\tau_k(n) = \sum_{d \mid n} \tau_{k-1}(d)$$
for all integers $k \ge 2$.

   (b) Show that
$$\sum_{n=1}^{\infty} \frac{\tau_k(n)}{n^s} = \zeta(s)^k$$
for all $k \in \mathbf{N}$ and $s \in \mathbf{C}$ with $\operatorname{Re} s > 1$.

   (c) Show that
$$\sum_{n \le x} \tau_3(n) = \frac{1}{2} x (\log x)^2 + c x \log x + c' x + O\left(x^{2/3} \log x\right)$$
for some constants $c, c' \in \mathbf{R}$.

   (d) Show, more generally, that for every fixed $k \in \mathbf{N}$,
$$\sum_{n \le x} \tau_k(n) = x P_k(\log x) + O\left(x^{1-1/k} (\log x)^{k-2}\right)$$
for all $x \ge 2$, where $P_k \in \mathbf{R}[y]$ is some polynomial of degree $k-1$ with leading coefficient $\frac{1}{(k-1)!}$.

6. We will continue using the notation from the previous problem. Assume that $k \geq 2$.

(a) Show that each τ_k is multiplicative.

(b) For each prime power p^a, show that

$$k \leq \tau_k(p^a) \leq \min\left\{k^a, (a+1)^{k-1}\right\}.$$

Deduce that

$$k^{\omega(n)} \leq \tau_k(n) \leq \min\left\{k^{\Omega(n)}, \tau(n)^{k-1}\right\}$$

for all $n \in \mathbf{N}$. (Recall that $\Omega(n)$ denotes the number of prime factors of n counted with multiplicity, so that if $n = p_1^{a_1} \cdots p_\ell^{a_\ell}$ is the prime factorization of n, then $\Omega(n) = \sum_{i=1}^{\ell} a_i$.)

(c) Show, for each prime power p^a, the exact formula

$$\tau_k(p^a) = \binom{a+k-1}{k-1}.$$

(d) Define, for each real number $y \geq 1$,

$$\Omega(n; y) = \sum_{\substack{p^a \| n \\ p > y}} a.$$

Show that $\Omega(n; y) \leq \frac{\log n}{\log y}$. (Recall that $p^a \parallel n$ means that p^a is the highest power of p dividing n, so that $p^a \mid n$ but $p^{a+1} \nmid n$.)

(e) Show that

$$\tau_k(n) \leq \prod_{\substack{p^a \| n \\ p \leq y}} (a+1)^{k-1} \prod_{\substack{p^a \| n \\ p > y}} k^a \leq (2\log n + 1)^{(k-1)y} k^{\log n / \log y}.$$

(f) Choose y appropriately in the above inequality to conclude that, for each fixed integer $k \geq 2$,

$$\tau_k(n) \leq n^{(\log k + o(1))/\log\log n}$$

as $n \to \infty$.

(g) Deduce from the above that, in particular, $\tau(n) \ll_\varepsilon n^\varepsilon$ for every fixed $\varepsilon > 0$.