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ON SUPERCUSPIDAL CHARACTERS OF GL_ℓ, ℓ A PRIME

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INTRODUCTION

Let F be a nonarchimedian local field of characteristic zero and G the group of all F-rational points of a connected reductive group defined over F. If π is an irreducible, admissible representation of G and Θ_π is the distribution character of π, then Harish-Chandra proved that Θ_π can be represented by a locally integrable function F_π on G which is locally constant on G^{reg}, the set of regular elements of G. That is, if f is a locally constant, compactly supported function on G, then we have

$$\Theta_\pi(f) = \int_G dg\, f(g) F_\pi(g)$$

where dg is a Haar measure on G. We shall abuse notation and denote both the distribution character and the function which represents it by Θ_π.

If π is a supercuspidal representation (that is, all of its matrix coefficients are compactly supported mod A_G, the split component of the center of G), then the character, $\Theta_\pi = F_\pi$, of π has been explicitly calculated only in the cases: $\mathrm{PGL}_2, \mathrm{SL}_2$, and GL_2 ([30], [28], and [29]). There are many reasons why only groups of semisimple rank one have had their supercuspidal characters worked out. The two leading reasons are: (1) we do not know how to construct supercuspidal representations in general; and (2) the familiar tools (see, for example, equation (0.0.1)) are ungainly to use in practice. However, recent advances have made this problem more tractable. Here we will show how these advances can be used to simplify the calculation of the characters of supercuspidal representations of GL_ℓ, with ℓ a prime. We now outline the content of this thesis.

If π is square integrable (that is, all of its matrix coefficients are square integrable mod A_G), then Rader and Silberger [23] have extended a result of Harish-Chandra [4] to show that for $\gamma \in G^{\mathrm{reg}}$

$$\Theta_\pi(\gamma) = \frac{\deg(\pi)}{\theta(1)} \int_{G/A_G} dg^* \int_K dk\, \theta({}^{gk}\gamma) \tag{0.0.1}$$

where θ is a matrix coefficient for π, dg^* is a G-invariant measure on G/A_G, and dk is the normalized Haar measure of a compact open subgroup K in G. (Equation (0.0.1) is valid in any characteristic.)

Let $\mathfrak{g}$ be the Lie algebra of G. Suppose that Λ is a nontrivial additive character of F and B is a G-invariant, symmetric, nondegenerate, bilinear form on $\mathfrak{g}$. If X is a regular element of $\mathfrak{g}$, then the Fourier transform of the orbital integral corresponding to X is represented by a locally integrable function on $\mathfrak{g}$ which is locally constant on $\mathfrak{g}^{\mathrm{reg}}$. We shall again abuse notation and denote both of these objects by $\widehat{\mu_X}$. If X happens to be regular elliptic, then an integral formula strikingly similar to (0.0.1) can be derived for $\widehat{\mu_X}$ (see [5, Lemma 19]). Indeed, if $Y \in \mathfrak{g}^{\mathrm{reg}}$, then

$$\widehat{\mu_X}(Y) = \int_{G/A_G} dg^* \int_K dk\, \Lambda\big(B(X, {}^{gk}Y)\big).$$

The similarity between these two integral formulas suggests the question: does there exist a relation between Θ_π and $\widehat{\mu_X}$ for some representation π and some elliptic X? Murnaghan has studied this question in some detail in, for example, [19, 20, 21, 22]. To simplify our discussion, we will just say that usually what happens is:

$$\Theta_\pi\big(\exp(X)\big) = \deg(\pi) \cdot \widehat{\mu_{X_\pi}}(X) \tag{0.0.2}$$

where X_π is a regular elliptic element of $\mathfrak{g}$ naturally associated to a supercuspidal representation π and the equality is valid for all regular X sufficiently near zero in $\mathfrak{g}$. This thesis began as an attempt to make this statement more precise for those supercuspidal representations constructed in [2] for $\mathrm{GL}_n(F)$. In particular, we wanted to identify a neighborhood of zero in $\mathfrak{g}^{\mathrm{reg}}$ where (0.0.2) would be valid. This program has been carried out in §3. The proofs there are just mild refinements of the arguments in [18].

Having established that in many cases Θ_π is determined in a large region by the Fourier transform of an elliptic orbital integral on $\mathfrak{g}$, it makes sense to study the latter object. Let us suppose that X is a regular element of $\mathfrak{g}$. There are at least two useful tools for obtaining information about $\widehat{\mu_X}$:

1. Near zero, the Harish-Chandra-Howe local expansion holds for $\widehat{\mu_X}$. That is, $\widehat{\mu_X}$ can be written as a finite linear combination of the Fourier transforms of nilpotent orbital integrals.
2. Far away from zero, Waldspurger [36, Proposition VIII.1] has provided an explicit formula for $\widehat{\mu_X}$.

However, in order for these results to be useful for our purposes, we need to have specific information about the neighborhoods in $\mathfrak{g}$ where they are valid.

Much of this thesis is concerned with the problem: where is the Harish-Chandra-Howe local expansion for $\widehat{\mu_X}$ valid? Waldspurger [34] has done much remarkable work towards answering this question (see §2.3 for a fuller, but not complete, description of this work); yet, for our purposes, it can be shown that in most cases this work does not yield "exact" enough results about where the Harish-Chandra-Howe expansion is valid. Moy and Prasad have a conjecture about where the expansion ought to be valid for the character of an admissible, irreducible representation of G (see §2.2), and we have made what we believe is an analogous conjecture about G-invariant distributions on $\mathfrak{g}$ (see Conjecture 2.3.2). It can be shown that, in some cases, the results of Waldspurger imply both Conjecture 2.3.2 and the conjecture of Moy and Prasad (see, for example, [35] and Remark 3.6.2). In §4.3 we verify Conjecture 2.3.2 in an additional case for $G = \mathrm{GL}_n(F)$. It seems unlikely that the proofs in [34] or §4.3 will generalize, so we have suggested an alternative approach to this problem (see §2.3). We have found the formalism of Moy and Prasad [16, 17] to be invaluable in this endeavor, and much of this thesis would be impossible to even think about if it were not for their work.

In this thesis, we have largely ignored the second problem: where is Waldspurger's formula valid? However, for regular elliptic X in the Lie algebra of $\mathrm{GL}_\ell(F)$ and under enough tameness restrictions on F, we show that "up to a single shell" the formula of Waldspurger and Conjecture 2.3.2 handle the entire description of the Fourier transform of a regular elliptic orbital integral. We also establish what happens on this single shell.

Besides carrying out the above program for the Fourier transform of a regular elliptic orbital integral of $\mathfrak{g}$, we also show that the conjecture of Moy and Prasad (see Conjecture 2.2.1) behaves well with respect to parabolic induction. We prove this in complete generality on the Lie algebra side for arbitrary G in §2.2. For the characters of parabolically induced representations, we obtain similar results on "admissible" domains for arbitrary G. Finally, if $G = \mathrm{GL}_n(F)$, we show that the characters of all parabolically induced representations behave well with respect to induction. Unfortunately, parabolically induced representations are not generally irreducible, and we cannot say anything about the irreducible components. However, in light of [17], these efforts can be seen as more evidence supporting the conjecture of Moy and Prasad.

Finally, we compute the remaining character values for supercuspidal representations of $\mathrm{GL}_\ell(F)$ in the tame case assuming the conjecture of Moy and Prasad and modulo the explicit computation of the constants appearing in the local character expansion. Note that for those supercuspidals of $\mathrm{GL}_\ell(F)$ for which the conjecture of Moy and Prasad is known to be true (see Remark 3.6.2), Murnaghan [22] has calculated the $c_{\mathcal{O}}$'s with some tameness restrictions. For GL_3, the conjecture is verified in the remaining cases by brute force and a few of the details can be found in §5.5. These computations for $\mathrm{GL}_3(F)$ are valid without any restrictions on F.

CHAPTER 1

MOY-PRASAD FILTRATIONS

1.1 Notation

Let F be a nonarchimedean local field of residual characteristic p with ring of integers R, a uniformizer ϖ, and prime ideal $\wp = \varpi R$. Suppose that $R/\wp \cong \mathbb{F}_q$. Fix an additive character $\Lambda\colon F^+ \longrightarrow \mathbb{C}^\times$ with conductor $\wp$. Let $\mathbf{G}$ be a connected reductive algebraic group defined over F. The Lie algebra of $\mathbf{G}$ will be denoted by $\boldsymbol{\mathfrak{g}}$.

For any field extension E of F, let $\mathbf{G}(E)$ denote the E-rational points of $\mathbf{G}$ and let $\boldsymbol{\mathfrak{g}}(E)$ denote the vector space of E-rational points of $\boldsymbol{\mathfrak{g}}$. We will let $G = \mathbf{G}(F)$ and $\mathfrak{g} = \boldsymbol{\mathfrak{g}}(F)$.

Let Ad denote the adjoint representation of G on $\mathfrak{g}$. We will often write ${}^gX$ instead of $\mathrm{Ad}(g)X$ for $X \in \mathfrak{g}$, and ${}^gh$ instead of $\mathrm{Int}(g)h = ghg^{-1}$ for $h \in G$. For an element $X \in \mathfrak{g}$, $C_G(X)$ will denote its centralizer in G, and $C_\mathfrak{g}(X)$ will denote its centralizer in $\mathfrak{g}$.

Let $\mathbf{X}_*^F(\mathbf{G})$ denote the set of 1-parameter subgroups of $\mathbf{G}$ defined over F. Call an element $X \in \mathfrak{g}$ *nilpotent* if there is some $\lambda \in \mathbf{X}_*^F(\mathbf{G})$ such that $\lim_{t\to 0} {}^{\lambda(t)}X = 0$. Let $\mathcal{N}$ denote the set of nilpotent elements. A more usual definition is that an element is nilpotent if the Zariski closure of its G-orbit contains zero. Let $\mathcal{N}''$ denote the set of elements in $\mathfrak{g}$ that are nilpotent in this sense. Let $\mathcal{N}'$ denote the set of elements in $\mathfrak{g}$ which contain zero in the topological (Hausdorff) closure of their G-orbits. It is clear that $\mathcal{N} \subseteq \mathcal{N}' \subseteq \mathcal{N}''$. By a theorem of Kempf [11, Corollary 4.3], $\mathcal{N} = \mathcal{N}' = \mathcal{N}''$ when F has characteristic zero. $\mathcal{N}'$ is a closed subset of $\mathfrak{g}$ (see Lemma 1.6.11).

Let $\mathcal{B}(\mathbf{G}, F)$ be the Bruhat-Tits building of $\mathbf{G}/F$. When there can be no confusion, we will let $\mathcal{B}(G)$ or just $\mathcal{B}$ denote this object.

We will call an element of G regular if it is regular semisimple. Let G^{reg} denote the set of regular elements in G. If $S \subset G$, then S^{reg} will denote the set $S \cap G^{\mathrm{reg}}$. Similar notation will be used for $\mathfrak{g}$.

For $x \in \mathbb{R}$, we will denote by $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) the ceiling function (resp. floor function). So, for example, $\lceil \pi \rceil = 4$, $\lfloor \pi \rfloor = 3$, and $\lceil x \rceil = \lfloor x \rfloor$ if and only if x is an integer.

For a set S, we will denote by $M_n(S)$ the set of $n \times n$ matrices with entries in S.

In chapters one, three, and four F is arbitrary. In chapter two, we assume that F has characteristic zero. In chapter five we make an assumption on p. Finally, there is, unfortunately, some duplicated notation: $\mathcal{O}$, ℓ, S, Z, and $\mathfrak{z}$ pull double duty. All five of these symbols are used in the papers of Moy and Prasad, and we do not wish to stray from their notation. On the other hand, it is standard to let $\mathcal{O}$ denote an orbit, ℓ a prime (as in GL_ℓ), S a set, Z the center of $\mathrm{GL}_n(F)$, and $\mathfrak{z}$ the center of $M_n(F)$ as a Lie algebra.

1.2 Notation for chapter one

It is necessary to recall some of the notation of [16, §6.1]. This notation will be used consistently throughout the remainder of this chapter.

Let K be a fixed maximal unramified extension of F. Let $\mathbf{S}$ be a maximal F-split torus of $\mathbf{G}$. Let $\mathbf{T}$ be a maximal K-split torus of $\mathbf{G}$ defined over F and containing $\mathbf{S}$. Let $\mathbf{Z}$ be the centralizer of $\mathbf{T}$ in $\mathbf{G}$. $\mathbf{Z}$ is a torus which is defined over F. Let L be the splitting field of $\mathbf{Z}$ over K and let $\ell = [L : K]$. Let ω be a valuation on L such that $\omega(L^\times) = \mathbb{Z}$.

Let $\mathcal{A}$ be the apartment of $\mathbf{T}$ in $\mathcal{B}(\mathbf{G}, K)$. Let $\mathbf{\Phi}$ be the set of roots of $\mathbf{G}$ relative to $\mathbf{T}$ and K, and let $\boldsymbol{\Psi}$ be the set of affine roots of $\mathbf{G}$ relative to $\mathbf{T}$, K, and ω. Fix a chamber $C \subset \mathcal{A}$. C determines a basis Δ for $\boldsymbol{\Psi}$. As usual, $\psi \in \boldsymbol{\Psi}$ is positive ($\psi > 0$) if ψ can be written as a nonnegative integral linear combination of elements in Δ, and negative ($\psi < 0$) if $-\psi$ is positive.

For each root $b \in \mathbf{\Phi}$, there is a unique (up to K-isomorphism) extension L_b/K in L such that the root group quotient $U_b(K)/U_{2b}(K)$ is isomorphic to the additive group of L_b. (If b is not multipliable, then write $U_{2b}(K) = \{1\}$.) For any $\psi \in \mathbf{\Psi}$ of gradient $b \in \mathbf{\Phi}$, let $\ell_\psi = \ell_b = [L \colon L_b]$.

1.3 The Moy-Prasad filtrations

Following [16], one can associate to any point x in the building $\mathcal{B}(\mathbf{G}, F)$ a parahoric subgroup $G_x = G_{x,0}$ of G, a filtration $\{G_{x,r}\}_{r \geq 0}$ of the parahoric, and a filtration $\{\mathfrak{g}_{x,r}\}_{r \in \mathbb{R}}$ of the Lie algebra $\mathfrak{g}$ of G.

Rather than repeat the definitions in [16], we offer a few illustrations in the case where $\mathbf{G} = \mathrm{SL}_2$. Figure 1 is a picture of $\mathcal{B}(\mathbf{G}, \mathbb{Q}_2)$. An apartment in this building is just the image of a one-to-one continuous map $\mathbb{R} \longrightarrow \mathcal{B}(\mathbf{G}, \mathbb{Q}_2)$ which maps $\mathbb{Z}$ into the set of vertices of $\mathcal{B}(\mathbf{G}, \mathbb{Q}_2)$. It is easy to see that given any $x, y \in \mathcal{B}(\mathbf{G}, \mathbb{Q}_2)$, there is an apartment which contains both of them.

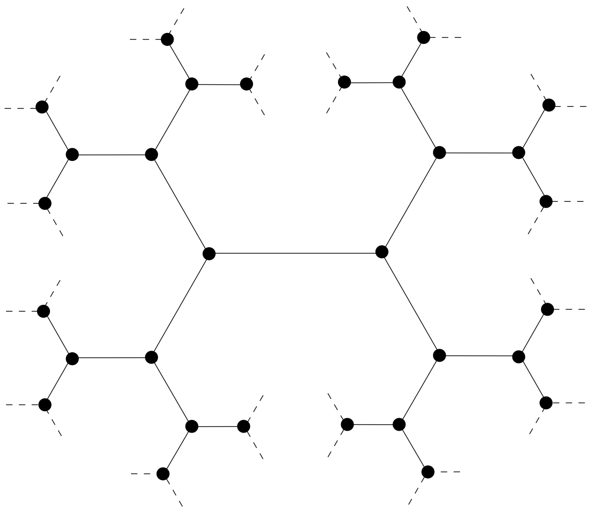

Figure 1: The building of $\mathrm{SL}_2(\mathbb{Q}_2)$

Let F again be arbitrary. Let us choose the apartment in $\mathcal{B} = \mathcal{B}(\mathbf{G}, F)$ corresponding to the diagonal torus $\mathbf{S}$ in SL_2. Since G is split, we have $\mathbf{S} = \mathbf{T} = \mathbf{Z}$;

so we will work over F in this example. In particular, let $S = \mathbf{S}(F)$. The group of characters of S is

$$\mathbf{X}^*(S) = \{\chi_m \colon S \to F^\times \mid m \in \mathbb{Z} \quad \text{and} \quad \chi_m \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) = t^m\}$$

and the group of 1-parameter subgroups for S is

$$\mathbf{X}_*(S) = \{\lambda_n \colon F^\times \to S \mid n \in \mathbb{Z} \quad \text{and} \quad \lambda_n(t) = \left(\begin{smallmatrix} t^n & 0 \\ 0 & t^{-n} \end{smallmatrix}\right)\}.$$

Note that $\chi_m \circ \lambda_n = (t \mapsto t^{mn})$, and so if we define

$$\langle \ \ , \ \ \rangle \colon \mathbf{X}^*(S) \times \mathbf{X}_*(S) \to \mathbb{Z}$$

by $\langle \chi_m, \lambda_n \rangle = mn$, then $\langle \ \ , \ \ \rangle$ is a nondegenerate, bilinear pairing. If we let $P_\emptyset$ be the set of upper triangular matrices in $\mathrm{SL}_2(F)$, then $P_\emptyset$ is a Borel subgroup in G and $S \leq P_\emptyset$. With respect to $P_\emptyset$, $\alpha = \chi_2$ is a basis for the root system $\mathbf{\Phi}$ of S, and $\mathbf{\Phi} = \{\alpha, -\alpha\}$. For $\beta \in \mathbf{\Phi}$, define the affine function β_n on $V = \mathbf{X}_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ by

$$\beta_n(\lambda_m \otimes s) = s \cdot \langle \beta, \lambda_m \rangle + n.$$

The set of affine roots of S (with respect to ω) is

$$\mathbf{\Psi} = \{\beta_n | \beta \in \mathbf{\Phi}\}.$$

Each $\psi \in \mathbf{\Psi}$ defines a hyperplane, H_ψ, in V. In our situation, these hyperplanes are just the points

$$\lambda_1 \otimes \frac{n}{2} \in V$$

for $n \in \mathbb{Z}$. The apartment $\mathcal{A}$ in $\mathcal{B}$ corresponding to S is the affine space underlying V, and the chambers of $\mathcal{A}$ are the maximal simplices in $V \smallsetminus \{H_\psi | \psi \in \mathbf{\Psi}\}$. Thus $\mathcal{A}$ is a line with a simplicial decomposition, as illustrated in Figure 2.

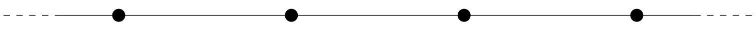

Figure 2: The standard apartment for $\mathrm{SL}_2(F)$

Let B_0 be the set of matrices of the form

$$\left(\begin{smallmatrix} R & R \\ \wp & R \end{smallmatrix}\right)$$

in $\mathrm{SL}_2(F)$. Note that B_0 is the usual Iwahori subgroup. Let C be the chamber in $\mathcal{A}$ with stabilizer B_0. Note that C is defined by the hyperplanes H_{α_0} and $H_{(-\alpha)_1}$, so the basis of $\mathbf{\Psi}$ corresponding to C is

$$\Delta = \{\alpha_0, (-\alpha)_1\}.$$

Since $\mathbf{S} = \mathbf{T} = \mathbf{Z}$, it follows that $\mathfrak{z} = \mathfrak{z}(F)$ is the Lie algebra of S; in an attempt to keep the notation in line with [16], we will write $\mathfrak{z}$ for the Lie algebra of S. Define a filtration of $\mathfrak{z}$ by

$$\mathfrak{z}_r = \varpi^{\lceil r \rceil} \cdot \{\left(\begin{smallmatrix} t & 0 \\ 0 & -t \end{smallmatrix}\right) | t \in R\}$$

for $r \in \mathbb{R}$. Define

$$\mathfrak{u}_{\alpha_n} = \left(\begin{smallmatrix} 0 & \wp^n \\ 0 & 0 \end{smallmatrix}\right)$$

and

$$\mathfrak{u}_{(-\alpha)_n} = \left(\begin{smallmatrix} 0 & 0 \\ \wp^n & 0 \end{smallmatrix}\right)$$

for $n \in \mathbb{Z}$. Note that $\mathfrak{u}_{\beta_n} \subset \mathfrak{g}_\beta$ for each $\beta \in \mathbf{\Phi}$, and so we have a filtration of the root spaces. Fix $x \in \mathcal{A}$ and $r \in \mathbb{R}$. Let $Q = \{\psi \in \mathbf{\Psi} | r + 1 > \psi(x) \geq r\}$. Define

$$\mathfrak{g}_{x,r} = \mathfrak{z}_r + \sum_{\psi \in Q} \mathfrak{u}_\psi.$$

The Moy-Prasad filtration lattices for $\mathfrak{sl}_2(F)$ are the sets $\mathfrak{g}_{x,r}$, up to conjugation. In Figure 3, we illustrate how the lattice $\mathfrak{g}_{x,r}$ varies as x and r vary for x near C. Since the apartment to which x belongs can be thought of as a copy of $\mathbb{R}$, this figure may be thought of as a picture of $\mathbb{R}^2$. Each diagonal dotted line is the graph of $r = \psi(x)$ for some affine root ψ, and the horizontal dotted lines correspond to the natural filtration of $\mathfrak{z}$.

The dotted lines divide the plane into convex polygons. Fix one such polygon D. If (x_1, r_1) and (x_2, r_2) are two points in the interior of D, then $\mathfrak{g}_{x_1,r_1} = \mathfrak{g}_{x_2,r_2}$.

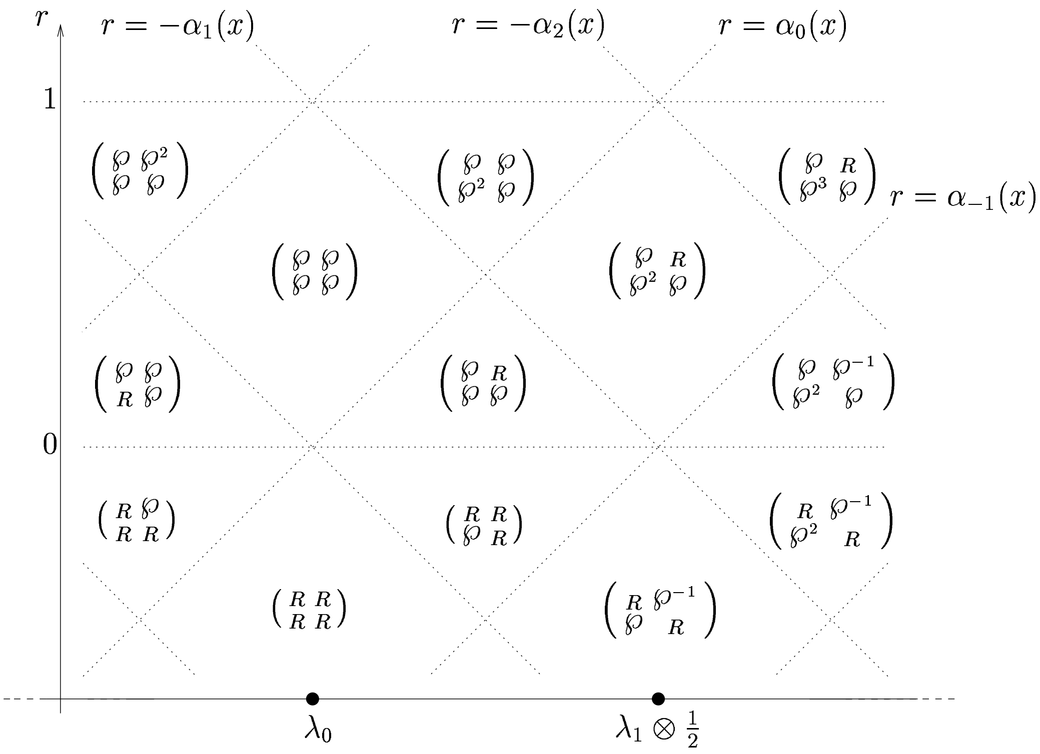

Figure 3: $(\mathfrak{sl}_2)_{x,r}$, x near C

Because of the conditions which define $\mathfrak{g}_{x,r}$ for arbitrary x and r, a point (y, s) on the boundary of D corresponds to the same filtration lattice as every other point in the interior of D if and only if there exists a point (y, r) in the interior of D with $r < s$.

Unfortunately, the Moy-Prasad filtration lattices which occur for $\mathfrak{sl}_2$ are all familiar. Unusual lattices do occur in, for example, $\mathfrak{sl}_3$.

Now let G be arbitrary. We list a few basic properties of the Moy-Prasad filtrations (see [16, 17] for proofs).

Proposition 1.3.1. *The Moy-Prasad filtrations have the following properties:*

(a) For any $g \in G$, let gx be the image of x under the action of G on $\mathcal{B}(G)$. Then $\operatorname{Int}(g)G_{x,r} = G_{gx,r}$ *and* $\operatorname{Ad}(g)\mathfrak{g}_{x,r} = \mathfrak{g}_{gx,r}$.

(b) If ϖ is a uniformizer in F, then $\varpi\mathfrak{g}_{x,r} = \mathfrak{g}_{x,r+\ell}$.

(c) *If M is a Levi component of a parabolic subgroup of G, then one can realize $\mathcal{B}(M)$ in $\mathcal{B}(G)$. If $\mathfrak{m}$ is the Lie algebra of M and $x \in \mathcal{B}(M) \subset \mathcal{B}(G)$, then $\mathfrak{g}_{x,r} \cap \mathfrak{m} = \mathfrak{m}_{x,r}$ and $G_{x,r} \cap M = M_{x,r}$.*

As a notational convenience, we write $G_{x,r^+} = \bigcup_{s>r} G_{x,s}$ and $\mathfrak{g}_{x,r^+} = \bigcup_{s>r} \mathfrak{g}_{x,s}$.

Moy and Prasad also define filtration lattices $\{\mathfrak{g}^*_{x,r}\}$ in the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ by

$$\mathfrak{g}^*_{x,r} = \left\{ \chi \in \mathfrak{g}^* \;\middle|\; \chi(\mathfrak{g}_{x,(-r)^+}) \subset \wp \right\}.$$

These lattices satisfy statements analogous to those in Proposition 1.3.1.

1.4 Optimal points

In [16, §6.1], *optimal* points are defined to be certain elements of $\mathcal{B}$ which have nice properties with respect to the Moy-Prasad filtrations of $\mathfrak{g}^*$. For the time being, we shall call these points $\mathfrak{g}^*$-*optimal*. In this section, we define $\mathfrak{g}$-optimal points, which have analogous properties with respect to the filtrations of $\mathfrak{g}$. We then show that we may assume that the set of $\mathfrak{g}^*$-optimal points is a subset of the set of $\mathfrak{g}$-optimal points.

Let

$$\Sigma = \left\{ \psi \in \boldsymbol{\Psi} \;\middle|\; \psi > 0 \text{ and } \psi - \ell < 0 \right\}.$$

This is a finite set. For each nonempty, $\mathrm{Gal}(K/F)$-invariant subset $\mathfrak{S} \subset \Sigma$, we can choose a point $x_{\mathfrak{S}} \in \bar{C}$ such that

i) $\min_{\psi \in \mathfrak{S}} \psi(x_{\mathfrak{S}}) \geq \min_{\psi \in \mathfrak{S}} \psi(y)$ for all $y \in \bar{C}$,

ii) $\psi(x_{\mathfrak{S}})$ is rational for all $\psi \in \boldsymbol{\Psi}$, and

iii) $x_{\mathfrak{S}}$ is $\mathrm{Gal}(K/F)$-invariant.

The existence of such a point follows by making the same arguments as those found in [16, §6.1]. For each nonempty, $\mathrm{Gal}(K/F)$-invariant subset $\mathfrak{S} \subset \Sigma$, fix a choice of

$x_{\mathfrak{S}} \in \bar{C}$ satisfying the conditions above, and let $\mathcal{O}$ be the finite set $\{x_{\mathfrak{S}}\}$. A point $x \in \mathcal{B}(G)$ is said to be $\mathfrak{g}$-*optimal for* $\mathfrak{S}$ if it is G-conjugate to $x_{\mathfrak{S}}$, and $\mathfrak{g}$-*optimal* if it is G-conjugate to some point in $\mathcal{O}$. The definition of $\mathfrak{g}^*$-optimal is the same, except that condition i) is replaced by

i′) $\min\limits_{\psi \in \mathfrak{S}} (\psi(x_{\mathfrak{S}}) - (\ell - \ell_\psi)) \geq \min\limits_{\psi \in \mathfrak{S}} (\psi(y) - (\ell - \ell_\psi))$ for all $y \in \bar{C}$.

Example 1.4.1. Recall the example of $\mathrm{SL}_2(F)$ in §1.3. Table 1 lists all the $\mathfrak{g}$-optimal points of $\mathcal{B}(\mathrm{SL}_2, F)$ up to conjugation. Since $\ell = 1$, these are, up to conjugation, all of the $\mathfrak{g}^*$-optimal points as well.

$\mathfrak{S}$	$x_{\mathfrak{S}}$
$\{\alpha_0\}$	$\lambda_1 \otimes \frac{1}{2}$
$\{(-\alpha)_1\}$	λ_0
$\{\alpha_0, (-\alpha)_1\}$	$\lambda_1 \otimes \frac{1}{4}$

Table 1: Optimal points for $\mathcal{B}(\mathrm{SL}_2, F)$

Lemma 1.4.2 (Adler and DeBacker). *Let* $\mathfrak{S}'$ *be a nonempty,* $\mathrm{Gal}(K/F)$*-invariant subset of* Σ*. There exists a nonempty,* $\mathrm{Gal}(K/F)$*-invariant subset* $\mathfrak{S}$ *of* Σ *such that for every* $x \in \bar{C}$*,* x *satisfies conditions i′), ii), and iii) for* $\mathfrak{S}'$ *if and only if* x *satisfies conditions i)–iii) for* $\mathfrak{S}$*.*

Proof. We first define a subset $\mathfrak{S}$ of Σ. One may assume without loss of generality that distinct elements of $\mathfrak{S}'$ have distinct gradients. Let

$$S = \left\{ \psi \in \mathfrak{S}' \,\middle|\, \psi + \ell_\psi \notin \Sigma \right\}.$$

Note that S is $\mathrm{Gal}(K/F)$-invariant. Let

$$\mathfrak{S} = \begin{cases} \left\{ \psi + \ell_\psi \,\middle|\, \psi \in \mathfrak{S}' \smallsetminus S \right\} & \text{if } S \subsetneq \mathfrak{S}', \\ \left\{ \psi + \ell_\psi - \ell \,\middle|\, \psi \in \mathfrak{S}' \right\} & \text{if } S = \mathfrak{S}'. \end{cases}$$

Then $\mathfrak{S}$ is a nonempty, $\mathrm{Gal}(K/F)$-invariant subset of Σ. It only remains to show that x satisfies condition i) for the set $\mathfrak{S}$ if and only if x satisfies condition i′) for $\mathfrak{S}'$.

Suppose $S \subsetneqq \mathfrak{S}'$. Since $\psi - (\ell - \ell_\psi) > 0$ for $\psi \in S$, we have for all $y \in \bar{C}$

$$\min_{\psi \in \mathfrak{S}'} \bigl(\psi(y) - (\ell - \ell_\psi)\bigr) = \min_{\psi \in \mathfrak{S}' \smallsetminus S} \bigl(\psi(y) - (\ell - \ell_\psi)\bigr) = \min_{\psi \in \mathfrak{S}} \psi(y) \ - \ell$$

so x satisfies i) for $\mathfrak{S}$ if and only if x satisfies i′) for $\mathfrak{S}'$.

Now suppose $S = \mathfrak{S}'$. Then for $y \in \bar{C}$

$$\min_{\psi \in \mathfrak{S}'} \bigl(\psi(y) - (\ell - \ell_\psi)\bigr) = \min_{\psi \in \mathfrak{S}} \psi(y),$$

so x satisfies i) for $\mathfrak{S}$ if and only if x satisfies i′) for $\mathfrak{S}'$. □

From this lemma we may assume that the set of $\mathfrak{g}^*$-optimal points is a subset of the $\mathfrak{g}$-optimal points. Therefore, it makes sense to call a point x of the set of $\mathfrak{g}$-optimal points *optimal* if it is G-conjugate to a point in $\mathcal{O}$.

We say that $r \in \mathbb{Q}$ is an *optimal number* if there is an optimal point x such that $\mathfrak{g}_{x,r} \neq \mathfrak{g}_{x,r^+}$. Since every optimal point is conjugate to a point in $\mathcal{O}$ and $\mathcal{O}$ is a finite set, the set of optimal numbers is discrete.

The following lemma has been extracted from the proof of [16, Proposition 6.3].

Lemma 1.4.3. *If $y \in \mathcal{B}$ and $r \in \mathbb{R}$, then there exist optimal points $x, z \in \mathcal{B}$ such that*

$$\mathfrak{g}_{x,r} \subset \mathfrak{g}_{y,r} \subset \mathfrak{g}_{z,r}.$$

Proof. We will only show the existence of z. The existence of x follows from a duality argument. We may and do assume that y is a $\mathrm{Gal}(K/F)$-invariant point of $\bar{C}$. Define $\mathfrak{S}'$ by

$$\mathfrak{S}' = \{\psi \in \mathbf{\Psi} | \psi(y) \geq r\}.$$

Let n be the least integer such that $\psi + n \cdot \ell > 0$ for all $\psi \in \mathfrak{S}'$. Define the nonempty, $\mathrm{Gal}(K/F)$-invariant subset $\mathfrak{S}$ of Σ by

$$\mathfrak{S} = \{\psi + n \cdot \ell | \psi \in \mathfrak{S}'\} \cap \Sigma.$$

Let $z = x_{\mathfrak{S}} \in \bar{C}$. Since $z \in \mathcal{B}$, we need only show that

$$\mathfrak{g}_{y,r}(K) \subset \mathfrak{g}_{z,r}(K).$$

In order to show this, it is enough to show that $\psi(z) \geq r$ for all $\psi \in \mathfrak{S}'$. But if $\psi \in \mathfrak{S}'$, then

$$\psi(z) + n \cdot \ell \geq \min_{\phi \in \mathfrak{S}} \phi(z) \geq \min_{\phi \in \mathfrak{S}} \phi(y).$$

Since $\mathfrak{S}$ is a finite set, there is a $\psi' \in \mathfrak{S}'$ such that $\psi(z) \geq \psi'(y)$; but $\psi'(y) \geq r$. $\square$

1.5 Some key results

This dissertation relies on three key results from the papers of Moy and Prasad. These results were originally shown to be true for $\mathfrak{g}^*$. However, with appropriate changes, the proofs carry over to $\mathfrak{g}$.

Proposition 1.5.1 (Moy and Prasad). *Let $y \in \mathcal{B}$. Let $r \in \mathbb{R}$ be such that $\mathfrak{g}_{y,r} \neq \mathfrak{g}_{y,r^+}$ and let $X \in \mathfrak{g}_{y,r} \cap \mathcal{N}$. Then there is an optimal point x such that*

$$X + \mathfrak{g}_{y,r^+} \subset \mathfrak{g}_{x,r^+}.$$

Proof. This is [16, Proposition 6.3]. $\square$

Proposition 1.5.2 (Moy and Prasad). *Let $y \in \mathcal{B}$. Let $r \in \mathbb{R}$ be such that $\mathfrak{g}_{y,r} \neq \mathfrak{g}_{y,r^+}$ and let $X \in \mathfrak{g}_{y,r}$ be such that $(X + \mathfrak{g}_{y,r^+}) \cap \mathcal{N} = \emptyset$. Then for all $x \in \mathcal{B}$*

$$(X + \mathfrak{g}_{y,r^+}) \cap \mathfrak{g}_{x,r^+} = \emptyset.$$

Proof. This is [16, Proposition 6.4]. $\square$

Suppose that $P = MN$ is a proper parabolic subgroup of G with unipotent radical N and a Levi factor M. Let $\overline{P} = M\overline{N}$ be the parabolic opposite P. Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ and $\overline{\mathfrak{p}} = \mathfrak{m} + \overline{\mathfrak{n}}$ be the associated Lie algebras. Recall that $\mathfrak{g}$ has the direct sum decomposition

$$\mathfrak{g} = \overline{\mathfrak{n}} + \mathfrak{m} + \mathfrak{n}.$$

If $X \in \mathfrak{g}$, then X can be written uniquely as $X_{\overline{\mathfrak{n}}} + X_{\mathfrak{m}} + X_{\mathfrak{n}}$ where $X_{\overline{\mathfrak{n}}} \in \overline{\mathfrak{n}}$, $X_{\mathfrak{m}} \in \mathfrak{m}$, and $X_{\mathfrak{n}} \in \mathfrak{n}$. Let $\mathcal{N}_{\mathfrak{m}}$ denote the set of nilpotent elements in $\mathfrak{m}$.

Proposition 1.5.3 (Moy and Prasad). *Suppose that $x \in \mathcal{B}(M) \subset \mathcal{B}(G)$ and $X \in \mathfrak{g}_{x,r} \smallsetminus \mathfrak{g}_{x,r^+}$. If $(X + \mathfrak{g}_{x,r^+}) \cap \mathcal{N} \neq \emptyset$, then there exists an element $p \in (N \cap G_x)(M \cap G_x)$ so that*

$$\left(({}^{p}X)_{\mathfrak{m}} + \mathfrak{m}_{x,r^+}\right) \cap \mathcal{N}_{\mathfrak{m}} \neq \emptyset.$$

Proof. This is [17, Proposition 4.7]. ☐

Corollary 1.5.4. *Suppose that $x \in \mathcal{B}(M)$ and $X \in \mathfrak{m}_{x,r} \smallsetminus \mathfrak{m}_{x,r^+}$.*

$$(X + \mathfrak{g}_{x,r^+}) \cap \mathcal{N} \neq \emptyset \;\; \textit{if and only if} \;\; (X + \mathfrak{m}_{x,r^+}) \cap \mathcal{N}_{\mathfrak{m}} \neq \emptyset.$$

Proof. Since $\mathcal{N}_{\mathfrak{m}} \subset \mathcal{N}$ and $X + \mathfrak{m}_{x,r^+} \subset X + \mathfrak{g}_{x,r^+}$, it is clear that $(X + \mathfrak{m}_{x,r^+}) \cap \mathcal{N}_{\mathfrak{m}} \neq \emptyset$ implies $(X + \mathfrak{g}_{x,r^+}) \cap \mathcal{N} \neq \emptyset$.

Now suppose that $(X + \mathfrak{g}_{x,r^+}) \cap \mathcal{N} \neq \emptyset$. From Proposition 1.5.3 there exists a $p = nm \in (N \cap G_x)(M \cap G_x)$ such that $\left(({}^{p}X)_{\mathfrak{m}} + \mathfrak{m}_{x,r^+}\right) \cap \mathcal{N}_{\mathfrak{m}} \neq \emptyset$. However, ${}^{p}X = {}^{m}X + Z$ where Z is an element of $\mathfrak{n}$. Therefore, $({}^{p}X)_{\mathfrak{m}} = {}^{m}X$ and so $({}^{m}X + \mathfrak{m}_{x,r^+}) \cap \mathcal{N}_{\mathfrak{m}} \neq \emptyset$. Since $\mathfrak{m}_{x,r^+}$ and $\mathcal{N}_{\mathfrak{m}}$ are m-stable, the proof is complete. ☐

1.6 Results on the Lie algebra filtrations

The results in this section do not rely on the structure of $\mathfrak{g}$ as a Lie algebra. Therefore, with appropriate changes, they are all valid for the filtrations of $\mathfrak{g}^*$.

Asymptotic results

If ω is a subset of $\mathfrak{g}$, then ${}^{G}\omega$ denotes the set

$$\left\{ {}^{g}X \;\middle|\; X \in \omega \text{ and } g \in G \right\}.$$

In [9, Lemma 2.4] it is shown that the $\mathrm{GL}_n(F)$-orbit of $\varpi^m M_n(R)$ is contained in $\varpi^m M_n(R) + \mathcal{N}$. It is also known (see, for example, [5, Lemma 14.2]) that for any compact set $\omega \subset \mathfrak{g}$, there exists a lattice $\mathcal{L} \subset \mathfrak{g}$ such that ${}^{G}\omega \subset \mathcal{L} + \mathcal{N}$. The Moy-Prasad filtration lattices allow us simultaneously to extend the first result and refine the second.

Lemma 1.6.1. *Let $x, y \in \mathcal{B}$, and let $r \in \mathbb{R}$. Then $\mathfrak{g}_{x,r} \subset \mathfrak{g}_{y,r} + \mathcal{N}$.*

Proof. Since $\mathfrak{g}_{x,r} \subset \mathfrak{g}_{y,r} + \mathcal{N}$ if and only if $\mathfrak{g}_{gx,r} \subset \mathfrak{g}_{gy,r} + \mathcal{N}$ for $g \in G = \mathbf{G}(F)$, we may assume that x and y are $\mathrm{Gal}(K/F)$-fixed points of $\mathcal{A}$.

Let $\mathfrak{z}$ be the Lie algebra of $\mathbf{Z}$. Let Q be the finite set of $\psi \in \mathbf{\Psi}$ such that

$$r + \ell_\psi > \psi(x) \geq r.$$

Then from [16], we have

$$\mathfrak{g}_{x,r}(K) = \mathfrak{z}_r(K) + \sum_{\psi \in Q} \mathfrak{u}_\psi(K)$$

and $\mathfrak{g}_{x,r}$ is the set of the $\mathrm{Gal}(K/F)$-fixed points of $\mathfrak{g}_{x,r}(K)$.

Fix $X \in \mathfrak{g}_{x,r}$. Write $X = X_0 + \sum_{\psi \in Q} X_\psi$, where $X_0 \in \mathfrak{z}_r(K)$, and X_ψ lies in $\mathfrak{u}_\psi(K)$. Let Q^+ denote the set of affine roots ψ in Q such that $\psi(x) > \psi(y)$. Then Q^+ is $\mathrm{Gal}(K/F)$-invariant. Therefore, $X_0 + \sum_{\psi \in Q \smallsetminus Q^+} X_\psi \in \mathfrak{g}_{y,r}$, and we only need to show that $\sum_{\psi \in Q^+} X_\psi \in \mathcal{N}$.

The set $\left\{ x + u\lambda \;\middle|\; u \in \mathbb{R},\ \lambda \in \mathbf{X}_*^F(\mathbf{T}) \right\}$ is dense in the $\mathrm{Gal}(K/F)$-fixed points of $\mathcal{A}$, so we can find $u < 0$ and $\lambda \in \mathbf{X}_*^F(\mathbf{T})$ such that for all affine roots ψ in Q^+,

$$\psi(x) > \psi(x + u\lambda) = \psi(x) + u\langle \dot{\psi}, \lambda \rangle$$

where $\dot{\psi}$ is the gradient of ψ. So for all $\psi \in Q^+$, $\langle \dot{\psi}, \lambda \rangle > 0$, and therefore

$$\lim_{t \to 0} \mathrm{Ad}\bigl(\lambda(t)\bigr) \sum_{\psi \in Q^+} X_\psi = 0. \quad \square$$

Corollary 1.6.2. ${}^G\mathfrak{g}_{x,r} \subset \mathfrak{g}_{x,r} + \mathcal{N}$.

G-domains and parabolic descent

Definition 1.6.3. $V \subset \mathfrak{g}$ is a *G-domain* if V is G-invariant, open, and closed.

The next few results are inspired by ideas from $\mathrm{GL}_n(F)$. Let P be a proper parabolic subgroup of G with a Levi decomposition $P = MN$ and Lie algebra $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$. Suppose that $\mathfrak{k}_0 = M_n(R)$ and $\mathfrak{m}$ is in standard form (i.e., $\mathfrak{m} = \prod M_{n_i}(F)$ is embedded in the usual way). If $V = {}^G\mathfrak{k}_0$, then it is clear that V is open and G-invariant. It is also true that V is closed. Therefore, V is a G-domain in $\mathfrak{g}$. Furthermore, if $V^{\mathfrak{m}} = V \cap \mathfrak{m}$, then $V^{\mathfrak{m}}$ is an M-domain in $\mathfrak{m}$ and $V^{\mathfrak{m}} = {}^M(\mathfrak{k}_0 \cap \mathfrak{m})$.

The following lemmas and corollaries extend these ideas to arbitrary G.

Lemma 1.6.4. *For all $r \in \mathbb{R}$,*

$$\bigcup_{x \in \mathcal{B}} \mathfrak{g}_{x,r} = \bigcap_{x \in \mathcal{B}} (\mathfrak{g}_{x,r} + \mathcal{N}).$$

Proof. Lemma 1.6.1 implies that the left-hand side is a subset of the right. Let us suppose that X is contained in the right-hand side, but not the left, and derive a contradiction.

For all $x \in \mathcal{B}$, there exists $s(x) < r$ such that $X \in \mathfrak{g}_{x,s(x)} \setminus \mathfrak{g}_{x,s(x)^+}$. Pick an optimal $x_1 \in \mathcal{B}$. Then $s(x_1)$ is optimal, and since $X \in \left(\mathfrak{g}_{x_1,r} + \mathcal{N}\right)$, the coset $X + \mathfrak{g}_{x_1,s(x_1)^+}$ contains nilpotent elements. From Proposition 1.5.1, there is some optimal $x_2 \in \mathcal{B}$ such that $X \in \mathfrak{g}_{x_2,s(x_1)^+}$. Thus, $s(x_2) > s(x_1)$. Continuing in this way, we get an infinite sequence of optimal numbers such that

$$s(x_1) < s(x_2) < \cdots < r,$$

which contradicts the fact that the set of optimal numbers is discrete. □

Definition 1.6.5. $V_r = \bigcup_{x \in \mathcal{B}} \mathfrak{g}_{x,r}$.

Corollary 1.6.6. *V_r is a G-domain.*

Proof. V_r is G-invariant and open, so we need only show that V_r is closed. But this follows from Lemma 1.6.4 and the fact that $\mathfrak{g}_{x,r} + \mathcal{N}$ is closed. $\square$

Example 1.6.7. If $G = \mathrm{GL}_n(F)$, then $V_0 = {}^G\mathfrak{k}_0$ (in the notation introduced prior to Lemma 1.6.4).

Example 1.6.8. Recall the example of $\mathrm{SL}_2(F)$ in §1.3. If $G = \mathrm{SL}_2(F)$, we have

$$V_0 = {}^G\left(\begin{smallmatrix} R & R \\ R & R \end{smallmatrix}\right) \cup {}^G\left(\begin{smallmatrix} R & \wp^{-1} \\ \wp & R \end{smallmatrix}\right)$$

and

$$V_{1/2} = {}^G\left(\begin{smallmatrix} \wp & R \\ \wp & \wp \end{smallmatrix}\right).$$

Note that up to scaling, these are the only two G-domains of the form V_r which occur.

The next two lemmas show that the G-domains V_r defined above are related to the Moy-Prasad filtrations in the sense of [5, §4 and Lemma 12].

Lemma 1.6.9.

$$\varpi\mathfrak{g}_{y,r} \subset \varpi V_r \subset {}^G(\mathfrak{g}_{y,r}).$$

Proof. We may suppose that $y \in \bar{C}$. It is enough to show that for all $x \in \bar{C}$

$$\mathfrak{g}_{x,r+\ell}(K) \subset \mathfrak{g}_{y,r}(K).$$

Suppose that this is not true. Then there exists an affine root ψ such that

$$\psi(x) \geq r + \ell \text{ and } \psi(y) < r.$$

So $\psi(x) - \psi(y) > \ell$, which contradicts the fact that both x and y lie in $\bar{C}$. $\square$

Lemma 1.6.10. $\mathcal{N}' = \displaystyle\bigcap_{r\in\mathbb{R}} V_r$.

Proof. Suppose $X \in \mathcal{N}'$. Then ${}^GX$ intersects $\mathfrak{g}_{x,r}$ for all x and all r, so $X \in \bigcap_{r\in\mathbb{R}} V_r$. Fix $y \in \mathcal{B}(G)$. If $X \in \bigcap_{r\in\mathbb{R}} V_r$, then $X \in \bigcap_{r\in\mathbb{R}}({}^G\mathfrak{g}_{y,r})$ from Lemma 1.6.9. Thus, $X \in \mathcal{N}'$. $\square$

Lemma 1.6.11. *$\mathcal{N}' = \mathrm{cl}\,\mathcal{N}$, the topological closure of $\mathcal{N}$.*

Proof. From Lemma 1.6.10, it is enough to show that $\mathrm{cl}(\mathcal{N}) = \bigcap_{r\in\mathbb{R}} V_r$. The latter object contains $\mathcal{N}$ and is closed, so it will be enough to show that $\bigcap_{r\in\mathbb{R}} V_r \subset \mathrm{cl}(\mathcal{N})$. Suppose $Y \in \bigcap_{r\in\mathbb{R}} V_r$ and fix $x \in \mathcal{B}$. Then $Y \in \mathfrak{g}_{x,r} + \mathcal{N}$ for all r, so $Y \in \mathrm{cl}(\mathcal{N})$. ☐

For $t \in \mathbb{R}$, define $V_{t^+} = \bigcup_{s>t} V_s$. The statement of the next lemma was derived from an unpublished result of Jeff Adler. Allen Moy has also proved a similar result.

Lemma 1.6.12. *To each $X \in \mathfrak{g} \smallsetminus \mathcal{N}'$ we can associate a unique rational number $m(X)$ so that $X \in V_{m(X)} \smallsetminus V_{m(X)^+}$. We call $m(X)$ the level of X.*

Proof. Uniqueness follows once existence is shown.

Since $X \notin \mathcal{N}'$, it follows from Lemma 1.6.10 that there is an $s \in \mathbb{R}$ such that $X \notin V_s$. On the other hand, there is an $r \in \mathbb{R}$ such that $X \in V_r$.

From Lemma 1.4.3, $V_t = \bigcup \left({}^G\mathfrak{g}_{x,t}\right)$ where the union is taken over the finite set $\mathcal{O}$. Consequently, if t is not an optimal number, we have $V_t = V_{t^+}$. Since there are only finitely many optimal numbers between r and s, the lemma follows. ☐

Definition 1.6.13. If $X \in \mathcal{N}'$, then $m(X) = \infty$.

Lemma 1.6.14. *The map $m\colon \mathfrak{g} \smallsetminus \mathcal{N}' \to \mathbb{Q}$ sending X to $m(X)$ is locally constant.*

Proof. Suppose that $m(X) = s$. Then there exists a point $x \in \mathcal{B}$ such that $X \in \mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+}$. Choose $Y \in \mathfrak{g}_{x,s^+}$. We claim that $m(X+Y) = s$.

Since $X + Y \in \mathfrak{g}_{x,s}$, we have that $m(X+Y) \geq s$. If $m(X+Y) > s$, then there exists a $z \in \mathcal{B}$ such that $X + Y \in \mathfrak{g}_{z,s^+}$. From Lemma 1.6.1 we have that $X + Y \in \mathcal{N} + \mathfrak{g}_{x,s^+}$ and so $X \in \mathcal{N} + \mathfrak{g}_{x,s^+}$. But then Proposition 1.5.1 implies that $X \in V_{s^+}$ and so $m(X) > s$, a contradiction. ☐

Jeff Adler conjectured that the following lemma was true.

Lemma 1.6.15. *Assume that F has characteristic zero. If $X \in \mathfrak{g}$ and X has Jordan decomposition $X = X_s + X_n$, then $m(X) = m(X_s)$.*

Proof. Fix $z \in \mathcal{B}(G)$ such that $X_s \in \mathfrak{g}_{z,m(X_s)} \setminus \mathfrak{g}_{z,m(X_s)+}$. Since $0 \in \mathrm{cl}({}^{C_G(X_s)}X_n)$, there exists a $g \in C_G(X_s)$ such that ${}^gX_n \in \mathfrak{g}_{z,m(X_s)+}$. Therefore,

$$X = {}^{g^{-1}}({}^gX) = {}^{g^{-1}}(X_s + {}^gX_n) \in {}^{g^{-1}}(\mathfrak{g}_{z,m(X_s)}) = \mathfrak{g}_{g^{-1}z,m(X_s)}.$$

Let $y = g^{-1}z$. Then $X \in \mathfrak{g}_{y,m(X_s)} \subset V_{m(X_s)}$ which implies that $m(X) \geq m(X_s)$. Note that $X_n \in \mathfrak{g}_{y,m(X_s)+}$.

On the other hand, if there is a point $x \in \mathcal{B}$ such that $X \in \mathfrak{g}_{x,m(X_s)+}$, then $X \in \mathfrak{g}_{y,m(X_s)+} + \mathcal{N}$ which implies that $X_s \in \mathfrak{g}_{y,m(X_s)+} + \mathcal{N}$. From Proposition 1.5.1 we have that $X_s \in V_{m(X_s)+}$, a contradiction. □

Let P be a proper parabolic subgroup of G with a Levi decomposition $P = MN$ and Lie algebra $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$. Recall that $\mathcal{N}_{\mathfrak{m}}$ denotes the set of nilpotent elements in $\mathfrak{m}$ and let $V_r^{\mathfrak{m}} = \bigcup_{x \in \mathcal{B}(M)} \mathfrak{m}_{x,r}$.

Lemma 1.6.16.

$$V_r \cap \mathfrak{m} = V_r^{\mathfrak{m}}.$$

Proof. It is clear that $V_r^{\mathfrak{m}} \subset V_r \cap \mathfrak{m}$.

Suppose that $X \in V_r \cap \mathfrak{m}$ and $X \notin V_r^{\mathfrak{m}}$. We will derive a contradiction. From Lemma 1.6.12 there exists an $s < r$ such that $X \in V_s^{\mathfrak{m}} \setminus V_{s+}^{\mathfrak{m}}$. So, there exists an $x \in \mathcal{B}(M)$ such that $X \in \mathfrak{m}_{x,s} \setminus \mathfrak{m}_{x,s+}$.

Since $X \in V_r$, we have $X \in \mathfrak{g}_{x,r} + \mathcal{N} \subset \mathfrak{g}_{x,s+} + \mathcal{N}$ from Lemma 1.6.4. Since $X + \mathfrak{g}_{x,s+}$ contains a nilpotent element, Corollary 1.5.4 says that $X \in \mathfrak{m}_{x,s+} + \mathcal{N}_{\mathfrak{m}}$. From Proposition 1.5.1, there exists a $y \in \mathcal{B}(M)$ such that $X \in \mathfrak{m}_{y,s+} \subset V_{s+}^{\mathfrak{m}}$. This is a contradiction. □

Example 1.6.17. Suppose that $G = \mathrm{GL}_n(F)$. In the notation introduced prior to Lemma 1.6.4, $V_0^{\mathfrak{m}} = {}^M(\mathfrak{k}_0 \cap \mathfrak{m})$.

Corollary 1.6.18.

$$\mathcal{N}' \cap \mathfrak{m} = \mathcal{N}'_{\mathfrak{m}}.$$

Proof.

$$\begin{aligned}\mathcal{N}' \cap \mathfrak{m} &= \Big(\bigcap_r V_r\Big) \cap \mathfrak{m} = \bigcap_r (V_r \cap \mathfrak{m}) \\ &= \bigcap_r V_r^{\mathfrak{m}} = \mathcal{N}'_{\mathfrak{m}}. \quad \square\end{aligned}$$

Corollary 1.6.19. *If $X \in \mathfrak{m}$ and $m_{\mathfrak{m}}(X)$ is the level of X in $\mathfrak{m}$, then $m_{\mathfrak{m}}(X) = m(X)$.*

On some results of Harish-Chandra

The following results came from attempts to understand the proof of [5, Lemma 18]. In that proof, we are asked to consider the eigenvalues of $\mathrm{ad}(X)$ for $X \in \mathfrak{g}^{\mathrm{reg}}$. The following two lemmas provide a different approach to that part of the proof.

Fix $x \in \mathcal{B}(G)$. Let $S = \mathfrak{g}_{x,0} \setminus \varpi\mathfrak{g}_{x,0}$. S is a compact, G_x-invariant subset of $\mathfrak{g}$. For all $X \in \mathfrak{g} \setminus \{0\}$, define $\lambda(X) \in \mathbb{Z}$ by

$$\varpi^{-\lambda(X)} X \in S.$$

Note that $\lambda(X)$ is uniquely determined by X.

Lemma 1.6.20. *Define $S_r = V_r \cap S$. The set $\{S_r | r \geq 0\}$ is a neighborhood basis of $\mathcal{N}' \cap S$ in S.*

Proof. We need to show that for all open sets U of S containing $\mathcal{N}' \cap S$, there exist s and t such that

$$S_s \subset U \subset S_t.$$

Since $S_0 = S$, we only need to find an s such that $S_s \subset U$.

Note that $\mathcal{N}' \cap S$ is compact. For each $X \in \mathcal{N}' \cap S$, choose $r_X > 0$ such that

$$X + \mathfrak{g}_{x,r_X} \subset U.$$

Since $\{X + \mathfrak{g}_{x,r_X} | X \in \mathcal{N}' \cap S\}$ is an open cover of $\mathcal{N}' \cap S$, there exists an $s > \ell$ such that

$$\mathcal{N}' \cap S \subset \bigcup_{X \in \mathcal{N} \cap S} (X + \mathfrak{g}_{x,s}) \subset U.$$

But,

$$\begin{aligned} S_s &= V_s \cap S \\ &= \Big(\bigcap_{y \in \mathcal{B}(G)} (\mathfrak{g}_{y,s} + \mathcal{N}) \Big) \cap S \\ &\subset (\mathfrak{g}_{x,s} + \mathcal{N}) \cap S \\ &= \mathfrak{g}_{x,s} + (\mathcal{N} \cap S) \subset U. \quad \square \end{aligned}$$

Lemma 1.6.21. *Let* $\mathfrak{h}$ *be a Cartan subalgebra of* $\mathfrak{g}$. *Let* ω_2 *be a compact subset of* $\mathfrak{h} \cap \mathfrak{g}^{\mathrm{reg}}$. *Fix* $N > 0$. *There exists an* $r > 0$ *with the property that if* $H_2 \in \omega_2$ *and* $g \in G$ *such that*

$$\varpi^{-\lambda({}^g H_2)} \cdot {}^g H_2 \in S_r,$$

then $\lambda({}^g H_2) < -N$.

Proof. Since ω_2 is compact, there exists a positive m such that $\omega_2 \cap V_m = \emptyset$. Choose $r \in \mathbb{R}$ such that $r - m > \ell \cdot N$. Fix $H_2 \in \omega_2$ and $g \in G$. Since $\omega_2 \cap V_m = \emptyset$, there exists a $z \in \mathcal{B}(G)$ such that ${}^g H_2 \notin \mathfrak{g}_{z,m} + \mathcal{N}$.

If

$$\varpi^{-\lambda({}^g H_2)} \cdot {}^g H_2 \in S_r,$$

then

$${}^g H_2 \in \varpi^{\lambda({}^g H_2)} (V_r \cap S).$$

This implies that

$${}^g H_2 \in \mathfrak{g}_{z,(r + \ell \cdot \lambda({}^g H_2))} + \mathcal{N}.$$

But ${}^g H_2 \notin \mathfrak{g}_{z,m} + \mathcal{N}$ which implies that $r + \ell \cdot \lambda({}^g H_2) < m$, or

$$\lambda({}^g H_2) < \frac{m - r}{\ell} < -N. \quad \square$$

CHAPTER 2

INVARIANT DISTRIBUTIONS ON THE LIE ALGEBRA

In this chapter, we assume that F has characteristic zero.

2.1 Notation for chapter two

Suppose that V is a finite-dimensional vector space over F with dual V^*. As usual, let $C_c^\infty(V)$ denote the space of locally constant, complex-valued functions on V with compact support. Let dv be a Haar measure on V. For any $f \in C_c^\infty(V)$, we define the Fourier transform $\hat{f} \in C_c^\infty(V^*)$ of f by

$$\hat{f}(\chi) = \int_V dv\, f(v)\Lambda\big(\chi(v)\big)$$

for $\chi \in V^*$. Let $d\chi$ be a Haar measure on V^*. For $f \in C_c^\infty(V^*)$ we use the usual identification of V^{**} with V and define the Fourier transform $\hat{f} \in C_c^\infty(V)$ by

$$\hat{f}(v) = \int_{V^*} d\chi\, f(\chi)\Lambda\big(\chi(v)\big)$$

for $v \in V$. In this chapter we will assume that measures are normalized so that for $v \in V$ and $f \in C_c^\infty(V)$

$$\hat{\hat{f}}(v) = f(-v).$$

If $\mathcal{M} \subset \mathcal{L}$ are lattices in V, then $C(\mathcal{L}/\mathcal{M})$ embeds in $C_c^\infty(V)$ in a natural way. The Moy-Prasad filtration lattices are defined so that $f \in C(\mathfrak{g}_{x,r}/\mathfrak{g}_{x,s}) \subset C_c^\infty(\mathfrak{g})$ if and only if $\hat{f} \in C(\mathfrak{g}^*_{x,(-s)+}/\mathfrak{g}^*_{x,(-r)+}) \subset C_c^\infty(\mathfrak{g}^*)$.

Let x be a special point in $\mathcal{B}$ (see, for example, [32, §§3.3.2 and 3.3.3]). Then, in the language of Harish-Chandra, G_x is a compact open subgroup of Bruhat-Tits [4, Theorem 5, p. 16]. Indeed, in the context of the Moy-Prasad filtrations, it is clear that $\mathfrak{g}_{x,r}$ is well adapted in the sense of [5, §12.2]. Moreover, for any proper parabolic subgroup P of G, we have $G = PG_x$. Let dk be the normalized Haar measure on G_x. If P has a Levi decomposition $P = MN$, let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ be the corresponding Lie algebra. Let dZ be a Haar measure on $\mathfrak{n}$.

Definition 2.1.1. For $f \in C_c^\infty(\mathfrak{g})$, define $f_P \in C_c^\infty(\mathfrak{m})$ by

$$f_P(Y) = \int_{\mathfrak{n}} dZ \int_{G_x} dk\, f\big({}^k(Y+Z)\big)$$

for $Y \in \mathfrak{m}$.

Remark 2.1.2. It is true [13] that if f has support in V_r, then f_P has support in $V_r \cap \mathfrak{m}$. We know from Lemma 1.6.16 that $V_r \cap \mathfrak{m} = V_r^{\mathfrak{m}}$.

Let $J = J(\mathfrak{g})$ denote the set of G-invariant distributions on $\mathfrak{g}$. Similarly, $J_{\mathfrak{m}}$ will denote the set of M-invariant distributions on $\mathfrak{m}$.

Suppose that $\mathcal{O}$ is an orbit in $\mathfrak{g}$, i.e., $\mathcal{O} = {}^G X$ for some $X \in \mathfrak{g}$. Let $C_G(X)$ denote the centralizer of X in G and let dg^* denote a G-invariant measure on $G/C_G(X)$. From [24] the integral

$$\mu_X(f) = \mu_{\mathcal{O}}(f) = \int_{G/C_G(X)} dg^*\, f({}^g X)$$

is well defined for $f \in C_c^\infty(\mathfrak{g})$. Therefore, $\mu_X \in J$.

$J(\mathcal{N})$ will denote the set of G-invariant distributions supported on $\mathcal{N}$. It is known [5, Lemma 5.3] that $J(\mathcal{N})$ is spanned by the nilpotent orbital integrals.

Let $\mathcal{O}(0)$ denote the set of all nilpotent orbits. Let $\mathcal{O}_{\mathfrak{m}}(0)$ denote the set of all nilpotent orbits in $\mathfrak{m}$.

2.2 Induced distributions

By way of introduction, we first look at the group side of things.

Suppose that π is an irreducible admissible representation of G. We can, as in [16], associate a nonnegative number $\rho(\pi)$—the depth of π—to π. In [16], one finds the following

Conjecture 2.2.1 (Moy and Prasad). *The Harish-Chandra-Howe local expansion for the character* Θ_π *of* π *is valid for all regular* $\gamma \in G_{x,\rho(\pi)^+}$ *for any point* x *in* $\mathcal{B}(G)$.

In [17], it is shown that parabolic induction preserves depth. Let us make this statement more precise. Suppose that $P = MN$ is a parabolic subgroup of G with unipotent radical N and a Levi factor M. Let σ be an admissible irreducible representation of M and let π be an irreducible subquotient of the induced representation $\operatorname{Ind}_{MN}^{G} \sigma$. Then $\rho(\pi) = \rho(\sigma)$. (This result does not depend on whether or not our induction is normalized.)

Given the above discussion, it is natural to ask: if we assume that the local character expansion for Θ_σ is valid on $\bigcup_{x \in \mathcal{B}(M)} M_{x,\rho(\sigma)^+}$, is the local character expansion of Θ_π valid on $\bigcup_{x \in \mathcal{B}(G)} G_{x,\rho(\sigma)^+}$?

In general, the exponential map is not defined everywhere that it would have to be in order to answer this question. So we will defer this question until §4.2 and first consider the analogous question for G-invariant distributions on the Lie algebra. In order to formulate this result, we need some more notation.

For this section only, let B be a nondegenerate, G-invariant, symmetric, bilinear form on $\mathfrak{g}$. Then B induces a G-equivariant isomorphism $\mathfrak{g}^* \longrightarrow \mathfrak{g}$. More to the point, if $f \in C_c^\infty(\mathfrak{g})$, then define $\hat{f} \in C_c^\infty(\mathfrak{g})$ by

$$\hat{f}(Y) = \int_{\mathfrak{g}} dX\, f(X)\Lambda\bigl(B(X,Y)\bigr).$$

Note that B need not respect the Moy-Prasad filtration structure, that is, $f \in C(\mathfrak{g}_{x,r}/\mathfrak{g}_{x,s})$ does not necessarily imply that $\hat{f} \in C(\mathfrak{g}_{x,(-s)^+}/\mathfrak{g}_{x,(-r)^+})$. If T is a G-invariant distribution on $\mathfrak{g}$, then we define the Fourier transform of T by

$$\widehat{T}(f) = T(\hat{f})$$

for all $f \in C_c^\infty(\mathfrak{g})$. Note that $\widehat{T}$ is also a G-invariant distribution on $\mathfrak{g}$.

Fix a proper parabolic subgroup P of G with a Levi decomposition $P = MN$. Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ be the corresponding Lie algebra.

If $\theta \in J_{\mathfrak{m}}$, then we can define (see, for example, [5, Lemma 3.12]) a G-invariant distribution $\Theta \in J$ by

$$\Theta(f) = \theta(f_P)$$

for all $f \in C_c^\infty(\mathfrak{g})$.

Lemma 2.2.2. *If $\xi \in \mathcal{O}_{\mathfrak{m}}(0)$, then there exist constants $c_{\mathcal{O}} \in \mathbb{C}$ indexed by $\mathcal{O} \in \mathcal{O}(0)$ such that for all $f \in C_c^\infty(\mathfrak{g})$,*

$$\mu_\xi(f_P) = \sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}}\, \mu_{\mathcal{O}}(f).$$

Furthermore, $c_{\mathcal{O}}$ is zero unless $r(\mathcal{O}) = r_{\mathfrak{m}}(\xi)$. Here $r(\mathcal{O})$ is the rank of $\mathcal{O}$, i.e., $r(\mathcal{O}) = \dim\bigl(C_{\mathfrak{g}}(X)\bigr)$ for any element $X \in \mathcal{O}$.

Proof. Fix $\xi \in \mathcal{O}_{\mathfrak{m}}(0)$ and let T be the distribution on $\mathfrak{g}$ defined by

$$T(f) = \mu_\xi(f_P)$$

for all $f \in C_c^\infty(\mathfrak{g})$. Since $\mathcal{N}_{\mathfrak{m}} + \mathfrak{n}$ lies in $\mathcal{N}$, T is in $J(\mathcal{N})$. Consequently, there exist constants $c_{\mathcal{O}}(T) \in \mathbb{C}$ indexed by $\mathcal{O} \in \mathcal{O}(0)$ such that

$$\mu_\xi(f_P) = T(f) = \sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}}(T)\, \mu_{\mathcal{O}}(f)$$

for all $f \in C_c^\infty(\mathfrak{g})$.

Since both sides have homogeneity properties, one can readily verify that $c_{\mathcal{O}}(T)$ is zero unless $r(\mathcal{O}) = r_{\mathfrak{m}}(\xi)$. □

Corollary 2.2.3. *Suppose that $\theta \in J_{\mathfrak{m}}$ has a local expansion on $V_r^{\mathfrak{m}}$. If $\Theta \in J(\mathfrak{g})$ is defined by $\Theta(f) = \theta(f_P)$, then Θ has a local expansion on V_r.*

Proof. Suppose that

$$\theta(h) = \sum_{\xi \in \mathcal{O}_{\mathfrak{m}}(0)} c_\xi(\theta)\, \widehat{\mu_\xi}(h)$$

for all $h \in C_c^\infty(V_r^{\mathfrak{m}})$.

Fix $f \in C_c^\infty(\mathfrak{g})$ with support in V_r. Then f_P has support in $V_r^{\mathfrak{m}}$. Recall from [5, §3.2] that $\widehat{f_P} = \hat{f}_P$. From Lemma 2.2.2, there exist constants $c_{\mathcal{O}} \in \mathbb{C}$ indexed by $\mathcal{O} \in \mathcal{O}(0)$ such that

$$\begin{aligned}\Theta(f) &= \theta(f_P)\\ &= \sum_{\xi \in \mathcal{O}_{\mathfrak{m}}(0)} c_\xi(\theta)\, \widehat{\mu_\xi}(f_P)\\ &= \sum_{\xi \in \mathcal{O}_{\mathfrak{m}}(0)} c_\xi(\theta)\, \mu_\xi(\widehat{f_P})\\ &= \sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}}\, \mu_{\mathcal{O}}(\hat{f})\\ &= \sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}}\, \widehat{\mu_{\mathcal{O}}}(f). \quad \square\end{aligned}$$

Note that it is entirely possible that the local expansion of a distribution could be valid in a larger region than these results would imply. However, it is also known that in some cases these results are sharp.

2.3 Towards Waldspurger-type results

Suppose that $\mathcal{L}$ is a lattice in $\mathfrak{g}$ and ω is a compact set in $\mathfrak{g}$. We will denote by $J(\omega)$ the set of G-invariant distributions T on $\mathfrak{g}$ with support in the closure of ${}^G\omega$. It makes sense to talk about $J(V_r)$ in this context since V_r is closed and since, from Lemma 1.4.3, we can write

$$V_r = {}^G\left(\cup \mathfrak{g}_{x,r}\right),$$

where the union is over a finite set of optimal points. If T is a distribution in $J(\mathfrak{g})$, then $j_{\mathcal{L}}T$ will denote its restriction to $C_c(\mathfrak{g}/\mathcal{L})$. It was first conjectured [7] for $\mathrm{GL}_n(F)$ without restrictions on F that

$$\dim j_{\mathcal{L}} J(\omega) < \infty. \tag{2.3.1}$$

This was proved in [8] and extended to arbitrary G in characteristic zero in [5, Theorem 14.1].

In [34] a more precise version of (2.3.1) is verified for many groups with few restrictions on F. In order to simplify this discussion, let us suppose for the moment that $G = \mathrm{GL}_n(F)$ and F is an arbitrary nonarchimedean local field whose residue field has q elements. Let $\mathfrak{k}_0 = M_n(R)$ and let $\mathfrak{b}_0$ be the inverse image under the "reduction mod $\wp$" map $\mathfrak{k}_0 \longrightarrow M_n(\mathbb{F}_q)$ of the standard Borel subalgebra in $M_n(\mathbb{F}_q)$. That is

$$\mathfrak{b}_0 = \left\{ X \in \mathfrak{k}_0 \,\middle|\, X_{ij} \in \wp \text{ if } i > j \right\}.$$

Then, the results of [34] imply that

$$j_{\mathfrak{b}_0} J(\mathfrak{k}_0) = j_{\mathfrak{b}_0} J(\mathcal{N}).$$

It is interesting to analyze this statement in terms of what we know about Moy-Prasad filtration lattices. For example, it follows from Lemma 1.6.1 that ${}^G\mathfrak{k}_0 \subset \mathfrak{b}_0 + \mathcal{N}$. As discussed in §1.6 prior to Lemma 1.6.4, ${}^G\mathfrak{k}_0$ is a G-domain in $M_n(F)$.

Now suppose that F has characteristic zero and let G be arbitrary. Combining the above observations, we think that an appropriate Lie algebra analogue of Conjecture 2.2.1 is:

Conjecture 2.3.2. *Suppose that $y \in \mathcal{B}(G)$ and $s \in \mathbb{R}$. Then*

$$j_{\mathfrak{g}_{y,s}} J(V_s) = j_{\mathfrak{g}_{y,s}} J(\mathcal{N}).$$

See Lemma 3.6.1 for more evidence for this analogy. If $s = 0$, this conjecture has been verified for many groups in [34]. If $G = \mathrm{GL}_n(F)$ and $s = (1/n)$, see §4.3. We have also verified the conjecture for $G = \mathrm{GL}_3(F)$ and $s = 2/3$. Note that in all these cases, the proofs do not place any restrictions on the characteristic of F. The proofs in [34] and §4.3 are quite complicated and it seems unlikely that they will generalize to arbitrary G and s. Here is an alternate approach. Recall that $\varpi \mathfrak{g}_{x,r} = \mathfrak{g}_{x,r+\ell}$.

Conjecture 2.3.3. *Fix an optimal point $x \in \mathcal{O}$ and an optimal $r \in \mathbb{Q}$ with $0 \leq r < \ell$. If $H \in V_r$ is elliptic and regular, then $j_{\mathfrak{g}_{x,r}} \mu_H \in j_{\mathfrak{g}_{x,r}} J(\mathcal{N})$.*

Lemma 2.3.4. *Conjectures 2.3.2 and 2.3.3 are equivalent.*

Proof. It is clear that Conjecture 2.3.2 implies Conjecture 2.3.3.

Suppose that Conjecture 2.3.3 is true.

We first show that we may replace y and s in Conjecture 2.3.2 with elements x and r as in Conjecture 2.3.3. From Lemma 1.4.3 there exists an optimal point w such that

$$\dim j_{\mathfrak{g}_{y,s}} J(V_s) \leq \dim j_{\mathfrak{g}_{w,s}} J(V_s).$$

Since $j_{\mathcal{L}} J(\mathcal{N}) \subset j_{\mathcal{L}} J(V_s)$ and $\dim j_{\mathcal{L}} J(\mathcal{N}) = |\mathcal{O}(0)|$ for all lattices $\mathcal{L} \subset \mathfrak{g}$, it will be enough to show that Conjecture 2.3.2 is true when y is optimal. Since V_s and $\mathcal{N}$ are G-invariant, we may assume that $y \in \mathcal{O}$. Moreover, because nilpotent orbital integrals have homogeneity properties, we need only consider s in the interval $[0, \ell)$. Finally, since V_s is the finite union of some ${}^{G}\mathfrak{g}_{z,s}$ with z optimal, Conjecture 2.3.2 only needs to be verified when s is an optimal number. Therefore, we may and do assume that $y = x$ and $s = r$ where x and r are as in Conjecture 2.3.3.

We shall show that Conjecture 2.3.2 is true by induction on the dimension of G.

First, suppose that G is a torus. Since every element of the Lie algebra of a torus is regular and elliptic, the result follows because orbital integrals are "dense" in $J(V_r)$. However, it is interesting to prove Conjecture 2.3.2 directly in this case: Note that $\mathfrak{g}_{z,r} = \mathfrak{g}_{y,r}$ for all z and y in $\mathcal{B}$. Consequently, $V_r = \mathfrak{g}_{x,r}$ and so $j_{\mathfrak{g}_{x,r}} J(V_r)$ is one-dimensional. Since $\mathcal{N} = \{0\}$, Conjecture 2.3.2 is clearly true.

Now drop the assumption that G is a torus. From Lemma 1.6.9 and [5, Corollary 6.2] we need to show that if $H \in V_r$ is regular but not elliptic, then

$$j_{\mathfrak{g}_{x,r}} \mu_H \in j_{\mathfrak{g}_{x,r}} J(\mathcal{N}).$$

Suppose that $H \in V_r$ is a regular semisimple element of $\mathfrak{g}$ and the Cartan subalgebra $\mathfrak{h}$ is the centralizer of H in $\mathfrak{g}$. Also assume that $\mathfrak{h}$ is not elliptic. Let A be the split component of the Cartan subgroup corresponding to $\mathfrak{h}$. Since $\mathfrak{h}$ is not elliptic, A is not in the center of G. Therefore, we can choose a parabolic subgroup P with Levi decomposition MN and Lie algebra $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ such that $H \in \mathfrak{m}$.

Fix $f \in C_c(\mathfrak{g}/\mathfrak{g}_{x,r})$. Then $\hat{f} \in C^\infty(\mathfrak{g}^*_{x,(-r)^+})$. Therefore,

$$\operatorname{supp}(\hat{f}_P) \subset \bigcup_{x \in \mathcal{B}(M)} \mathfrak{m}^*_{x,(-r)^+}.$$

If we define f_P in the natural way for $f \in C_c^\infty(V^*)$, then, as in [5, §3.2], $\hat{f}_P = \widehat{f_P}$. Define $g \in C_c^\infty(\mathfrak{m}^*)$ by $g(\chi) = \widehat{f_P}(-\chi)$ for $\chi \in \mathfrak{m}^*$. Since g has compact support, there exist $y_1, y_2, \ldots, y_N \in \mathcal{B}(M)$ such that

$$\operatorname{supp}(g) \subset \bigcup_{1 \le i \le N} \mathfrak{m}^*_{y_i,(-r)^+}.$$

We follow [5, p. 31]. Let F_i denote the characteristic function of $\mathfrak{m}^*_{y_i,(-r)^+}$. Let S denote the set $\{1, 2, \ldots, N\}$. For every nonempty subset I of S define

$$F_I = \prod_{i \in I} F_i \quad \text{and} \quad V_I = \bigcap_{i \in I} \mathfrak{m}^*_{y_i,(-r)^+}.$$

Let $[I]$ denote the cardinality of the set $I \subset S$. If $g_I = F_I \cdot g$, then

$$f_P = -\sum_{\emptyset \neq I \subset S} (-1)^{[I]}\, \widehat{g_I}.$$

For $I \neq \emptyset$, g_I lies in $C^\infty(\mathfrak{m}^*_{y_i,(-r)^+})$ for some $i \in S$, and therefore $\widehat{g_I} \in C_c(\mathfrak{m}/\mathfrak{m}_{y_i,r})$. By induction, there exist $c_\xi \in \mathbb{C}$ indexed by $\xi \in \mathcal{O}_{\mathfrak{m}}(0)$ such that

$$\mu_H^M(\widehat{g_I}) = \sum_{\xi \in \mathcal{O}_{\mathfrak{m}}(0)} c_\xi\, \mu_\xi(\widehat{g_I}).$$

For $I \neq \emptyset$, the c_ξ are independent of I. Therefore

$$\begin{aligned}
\mu_H^G(f) &= \mu_H^M(f_P) \\
&= -\sum_{\emptyset \neq I \subset S} (-1)^{[I]}\, \mu_H^M(\widehat{g_I}) \\
&= -\sum_{\emptyset \neq I \subset S} (-1)^{[I]} \sum_{\xi \in \mathcal{O}_{\mathfrak{m}}(0)} c_\xi\, \mu_\xi(\widehat{g_I}) \\
&= \sum_{\xi \in \mathcal{O}_{\mathfrak{m}}(0)} c_\xi\, \mu_\xi(f_P) \\
&= \sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}}\, \mu_{\mathcal{O}}(f)
\end{aligned}$$

for some constants $c_{\mathcal{O}} \in \mathbb{C}$, from Lemma 2.2.2. $\square$

CHAPTER 3

SINGLE ORBIT THEORY FOR SOME SUPERCUSPIDALS OF GL_n

We now assume that $G = \mathrm{GL}_n(F)$. F is an arbitrary nonarchimedian local field.

3.1 Notation for GL_n calculations

In the remainder of this dissertation, we will realize $\mathrm{GL}_n(F)$ as the subset of $\mathfrak{g} = M_n(F)$ consisting of $n \times n$ matrices with nonzero determinant.

For $f \in C_c^\infty(\mathfrak{g})$, we define the Fourier transform of f by

$$\hat{f}(X) = \int_{\mathfrak{g}} dY\, f(Y)\Lambda\big(\mathrm{tr}(X \cdot Y)\big)$$

for $X \in \mathfrak{g}$. One can verify that if $f \in C(\mathfrak{g}_{x,t}/\mathfrak{g}_{x,s})$, then $\hat{f} \in C(\mathfrak{g}_{x,(-s)^+}/\mathfrak{g}_{x,(-t)^+})$.

Let $\mathfrak{k}_0 = M_n(R)$. Let $\mathfrak{b}_0$ be the inverse image under the "reduction mod $\wp$" map $\mathfrak{k}_0 \longrightarrow M_n(\mathbb{F}_q)$ of the standard Borel subalgebra in $M_n(\mathbb{F}_q)$. That is

$$\mathfrak{b}_0 = \left\{ X \in \mathfrak{k}_0 \;\middle|\; X_{ij} \in \wp \text{ if } i > j \right\}.$$

Define

$$\Pi = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ \varpi & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

and let $\mathfrak{b}_i = \Pi^i \cdot \mathfrak{b}_0$ for all integers i. As usual we define the compact open subgroup B_0 of G to be the set of invertible elements of $\mathfrak{b}_0$, and for a positive integer i we define the congruence subgroup $B_i = 1 + \mathfrak{b}_i$, which is normal in B_0. Note that $V_{1/n} = {}^G\mathfrak{b}_1$.

Let $\mathfrak{k}_i = \varpi^i \cdot \mathfrak{k}_0$ for all integers i. As usual we will define the compact open subgroup K_0 of G to be the set of invertible elements of $\mathfrak{k}_0$. For a positive integer i we define the congruence subgroup $K_i = 1 + \mathfrak{k}_i$, which is normal in K_0. Note that $V_0 = {}^G\mathfrak{k}_0$.

3.2 The representations being considered

In this section we briefly review how to construct some irreducible supercuspidal representations of $\mathrm{GL}_n(F)$. If n is a prime, then up to twisting by a character of $\mathrm{GL}_n(F)$ these are all of the irreducible supercuspidal representations.

Generic elements

Let S be the maximal split torus formed by the diagonal matrices in G. Suppose that $s = \mathrm{Diag}(s_1, s_2, \ldots, s_n) \in S$ and define the root $\alpha_i(s) = s_i/s_{(i+1)}$ for $1 \leq i \leq (n-1)$. Let $\tilde{\alpha}(s) = s_1/s_n$. ($\tilde{\alpha}$ is the highest root with respect to the standard Borel subgroup containing S.) Let $\mathcal{A}$ be the apartment of S in $\mathcal{B}(G)$ and let C be the chamber in $\mathcal{A}$ stabilized by the Iwahori subgroup B_0. If $\boldsymbol{\Psi}$ is the set of affine roots of G relative to S, then the basis of $\boldsymbol{\Psi}$ determined by C is

$$\Delta = \{\psi_i \,|\, 1 \leq i \leq n\}$$

where $\psi_i = \alpha_i$ if $1 \leq i \leq (n-1)$ and $\psi_n = -\tilde{\alpha} + 1$.

Suppose n factors as $e \cdot f$ for positive integers e and f. In the notation of §1.4, we define

$$\mathfrak{S}_e = \{\psi_e, \psi_{2e}, \ldots, \psi_{(f-1)\cdot e}, \psi_n\}$$

and denote by x_e the optimal point $x_{\mathfrak{S}_e}$. In the language of [2, §2.1 and §2.6], we have $G_{x_e} = K_e$, $G_{x_e,s} = K_e^{\lceil e\cdot s\rceil}$, and $\mathfrak{g}_{x_e,s} = A_e^{\lceil s\cdot e\rceil}$ for all appropriate s. In particular, we have $G_{x_1,t} = K_{\lceil t\rceil}$ and $G_{x_n,t} = B_{\lceil n\cdot t\rceil}$ for all nonnegative t.

Fix e dividing n and $x = x_e \in \mathcal{B}(G)$ as above. We now introduce some special elements of $\mathfrak{g}$. Following [14] we call these elements generic, or, more descriptively,

x-generic. Choose $s \in \mathbb{Q}$ so that $e \cdot s \in \mathbb{Z}$ and $(e \cdot s, n) = 1$. In the language of [2, §3] an element $b \in \mathfrak{g}$ is x-generic if its image in $\mathfrak{g}_{x,s}/\mathfrak{g}_{x,s^+} = A_e^{s \cdot e}/A_e^{s \cdot e+1}$ is *(e -) cuspidal.* Rather than repeat the definition, we will simply recall some of the properties of x-generic elements. Fix an x-generic element $b \in \mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+}$.

Lemma 3.2.1. *With the notation of the previous paragraph, b has the following properties:*

1. *$E_b = F[b]$ is a degree n extension of F with ramification degree e,*
2. *the usual filtrations of E_b and $E_b^\times$ agree with the filtrations $\mathfrak{g}_{x,t}$ and $G_{x,t}$ for all appropriate t,*
3. *$E_b^\times$ normalizes $\mathfrak{g}_{x,t}$ and $G_{x,t}$ for all appropriate t, and*
4. *for all $t \in \mathbb{R}$, if $Y \in \big((E_b \cap \mathfrak{g}_{x,t}) + \mathfrak{g}_{x,t^+}\big) \smallsetminus \mathfrak{g}_{x,t^+}$, then $(Y + \mathfrak{g}_{x,t^+}) \cap \mathcal{N} = \emptyset$.*

Proof. The first three results may be found in [2, Proposition 3.3 and §2.5]. The final property is an easy consequence of the fact [2, §2.7] that for all $t, u \in \mathbb{R}$, $\mathfrak{g}_{x,t} \cdot \mathfrak{g}_{x,u} = \mathfrak{g}_{x,t+u}$. ☐

Moreover, any element of $b + \mathfrak{g}_{x,s^+}$ is again x-generic and has the abovementioned properties. Let $\mathfrak{g}'$ $(= E_b)$ denote the centralizer of b in $\mathfrak{g}$. Since the natural filtration of E_b is compatible with the Moy-Prasad filtration with respect to x, we have $\wp_{E_b}^m = \mathfrak{g}' \cap \mathfrak{g}_{x,m/e}$ for all integers m. Therefore, it makes sense to define $\mathfrak{g}'_t = \mathfrak{g}' \cap \mathfrak{g}_{x,t}$ for all real numbers t. Note that $\mathfrak{g}'_t = \wp_{E_b}^{\lceil t \cdot e \rceil}$. Similar statements can be made for $G' = C_G(b)$ $(= E_b^\times)$. In particular, $G'_t = G' \cap G_{x,t} = 1 + \wp_{E_b}^{\lceil t \cdot e \rceil}$ for all positive t.

For our immediate purposes, the following lemma is the key fact about x-generic elements.

Lemma 3.2.2 (Carayol). *Suppose that $Y \in \mathfrak{g}_{x,m} \smallsetminus \mathfrak{g}_{x,m^+}$ and $W \in b + \mathfrak{g}_{x,s^+}$. If $[W, Y] \in \mathfrak{g}_{x,(m+s)^+}$, then $Y \in E_b + \mathfrak{g}_{x,m^+}$.*

Proof. This is [2, Lemme 3.5, p. 201]. ☐

Example 3.2.3. Suppose that n is prime. Then $Y \in \mathfrak{b}_m \smallsetminus \mathfrak{b}_{(m+1)}$ is x_n-generic if and only if $(m,n) = 1$ and $(Y + \mathfrak{b}_{m+1}) \cap \mathcal{N} = \emptyset$. (This follows from [14, Proposition 1.5(2)].)

Depth zero representations

Let Z denote the center of G. Suppose that σ is an irreducible representation of ZK_0 such that

1. $K_1 \subset \ker \sigma$
2. $\sigma|_{K_0}$ is cuspidal as a representation of $K_0/K_1 \cong \mathrm{GL}_n(\mathbb{F}_q)$.

If $\pi_\sigma = \mathrm{Ind}_{ZK_0}^G \sigma$, then π_σ is an irreducible supercuspidal representation of G and $\rho(\pi_\sigma) = 0$. Let χ_σ denote the character of σ. Green first gave a complete description of χ_σ on K_0/K_1 in [6]. It follows from [10] that there exists an x_1-generic regular unramified elliptic element X_π of $\mathfrak{k}_0 \smallsetminus \mathfrak{k}_1$ such that for $X \in \mathfrak{b}_1$ and $g \in G$, if ${}^gX \in \mathfrak{k}_0$, then

$$\chi_\sigma(1 + {}^gX) = \deg(\sigma) \cdot \int_{K_0} dk\, \Lambda\big(\mathrm{tr}(X_\pi \cdot {}^{kg}X)\big).$$

Here dk is the normalized Haar measure on K_0.

Some positive depth representations

We now discuss the positive depth supercuspidals constructed in [2]. Fix a positive integer e which divides n. Let $x = x_e \in \mathcal{B}(G)$ as above. Let b be an x-generic element of $\mathfrak{g}$ and suppose that $b \in \mathfrak{g}_{x,-r} \smallsetminus \mathfrak{g}_{x,(-r)^+}$ with $r > 0$. Define filtrations on $\mathfrak{g}' = E_b$ and $G' = E_b^\times$ as above. We will outline how to construct a supercuspidal representation of G using b.

Since $b \in \mathfrak{g}'_{-r} \smallsetminus \mathfrak{g}'_{(-r)^+}$, we can find a character ϕ of G' such that the restriction of ϕ to $G'_{(r/2)^+}$ is given by

$$t \mapsto \Lambda\big(\mathrm{tr}(b \cdot (t-1))\big).$$

Extend ϕ to a character of $G'G_{x,(r/2)^+}$ by defining

$$\phi(x) = \Lambda\big(\mathrm{tr}(b \cdot (x-1))\big)$$

for $x \in G_{x,(r/2)^+}$. Fix a regular elliptic element $X_\pi \in b + \mathfrak{g}_{x,-r/2}$. Note that

$$\phi(x) = \Lambda\big(\mathrm{tr}(X_\pi \cdot (x-1))\big)$$

for $x \in G_{x,(r/2)^+}$. If F has characteristic zero or $(p,n)=1$, then we can take $X_\pi = b$.

Note that $M = E_{X_\pi}^\times = C_G(X_\pi)$ is an elliptic Cartan subgroup and $\mathfrak{m} = E_{X_\pi} = C_{\mathfrak{g}}(X_\pi)$ is an elliptic Cartan subalgebra of $\mathfrak{g}$. Since X_π is also x-generic, M and $\mathfrak{m}$ have all the properties of G' and $\mathfrak{g}'$ discussed in and following Lemma 3.2.1. In particular, $(\mathfrak{m}_s + \mathfrak{g}_{x,s^+}) \smallsetminus \mathfrak{g}_{x,s^+}$ contains no nilpotent elements for all $s \in \mathbb{R}$. We now show how to construct the representations we are interested in.

If $\mathfrak{g}_{x,r/2} = \mathfrak{g}_{x,(r/2)^+}$, then let $\sigma = \phi$.

Suppose that $\mathfrak{g}_{x,r/2} \neq \mathfrak{g}_{x,(r/2)^+}$. Let χ be ϕ restricted to $ZG'_{0^+}G_{x,(r/2)^+}$. From [2, §5] there is a unique extension of χ to an irreducible representation r_χ of $ZG'_{0^+}G_{x,r/2}$ such that the restriction of r_χ to $ZG'_{0^+}G_{x,(r/2)^+}$ contains χ. In fact, r_χ is the only irreducible component of

$$\mathrm{Ind}_{ZG'_{0^+}G_{x,(r/2)^+}}^{ZG'_{0^+}G_{x,(r/2)}} \chi$$

and occurs $\dim(r_\chi)$ times. The representation r_χ can be extended to an irreducible representation σ of $G'G_{x,r/2}$ and these extensions of r_χ are indexed by the characters of $G'G_{x,r/2}/ZG'_{0^+}G_{x,r/2}$.

Remark 3.2.4. Since any irreducible representation of $G'G_{x,r/2}$ whose restriction to $ZG'_{0^+}G_{x,(r/2)^+}$ contains χ must be an extension of r_χ as above, it follows that any irreducible component of

$$\mathrm{Ind}_{G'G_{x,(r/2)^+}}^{G'G_{x,r/2}} \phi$$

must be such an extension.

Let $\pi_\sigma = \mathrm{Ind}_{G'G_{x,r/2}}^{G} \sigma$. Then π_σ is an irreducible supercuspidal representation of G and $\rho(\pi_\sigma) = r$.

We will need the following information about the character χ_σ of σ.

Lemma 3.2.5. *If* $X \in \mathfrak{g}_{x,r/2}$*, then*

$$\chi_\sigma(1+X) = \begin{cases} 0 & \text{if } X \in \mathfrak{g}_{x,r/2} \smallsetminus \mathfrak{g}_{x,(r/2)^+} \\ \deg(\sigma) \cdot \Lambda\bigl(\operatorname{tr}(X_\pi \cdot X)\bigr) & \text{if } X \in \mathfrak{g}_{x,(r/2)^+}. \end{cases}$$

Proof. If $\mathfrak{g}_{x,r/2} = \mathfrak{g}_{x,(r/2)^+}$, then there is nothing to prove. Otherwise, σ is formed from an extension, as described above. Consequently, $\sigma|_{ZG'_{0+}G_{x,(r/2)}}$ occurs $\deg(\sigma)$ times in

$$\operatorname{Ind}_{ZG'_{0+}G_{x,(r/2)^+}}^{ZG'_{0+}G_{x,r/2}} \chi$$

and it is the only irreducible component of this representation. The result now follows from the Frobenius character formula and the fact that

$$\Lambda\bigl(\operatorname{tr}(X_\pi \cdot X)\bigr) = \Lambda\bigl(\operatorname{tr}(b \cdot X)\bigr)$$

if $X \in \mathfrak{g}_{x,(r/2)^+}$. □

3.3 Introduction to single orbit theory

Suppose $\pi = \pi_\sigma = \operatorname{Ind}_{G'G_{x,r/2}}^{G} \sigma$ is a supercuspidal representation as in §3.2. If χ_σ is the character of σ, then we will let $\dot{\chi}_\sigma$ be the function on G defined by

$$\dot{\chi}_\sigma(\gamma) = \begin{cases} \chi_\sigma(\gamma) & \text{if } \gamma \in G'G_{x,r/2}, \\ 0 & \text{otherwise} \end{cases}$$

for $\gamma \in G$.

Recall that Z is the center of G, and let dg^* be a G-invariant measure on G/Z. Let both dk and dk_1 denote the normalized Haar measure on G_x.

From [27], we know that $\dot{\chi}_\sigma$ is a matrix coefficient of π. Recall that to π we can associate a regular elliptic element X_π. We can cast the integrals discussed in the introduction into a more useful form as follows:

Lemma 3.3.1 (Murnaghan).

1. *For all* $\gamma \in G^{\mathrm{reg}}$*,*

$$\Theta_\pi(\gamma) = \frac{\deg(\pi)}{\deg(\sigma)} \cdot \int_{G/Z} dg^* \int_{G_x} dk \int_{G_x} dk_1 \, \dot{\chi}_\sigma({}^{k_1 g k}\gamma).$$

2. *For all* $X \in \mathfrak{g}^{\mathrm{reg}}$*,*

$$\widehat{\mu_{X_\pi}}(X) = \int_{G/Z} dg^* \int_{G_x} dk \int_{G_x} dk_1 \, \Lambda\big(\mathrm{tr}(X_\pi \cdot {}^{k_1 g k}X)\big).$$

Proof. This is [22, Lemma 4.1]. □

Harish-Chandra's proofs of the results behind Lemma 3.3.1 (2) assume that the characteristic of F is zero. However, there is no difficulty in extending these results to $\mathrm{GL}_n(F)$ for arbitrary characteristic. Since an element of $M_n(F)$ is regular if and only if it has distinct eigenvalues, this is not difficult to check.

In this chapter, we will show that for those representations discussed in §3.2 we have the following:

Theorem 3.3.2.

$$\Theta_\pi(1 + X) = \deg(\pi) \cdot \widehat{\mu_{X_\pi}}(X)$$

for all $X \in V^{\mathrm{reg}}_{(\rho(\pi)/2)+}$*.*

The lemmas which comprise the proof of this theorem are found in the two following sections. The proofs of these lemmas are minor refinements of the work found in [18]. There, the proof of a slightly weaker version of Theorem 3.3.2 is carried out for those unramified supercuspidal representations discussed in §3.2.

Finally, we will often make use of the following easily verified lemma.

Lemma 3.3.3. *Let* $r = \rho(\pi)$*. Fix* $m < r$*. Let* $s = (r - m)/2$*. If* $Z \in \mathfrak{g}_{x,s+}$ *and* $Y \in \mathfrak{g}_{x,m}$*, then*

$$\Lambda\big(\mathrm{tr}(X_\pi \cdot {}^{(1+Z)}Y)\big) = \Lambda\big(\mathrm{tr}(X_\pi \cdot (Y + [Z, Y]))\big).$$

3.4 Single orbit theory at depth zero

None of the work in this section is my own; it is the work of Murnaghan [18]. We include it here because: (1) we want to provide as complete a picture as possible of the supercuspidal characters of $\mathrm{GL}_\ell(F)$; (2) the proof has never been published; and (3) it is relatively short and of the same flavor as the remainder of this dissertation.

Let dk and dk_0 both be the normalized Haar measure on K_0. Let π_σ be a depth zero representation of G with inducing data σ and X_π as in §3.2.

Lemma 3.4.1. *If $\gamma \in B_1^{\mathrm{reg}}$, then*

$$\Theta_\pi(\gamma) = \deg(\pi) \cdot \widehat{\mu_{X_\pi}}(\gamma - 1).$$

Recall that $V_{0^+} = {}^G\mathfrak{b}_1$, so Lemma 3.4.1 implies Theorem 3.3.2 for π because $\rho(\pi) = 0$. Write $\gamma = 1 + X$ with $X \in \mathfrak{b}_1^{\mathrm{reg}}$. From Lemma 3.3.1 we have that

$$\Theta_\pi(\gamma) = \frac{\deg(\pi)}{\deg(\sigma)} \cdot \int_{G/Z} dg^* \int_{K_0} dk_0 \int_{K_0} dk\, \dot{\chi}_\sigma({}^{kgk_0}\gamma).$$

On the other hand, we have

$$\int_{K_0} dk\, \dot{\chi}_\sigma({}^{kgk_0}\gamma) = \begin{cases} \deg(\sigma) \cdot \displaystyle\int_{K_0} dk\, \Lambda\bigl(\mathrm{tr}(X_\pi \cdot {}^{kgk_0}X)\bigr) & \text{if } {}^{gk_0}X \in \mathfrak{k}_0 \\ 0 & \text{otherwise.} \end{cases}$$

So, Lemma 3.4.1 would follow from Lemma 3.3.1 along with

Lemma 3.4.2 (Murnaghan).

$$\int_{K_0} dk\, \Lambda\bigl(\mathrm{tr}(X_\pi \cdot {}^{kgk_0}X)\bigr) = 0$$

unless ${}^{gk_0}X \in \mathfrak{k}_0$.

Before we begin the proof, note that a more general version of this lemma occurs as Lemma 3.5.3.

Proof. Let $Y = {}^{gk_0}X$ and suppose that $Y \in \mathfrak{k}_t \setminus \mathfrak{k}_{t+1}$ for a negative integer t. Let $s = 1 + \lfloor \frac{-t}{2} \rfloor$. Then

$$\begin{aligned}\int_{K_0} dk\, \Lambda\big(\mathrm{tr}(X_\pi \cdot {}^kY)\big) &= \mathrm{const} \cdot \int_{K_0} dk \int_{K_s} dh\, \Lambda\big(\mathrm{tr}(X_\pi \cdot {}^{hk}Y)\big) \\ &\qquad \text{(from Lemma 3.3.3)} \\ &= \mathrm{const} \cdot \int_{K_0} dk \int_{\mathfrak{k}_s} dH\, \Lambda\big(\mathrm{tr}(X_\pi \cdot ({}^kY + [{}^kY, H]))\big) \\ &= \mathrm{const} \cdot \int_{K_0} dk\, \Lambda\big(\mathrm{tr}(X_\pi \cdot {}^kY)\big) \int_{\mathfrak{k}_s} dH\, \Lambda\big(\mathrm{tr}([X_\pi, {}^kY] \cdot H)\big).\end{aligned}$$

The inner integral is zero unless $[X_\pi, {}^kY] \in \mathfrak{k}_{1-s}$. Write ${}^kY = Y' + Y''$ with $Y' \in E = C_{\mathfrak{g}}(X_\pi)$ and $Y'' \in \mathfrak{k}_u \setminus \big((E \cap \mathfrak{k}_u) + \mathfrak{k}_{u+1}\big)$ for some $u \geq t$. (If ${}^kY \in E$, then $Y'' = 0$ and $u = \infty$.) From Lemma 3.2.2,

$$[X_\pi, {}^kY] \in \mathfrak{k}_u \setminus \mathfrak{k}_{u+1}.$$

In order for the inner integral not to be zero, we would need

$$u \geq 1 - s = \left\lfloor \frac{1+t}{2} \right\rfloor > t.$$

This would imply that ${}^kY \in (\wp_E^t \setminus \wp_E^{t+1}) + \mathfrak{k}_{t+1}$. But ${}^kY = {}^{kgk_0}X \in \mathfrak{k}_{t+1} + \mathcal{N}$ from Corollary 1.6.2. This is a contradiction. □

3.5 Single orbit theory at positive depth

The key to extending [18] from the unramified supercuspidals to all of the supercuspidals constructed in [2] is Lemma 1.6.1. To a large extent, this is the only difference between the material found in [18] and what appears below.

Let π be a positive depth representation of G with inducing data σ, $x = x_e$, b, and X_π as in §3.2. Recall that $\mathfrak{m} = C_{\mathfrak{g}}(X_\pi)$, $\mathfrak{g}' = C_{\mathfrak{g}}(b)$, $G' = E_b^\times = F[b]^\times$, etc.

Lemma 3.5.1. *If*

$$W \in (\mathfrak{g}_{x,m} \setminus \mathfrak{g}_{x,m^+}) \cap (\mathcal{N} + \mathfrak{g}_{x,m^+}),$$

then

$$W \in \mathfrak{g}_{x,m} \smallsetminus (\mathfrak{m}_m + \mathfrak{g}_{x,m^+}).$$

Proof. Recall that $(\mathfrak{m}_m + \mathfrak{g}_{x,m^+}) \smallsetminus \mathfrak{g}_{x,m^+}$ contains no nilpotent elements. $\square$

Lemma 3.5.2. *Fix $z \in \mathcal{B}$ and $m > 0$. Suppose $Y \in \mathfrak{g}_{z,m}$. If $Y \notin \mathfrak{g}_{x,m}$, then $(1+Y) \notin G'G_{x,m}$.*

Proof. Suppose that $(1+Y) \in G'G_{x,m}$. Then we can find an $\alpha \in \mathfrak{g}'$ and a $W \in \mathfrak{g}_{x,m}$ such that

$$(1+Y) = (1+\alpha)\cdot(1+W) = 1 + \alpha + (1+\alpha)\cdot W.$$

Suppose that $\alpha \in \mathfrak{g}'_t \smallsetminus \mathfrak{g}'_{t^+}$. Since $Y \notin \mathfrak{g}_{x,m}$, we must have that $t < m$. Since $m > 0$, it follows that $(1+\alpha)\cdot W \in \mathfrak{g}_{x,t^+}$ and so $\alpha \equiv Y$ modulo $\mathfrak{g}_{x,t^+}$. Consequently, $\alpha \in \mathcal{N} + \mathfrak{g}_{x,t^+}$ from Lemma 1.6.1. Since, from Lemma 3.2.1, the coset $\alpha + \mathfrak{g}_{x,t^+}$ cannot contain any nilpotent elements, this is a contradiction. $\square$

Lemma 3.5.3. *Fix $\rho < r$ and $z \in \mathcal{B}$. Let $Y \in \mathfrak{g}_{z,\rho^+}$. If $Y \notin \mathfrak{g}_{x,\rho^+}$, then*

$$\int_{G_x} dk\, \Lambda\big(\mathrm{tr}(X_\pi \cdot {}^kY)\big) = 0.$$

Proof. Write $Y \in \mathfrak{g}_{x,m} \smallsetminus \mathfrak{g}_{x,m^+}$, with $m \le \rho < r$. Let $s = (r-m)/2$. Then

$$\int_{G_x} dk\, \Lambda\big(\mathrm{tr}(X_\pi \cdot {}^kY)\big) = \mathrm{const} \cdot \int_{G_x} dk \int_{G_{x,s^+}} dk'\, \Lambda\big(\mathrm{tr}(X_\pi \cdot {}^{k'k}Y)\big).$$

It is enough to show that the inner integral is equal to zero for any $k \in G_x$. Fix $k \in G_x$. Let $W = {}^kY \in \mathfrak{g}_{x,m} \smallsetminus \mathfrak{g}_{x,m^+}$. Then W also lies in $\mathfrak{g}_{kz,\rho^+} \subset \mathfrak{g}_{x,\rho^+} + \mathcal{N} \subset \mathfrak{g}_{x,m^+} + \mathcal{N}$, from Lemma 1.6.1. From Lemma 3.5.1, we have that $W \in \mathfrak{g}_{x,m} \smallsetminus (\mathfrak{m}_m + \mathfrak{g}_{x,m^+})$.

Suppose that dV is the G-invariant measure on $\mathfrak{g}$ such that

$$\int_{G_x} dk\, f(k) = \int_{\mathfrak{g}} dV\, f(1+V)$$

for all $f \in C^\infty(G_{x,0^+})$. The inner integral becomes

$$\int_{G_{x,s^+}} dk'\, \Lambda\big(\mathrm{tr}(X_\pi \cdot {}^{k'}W)\big) = \int_{\mathfrak{g}_{x,s^+}} dV\, \Lambda\big(\mathrm{tr}(X_\pi \cdot {}^{(1+V)}W)\big).$$

From Lemma 3.3.3 the right-hand side becomes

$$\int_{\mathfrak{g}_{x,s+}} dV\,\Lambda\bigl(\operatorname{tr}(X_\pi\cdot(W+[V,W]))\bigr)=\Lambda\bigl(\operatorname{tr}(X_\pi\cdot W)\bigr)\cdot\int_{\mathfrak{g}_{x,s+}} dV\,\Lambda\bigl(\operatorname{tr}([W,X_\pi]\cdot V)\bigr).$$

From Lemma 3.2.2, $[W,X_\pi]\in\mathfrak{g}_{x,m-r}\smallsetminus\mathfrak{g}_{x,(m-r)^+}$. But $m-r<(m-r)/2=-s$, so the integral on the right is zero. ☐

Lemma 3.5.4. *Suppose that $g\in G$ and $y\in\mathcal{B}$. Then for $X\in\mathfrak{g}_{y,(r/2)^+}$ we have*

$$\int_{G_x} dk\,\dot\chi_\sigma\bigl({}^{kg}(1+X)\bigr)=\deg(\sigma)\cdot\int_{G_x} dk\,\Lambda\bigl(\operatorname{tr}(X_\pi\cdot{}^{kg}X)\bigr).$$

Proof. Fix a $k\in G_x$. Since $X\in\mathfrak{g}_{y,(r/2)^+}$, it follows from Lemma 3.5.2 (with $z=kgy$) that if ${}^{kg}(1+X)\in G'G_{x,r/2}$, then ${}^{kg}X\in\mathfrak{g}_{x,r/2}$. Consequently, $\dot\chi_\sigma\bigl({}^{kg}(1+X)\bigr)=0$ unless ${}^{g}X\in\mathfrak{g}_{x,r/2}$. Therefore,

$$\begin{aligned}\int_{G_x} dk\,\dot\chi_\sigma\bigl({}^{kg}(1+X)\bigr)&=\begin{cases}\displaystyle\int_{G_x} dk\,\chi_\sigma\bigl({}^{kg}(1+X)\bigr) & \text{if } {}^{g}X\in\mathfrak{g}_{x,r/2}\\ 0 & \text{otherwise}\end{cases}\\ &\qquad\text{(from Lemma 3.2.5)}\\ &=\begin{cases}\deg(\sigma)\cdot\displaystyle\int_{G_x} dk\,\Lambda\bigl(\operatorname{tr}(X_\pi\cdot{}^{kg}X)\bigr) & \text{if } {}^{g}X\in\mathfrak{g}_{x,(r/2)^+}\\ 0 & \text{otherwise}\end{cases}\\ &\qquad\text{(from Lemma 3.5.3, with } \rho=r/2 \text{ and } z=gy)\\ &=\deg(\sigma)\cdot\int_{G_x} dk\,\Lambda\bigl(\operatorname{tr}(X_\pi\cdot{}^{kg}X)\bigr).\qquad\square\end{aligned}$$

Theorem 3.3.2 now follows from Lemma 3.3.1 and Lemma 3.5.4.

3.6 Towards a quantitative description of the characters

Theorem 3.3.2 allows us to transfer any results we may be able to prove about $\widehat{\mu_{X_\pi}}$ on $V_{(\rho(\pi)/2)^+}$ to the associated character Θ_π. For example:

Lemma 3.6.1. *Conjecture 2.3.2 implies Conjecture 2.2.1 for the supercuspidal representations discussed in §3.2.*

Proof. From Theorem 3.3.2, it will be enough to show that for any $y \in \mathcal{B}$,

$$\widehat{\mu_{X_\pi}}(f) = \sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}}(X_\pi)\, \widehat{\mu_{\mathcal{O}}}(f)$$

for all $f \in C^\infty(\mathfrak{g}_{y,r^+})$. This is true if and only if for any $y \in \mathcal{B}$,

$$\mu_{X_\pi}(f) = \sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}}(X_\pi)\, \mu_{\mathcal{O}}(f)$$

for all $f \in C_c(\mathfrak{g}/\mathfrak{g}_{y,-r})$. Since $X_\pi \in \mathfrak{g}_{x,-r}$, this is exactly the content of Conjecture 2.3.2. ☐

Remark 3.6.2. Note that the above result, [34] and §4.3 imply that the conjecture of Moy and Prasad is true for $(2/\ell)^{\text{ths}}$ of the supercuspidal representations of $\mathrm{GL}_\ell(F)$ where ℓ is a prime.

3.7 The map $X \mapsto 1 + X$

Assume that the characteristic of F is zero. We have made extensive use of the map $X \mapsto (1 + X)$ from $\mathfrak{g}$ to G in this dissertation. It is interesting to ask: How does this map compare with the traditional exponential map? In particular, how does it compare with respect to the Harish-Chandra-Howe local character expansion?

Suppose that π is an irreducible representation of G. In [5] it is shown that there exist constants $c_{\mathcal{O}}(\pi) \in \mathbb{C}$ such that

$$\Theta_\pi(\gamma) = \sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}}(\pi)\, \widehat{\mu_{\mathcal{O}}}\big(\log(\gamma)\big)$$

for all regular γ sufficiently near the identity in G. It is established in [8] that there exist constants $c'_{\mathcal{O}}(\pi)$ such that

$$\Theta_\pi(\gamma) = \sum_{\mathcal{O} \in \mathcal{O}(0)} c'_{\mathcal{O}}(\pi) \widehat{\mu_{\mathcal{O}}}(\gamma - 1)$$

for all regular γ sufficiently near the identity in G. Since the functions $\widehat{\mu_{\mathcal{O}}}$ are linearly independent when restricted to any open set containing zero in $\mathfrak{g}$, it would seem that we must have

$$\widehat{\mu_{\mathcal{O}}}\big(\log(\gamma)\big) = \widehat{\mu_{\mathcal{O}}}(\gamma - 1)$$

for all regular γ sufficiently near the identity in G and all $\mathcal{O} \in \mathcal{O}(0)$. As is explained in this section, this relationship does hold.

Fix a regular, topologically nilpotent element X in $M_n(F)$ and $\mathcal{O} \in \mathcal{O}(0)$. It follows from [22, Lemma 5.1] that the function $\widehat{\mu_{\mathcal{O}}}$ is a rational function in the discriminants of the Levi components of parabolic subalgebras of $\mathfrak{g}$. In turn, these discriminants are functions of the norms of the differences of distinct eigenvalues of X. Suppose that X is close to zero (so, in particular, $\log(1+X)$ is defined). Since discriminants are independent of the base field, we will work over a finite extension E of F where X splits, and we will assume that X is diagonal with distinct eigenvalues $\alpha_1, \alpha_2, \dots, \alpha_n$. It will be enough to verify that for $i \neq j$,

$$\left|\log(1+\alpha_i) - \log(1+\alpha_j)\right|_E = \left|\alpha_i - \alpha_j\right|_E .$$

However, this follows immediately from the definition of log since all of the eigenvalues of X live in $\wp_E$.

CHAPTER 4

SOME RESULTS FOR INVARIANT DISTRIBUTIONS ON GL_n

Let $G = \mathrm{GL}_n(F)$. F is a nonarchimedean local field.

4.1 Notation for chapter four

We recall some basic notation. Let t be an indeterminate and let k denote the rank of G. For $\gamma \in G$, define $D_G(\gamma)$ by

$$\det\bigl(t + 1 - \mathrm{Ad}(\gamma)\bigr) = D_G(\gamma) \cdot t^k + \cdots \text{ (terms of higher order)},$$

and for $X \in \mathfrak{g}$, define $\eta_{\mathfrak{g}}(X)$ by

$$\det\bigl(t - \mathrm{ad}(X)\bigr) = \eta_{\mathfrak{g}}(X) \cdot t^k + \cdots \text{ (terms of higher order)}.$$

An element γ of G is regular if and only if $D_G(\gamma) \neq 0$. An element $X \in \mathfrak{g}$ is regular if and only if $\eta_{\mathfrak{g}}(X) \neq 0$.

Suppose P is a proper parabolic subgroup of G with a Levi decomposition $P = MN$. Let δ_P be the modular function for P defined by $d(pq^{-1}) = \delta_P(q) \cdot dp$ for the left Haar measure dp on P.

Finally, an element $X \in \mathfrak{g}$ is said to be *topologically nilpotent* if $X \in V_{0+}$.

4.2 Induced distributions

Let P be a proper parabolic subgroup with a Levi decomposition MN and let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ be the corresponding Lie subalgebra of $\mathfrak{g}$.

Lemma 4.2.1. *For all $X \in V_{0+}$ we have:*

1. *$|D_G(1+X)| = |\eta_{\mathfrak{g}}(X)|$ and*
2. *if $X \in \mathfrak{p}$, then $\delta_P(1+X) = 1$*

Proof. It is easy to see that (2) is valid.

If X is not regular, then both sides of (1) are zero. So suppose that X is regular. Let $\{\alpha_i\}_{i=1}^n$ be the distinct eigenvalues of X. Since D_G and $\eta_{\mathfrak{g}}$ are independent of the base field, we may assume that X is split over F. Since X is topologically nilpotent, all of the eigenvalues of X lie in $\wp$. We have

$$\begin{aligned} |D_G(1+X)| &= \prod_{i\neq j} \left| \frac{1+\alpha_i}{1+\alpha_j} - 1 \right| \\ &= \prod_{i\neq j} \frac{|1+\alpha_i - 1 - \alpha_j|}{|1+\alpha_j|} \\ &= \prod_{i\neq j} |\alpha_i - \alpha_j| \\ &= |\eta_{\mathfrak{g}}(X)|. \quad \square \end{aligned}$$

The Iwasawa decomposition gives us $G = K_0NM$. Choose Haar measures dx, dk, dn, and dm so that $dx = dk\,dn\,dm$.

Definition 4.2.2. If $f \in C_c^\infty(G)$, then

$$g_f(m) = \delta_P^{1/2}(m) \int_N dn \int_{K_0} dk\, f\left({}^k(mn)\right)$$

for $m \in M$.

Note that, as on the Lie algebra side, if $f \in C_c^\infty(V)$ for a G-domain V in G, then $\operatorname{supp}(g_f) \subset V \cap M$.

Definition 4.2.3. For $f \in C_c^\infty(\cup_{x\in\mathcal{B}(G)} G_{x,0^+})$, define $\tilde{f} \in C_c^\infty(V_{0^+})$ by $\tilde{f}(X) = f(1+X)$.

Lemma 4.2.4. *Suppose that σ is an admissible irreducible representation of M, that $\pi = \operatorname{Ind}_{MN}^G \sigma$ (normalized induction), and that for some $t > 0$ there exist constants $c_\xi(\sigma)$, indexed by $\xi \in \mathcal{O}_{\mathfrak{m}}(0)$, such that*

$$\Theta_\sigma(f) = \sum_{\xi\in\mathcal{O}_{\mathfrak{m}}(0)} c_\xi(\sigma)\widehat{\mu}_\xi(\tilde{f})$$

for all $f \in C_c^\infty(\cup_{x\in\mathcal{B}(M)} M_{x,t})$. *Then there exist constants* $c_{\mathcal{O}}(\pi)$, *indexed by* $\mathcal{O} \in \mathcal{O}(0)$, *such that*

$$\Theta_\pi(f) = \sum_{\mathcal{O}\in\mathcal{O}(0)} c_{\mathcal{O}}(\pi)\widehat{\mu_{\mathcal{O}}}(\tilde{f})$$

for all $f \in C_c^\infty(\cup G_{x,t})$.

Proof. We begin by showing that

$$\widetilde{g_f}(X) = (\tilde{f})_P(X)$$

for all $f \in C_c^\infty(\cup G_{x,t})$ and $X \in \mathfrak{m} \cap \mathfrak{g}^{\mathrm{reg}}$.

Fix such an X and f.

Note immediately that both $\widetilde{g_f}$ and $(\tilde{f})_P$ are zero if $X \notin V_t^{\mathfrak{m}}$. Consequently, we may assume that $X \in V_t^{\mathfrak{m}}$ and so $\delta_P(1+X) = 1$. Let dZ be a Haar measure on $\mathfrak{n}$ such that dZ goes into dn under the $X \mapsto (1+X)$ mapping. Following [31, Lemma 4.7.7, p. 205], for $\gamma \in M \cap G^{\mathrm{reg}}$, define

$$\Delta(\gamma) = \left|\frac{D_G(\gamma)}{D_M(\gamma)}\right|^{1/2}.$$

Then

$$\begin{aligned}
\widetilde{g_f}(X) &= \int_N dn \int_{K_0} dk\, f\big({}^k((1+X)n)\big) \\
&\qquad \text{from [33, Lemma 8]} \\
&= \Delta(1+X)\int_N dn \int_{K_0} dk\, f\big({}^{kn}(1+X)\big) \\
&= \Delta(1+X)\int_N dn \int_{K_0} dk\, \tilde{f}({}^{kn}X) \\
&\qquad \text{from [5, Lemma 3.6]} \\
&= \frac{\Delta(1+X)}{|\det(\operatorname{ad} X)_{\mathfrak{n}}|}\int_{\mathfrak{n}} dZ \int_{K_0} dk\, \tilde{f}\big({}^k(X+Z)\big) \\
&= \frac{\Delta(1+X)}{|\det(\operatorname{ad} X)_{\mathfrak{n}}|}(\tilde{f})_P(X).
\end{aligned}$$

From [5, p. 12] we have

$$|\det(\operatorname{ad} X)_{\mathfrak{n}}| = \left|\frac{\eta_{\mathfrak{g}}(X)}{\eta_{\mathfrak{m}}(X)}\right|^{1/2}.$$

Therefore,

$$\frac{\Delta(1+X)}{|\det(\operatorname{ad}X)_{\mathfrak{n}}|} = 1.$$

We now complete the proof. From [33, Theorem 2], we have

$$\begin{aligned}\Theta_\pi(f) &= \Theta_\sigma(g_f)\\ &= \sum_{\xi\in\mathcal{O}_{\mathfrak{m}}(0)} c_\xi\,\widehat{\mu_\xi}(\tilde{g_f})\\ &= \sum_{\xi\in\mathcal{O}_{\mathfrak{m}}(0)} c_\xi\,\widehat{\mu_\xi}\big((\tilde{f})_P\big)\\ &= \sum_{\mathcal{O}\in\mathcal{O}(0)} c_{\mathcal{O}}\,\widehat{\mu_{\mathcal{O}}}(\tilde{f}). \quad \square\end{aligned}$$

Remark 4.2.5. Suppose F has characteristic zero and G is arbitrary. Let P be a proper parabolic subgroup with a Levi decomposition MN and Lie algebra $\mathfrak{p}=\mathfrak{m}+\mathfrak{n}$. Let V be a G-domain in $\mathfrak{g}$ such that $\exp\colon V\to G$ is defined. Suppose that

1. $\big|\eta_{\mathfrak{g}}(X)\big| = \big|D_G\big(\exp(X)\big)\big|$ for all $X\in V$ and
2. $\big|\eta_{\mathfrak{m}}(X)\big| = \big|D_M\big(\exp(X)\big)\big|$ for all $X\in V\cap\mathfrak{m}$.

If $\mathfrak{g}_{x,t}\subset V$, then exp is a bijective map from $\mathfrak{g}_{x,t}$ to $G_{x,t}$. With appropriate changes, Lemma 4.2.4 is valid after imposing the additional restriction: $V_t\subset V$.

4.3 A Waldspurger-type result

In section 2.3, we discussed some work of Waldspurger and its relation to the conjecture of Moy and Prasad. In this section, we will show that for $G=\mathrm{GL}_n(F)$, F an arbitrary nonarchimedean local field, and $s=1/n$,

$$j_{\mathfrak{g}_{x,s}}J(V_s) = j_{\mathfrak{g}_{x,s}}J(\mathcal{N}). \tag{4.3.1}$$

Note that $1/n$ is the first positive optimal number for GL_n. Let C be the chamber in $\mathcal{B}(\mathrm{GL}_n,F)$ corresponding to B_0. As noted in §1.6,

$$V_s = \bigcup \left({}^G\mathfrak{g}_{y,s}\right)$$

where the union is over a finite set of points in $\bar{C}$. It follows from the proof of Lemma 2.3.4 that we can restrict our attention to optimal x's. For any y in $\bar{C}$, the filtration lattice $\mathfrak{g}_{y,s}$ is contained in $\mathfrak{b}_1$. On the other hand, $\mathfrak{g}_{x,(-s)^+} = \mathfrak{g}_{x,0}$. This implies that $\mathfrak{k}_1$ is contained in some conjugate of $\mathfrak{g}_{x,s}$. Therefore, the validity of equation (4.3.1) follows from the following theorem.

Theorem 4.3.2.

$$j_{\mathfrak{k}_1} J(\mathfrak{b}_1) = j_{\mathfrak{k}_1} J(\mathcal{N}).$$

The proof is rather long, so we break it into pieces. If $X \in \mathfrak{g}$ and $\mathcal{L}$ is a lattice in $\mathfrak{g}$, then let $[X + \mathcal{L}]$ denote the characteristic function of the coset $X + \mathcal{L}$.

Lemma 4.3.3. *Fix a distribution $D \in J(\mathfrak{b}_1)$, a negative integer j, and $X \in \mathcal{N} \cap (\mathfrak{k}_j \smallsetminus \mathfrak{k}_{j+1})$. If*

$$D|_{C(\mathfrak{k}_{j+1}/\mathfrak{k}_1)} = 0,$$

then $D([X + \mathfrak{k}_1]) = 0$.

Lemma 4.3.4.

$$\dim \left(J(\mathfrak{b}_1)|_{C(\mathfrak{k}_0/\mathfrak{k}_1)} \right) \leq |\mathcal{O}(0)|\,.$$

We will also need the following corollary of Lemma 1.6.1.

Corollary 4.3.5.

$$^{G}\mathfrak{b}_1 \subset \mathfrak{k}_1 + \mathcal{N}.$$

Proof. In the notation of §3.2, let $x = x_1$ and $y = x_n$. If $r = 1/n$, then the result follows from Lemma 1.6.1. □

Proof of Theorem 4.3.2. Corollary 4.3.5 and Lemma 4.3.3 imply that $\dim(j_{\mathfrak{k}_1} J(\mathfrak{b}_1)) = \dim \left(J(\mathfrak{b}_1)|_{C(\mathfrak{k}_0/\mathfrak{k}_1)} \right)$. Since $j_{\mathfrak{k}_1} J(\mathcal{N}) \subset j_{\mathfrak{k}_1} J(\mathfrak{b}_1)$, the theorem follows from Lemma 4.3.4. □

Proof of Lemma 4.3.4. Fix $D \in J(\mathfrak{b}_1)$.

Since $\mathfrak{k}_0$ and $\mathfrak{k}_1$ are K_0-invariant, it follows from Corollary 4.3.5 that $D|_{C(\mathfrak{k}_0/\mathfrak{k}_1)}$ is determined by its values on the functions $[n+\mathfrak{k}_1]$ with $n \in \mathcal{N} \cap \mathfrak{k}_0$. Of course,

$$D([n+\mathfrak{k}_1]) = D([{}^k n + \mathfrak{k}_1])$$

for $n \in \mathcal{N} \cap \mathfrak{k}_0$ and $k \in K_0$. Therefore, the dimension of $J(\mathfrak{b}_1)|_{C(\mathfrak{k}_0/\mathfrak{k}_1)}$ is less than or equal to the number of nilpotent $\mathrm{GL}_n(\mathbb{F}_q)$-orbits in $M_n(\mathbb{F}_q)$. But the latter number is $|\mathcal{O}(0)|$. ☐

Fix j, D, and X as in the statement of Lemma 4.3.3. Before we begin the proof of this lemma, we need some additional notation and a simple result.

Let $P_\emptyset \leq G$ denote the standard Borel subgroup, i.e., the set of all invertible upper triangular matrices. Let $\mathfrak{u}$ denote the nilradical of the Lie algebra of $P_\emptyset$, i.e., the set of strictly upper triangular matrices in $M_n(F)$. Let $N_\emptyset$ denote the unipotent radical of $P_\emptyset$.

If $W \in \mathfrak{k}_0$, then $\overline{W}$ denotes the image of W in $M_n(\mathbb{F}_q) = \mathfrak{k}_0/\mathfrak{k}_1$. If $W \in \mathfrak{k}_j$, then we define

$$\overline{\mathrm{rank}}(W) = \mathrm{rank}_{\mathbb{F}_q}(\overline{\varpi^{-j}W}|_{\mathbb{F}_q^n}).$$

Let $m = \overline{\mathrm{rank}}(X)$. Note that, by hypothesis, $0 < m < n$. An element $Y \in \mathfrak{g}$ will be called *good* if

1. $Y \in \mathfrak{u} \cap \mathfrak{k}_j$,
2. $\overline{\mathrm{rank}}(Y) = m$, and
3. there exists a set $S_Y \subset \{2, 3, \dots n\}$ of size m such that if $k \notin S_Y$, then the k^{th} column of $\overline{\varpi^{-j}Y}$ is zero.

Definition 4.3.6. If Y is good, we will denote by $\delta(Y)$ the greatest element of the set S_Y.

Lemma 4.3.7. *Suppose that* $Y \in \mathfrak{u} \cap \mathfrak{k}_j$ *and* $\overline{\mathrm{rank}}(Y) = m > 0$. *There exists an* $n \in K_0 \cap N_\emptyset$ *such that* ${}^nY$ *is good.*

Proof. For $\alpha \in R$ and $s < t$, we will let $e_{st}(\alpha) \in N_\emptyset \cap K_0$ denote the matrix

$$\big(e_{st}(\alpha)\big)_{cd} = \begin{cases} 1 & \text{if } c = d \\ \alpha & \text{if } c = s \text{ and } d = t \\ 0 & \text{otherwise.} \end{cases}$$

Note that conjugating a matrix in $\mathfrak{g}$ by $e_{st}(\alpha)$

1. adds α times the t^{th} row to the s^{th} row, and
2. adds $-\alpha$ times the s^{th} column to the t^{th} column.

Since $\overline{\operatorname{rank}}(Y) = m$, the linear span of the columns of $\overline{\varpi^{-j}Y}$ has dimension m. Therefore, by conjugating Y by elements of the form $e_{st}(\alpha)$ with $s < t$ and $\alpha \in R$, we can obtain a good element. $\square$

Proof of Lemma 4.3.3. We begin with a warning about notation. Since $\mathfrak{k}_1$ is K_0-invariant, we have

$$D([X + \mathfrak{k}_1]) = D([{}^kX + \mathfrak{k}_1])$$

for all $k \in K_0$. Therefore, we will often ignore $\mathfrak{k}_1$ when conjugating by elements of K_0 and deal only with X.

Since G has an Iwasawa decomposition ($G = P_\emptyset K_0$), we may assume that $X \in \mathfrak{u}$. Since $X \in \mathfrak{u}$, we may assume that X is good from Lemma 4.3.7.

The proof is by induction on $\overline{\operatorname{rank}}(X)$. Here is the plan. We will produce a finite collection of $X_i \in \mathcal{N} \cap \mathfrak{k}_j$ such that

$$D([X + \mathfrak{k}_1]) = \sum_i c_i \cdot D([X_i + \mathfrak{k}_1])$$

for constants $c_i \in \mathbb{Q}$ and either

1. $\overline{\operatorname{rank}}(X_i) < m$ for all i or
2. X_i is good and $\delta(X_i) > \delta(X)$ for all i.

At the end of this proof, it will be clear that if $\delta(X) = n$, then the first outcome must occur. Therefore, repeating the steps below a finite number of times will produce a finite collection of $X_i \in \mathcal{N} \cap \mathfrak{k}_j$ and $c_i \in \mathbb{Q}$ such that

$$D([X + \mathfrak{k}_1]) = \sum_i c_i \cdot D([X_i + \mathfrak{k}_1])$$

and $\overline{\mathrm{rank}}(X_i) < m$ for all i. The lemma follows.

Step I. We have that

$$\ker(\overline{\varpi^{-j} X}|_{\mathbb{F}_q^n}) = \{(\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{F}_q^n \mid \xi_\beta = 0 \text{ if } \beta \in S_X\}.$$

Let L be the lift of this kernel in R^n, that is,

$$L = \{(\alpha_1, \alpha_2, \dots, \alpha_n) \in R^n | \alpha_i \in \wp \text{ if } i \in S_X\}.$$

Let

$$\mathfrak{C} = \{Z \in \mathfrak{k}_0 \mid Z \cdot L \subset \varpi \cdot R^n = \wp + \wp + \cdots + \wp \text{ and } Z \cdot R^n \subset L\}.$$

If $\mathfrak{c} \in \mathfrak{C}$, then

$$\mathfrak{c}_{rs} \in \begin{cases} \wp & \text{if } r \in S_X, \\ \wp & \text{if } r \notin S_X \text{ and } s \notin S_X, \textit{and} \\ R & \text{if } r \notin S_X \text{ and } s \in S_X. \end{cases}$$

From [12, pp. 97-98] we have the following lemma.

Lemma 4.3.8. *Choose $\mathfrak{c} \in \mathfrak{C}$. There exists a $Z \in \varpi^{-j}\mathfrak{k}_0$ such that*

$${}^{(1+Z)}X \equiv X + \mathfrak{c} \mod \mathfrak{k}_1.$$

From this lemma it follows that

$$D([X + \mathfrak{k}_1]) = \text{const} \cdot D([X + \mathfrak{C}]).$$

Step II. Let $a \in G$ be the diagonal matrix

$$a_{rs} = \begin{cases} 1 & \text{if } r = s \notin S_X, \\ \varpi^{-1} & \text{if } r = s \in S_X, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$^{a}(X + \mathfrak{C}) = \bigcup_{\alpha} (X' + \alpha + \mathfrak{k}_1)$$

where

$$X'_{cd} = \begin{cases} \varpi X_{cd} & \text{if } c \notin S_X \text{ and } d \in S_X, \\ \varpi^{-1} X_{cd} & \text{if } c \in S_X \text{ and } d \notin S_X, \text{ and} \\ X_{cd} & \text{otherwise,} \end{cases}$$

and $\alpha \in \mathfrak{k}_0$ with $\alpha_{cd} = 0$ unless $c \in S_X$ and $d \notin S_X$.

We now have

$$\begin{aligned} D([X + \mathfrak{k}_1]) &= \text{const} \cdot D([X + \mathfrak{C}]) \\ &= \text{const} \cdot \sum_{\alpha} D([X' + \alpha + \mathfrak{k}_1]) \end{aligned}$$

where the sum is over those α as above such that $X' + \alpha \in \mathfrak{k}_1 + \mathcal{N}$ (because the support of D is contained in $\mathfrak{k}_1 + \mathcal{N}$).

Step III. Note that $\overline{\text{rank}}(X') = \overline{\text{rank}}(X' + \alpha)$ and $\overline{\text{rank}}(X') \leq \overline{\text{rank}}(X) = m$ because $\overline{\varpi^{-j} X'}$ has at most m rows with nonzero entries. (If the i^{th} row of $\overline{\varpi^{-j} X'}$ has nonzero entries, then $i \in S_X$.) Moreover, we may assume that $X' + \alpha \in \mathfrak{k}_j \cap \mathcal{N}$. If $\overline{\text{rank}}(X') < m$, then we are done. If $\delta(X) = n$, then the bottom row of $\overline{\varpi^{-j} X'}$ has no nonzero entries and so $\overline{\text{rank}}(X') < m$.

Otherwise, let us assume that $\overline{\text{rank}}(X') = m$. Let P be the proper parabolic of G containing $P_{\emptyset}$ and having a Levi decomposition MN where $M \cong \text{GL}_{\delta(X)}(F) \times F^{\times} \times F^{\times} \times \cdots \times F^{\times}$ is embedded in G in the obvious way. (Since $\overline{\text{rank}}(X') = m$, we have that $\delta(X) < n$.) Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ be the corresponding Lie algebra.

Fix an α as in step (II) for which $W \equiv X' + \alpha \in \mathcal{N} + \mathfrak{k}_1$. We may assume that $W \in \mathfrak{p} \cap \mathfrak{k}_j \cap \mathcal{N}$ and if the i^{th} row of $\overline{\varpi^{-j} W}$ has nonzero entries, then $i \leq \delta(X)$.

We can write $W = W_{\mathfrak{m}} + W_{\mathfrak{n}}$ with $W_{\mathfrak{m}} \in \mathfrak{m}$ and $W_{\mathfrak{n}} \in \mathfrak{n}$. Since $W_{\mathfrak{m}}$ is nilpotent in $\mathfrak{m}$, there exists an $m \in \mathrm{GL}_{\delta(X)}(R) \times \{1\} \times \{1\} \times \cdots \times \{1\} \subset M$ such that ${}^{m}W \in \mathfrak{u}$. So without loss of generality, we will assume that $W \in \mathfrak{u} \cap \mathfrak{k}_j$ and if the i^{th} row of $\overline{\varpi^{-j} W}$ has nonzero entries, then $i \leq \delta(X)$.

From Lemma 4.3.7 there exists a $u \in K_0 \cap N_{\emptyset}$ such that ${}^{u}W$ is good. Since $\overline{\varpi^{-j} W_{\mathfrak{m}}}$ has at most $(m-1)$ nonzero rows, $\overline{\varpi^{-j} (\mathrm{Ad}(u) W)_{\mathfrak{m}}}$ has at most $(m-1)$ linearly independent columns. Consequently, $\delta({}^{u}W) > \delta(X)$. $\square$

CHAPTER 5

RESULTS FOR GL_ℓ

In this chapter, we will let $G = \mathrm{GL}_\ell(F)$ where ℓ is a prime. Moreover, we will assume that $p > \ell$.

5.1 Notation for GL_ℓ calculations

Let $\mathfrak{z} \cong F$ denote the center of $\mathfrak{g}$. Define $\mathfrak{z}_s = V_s \cap \mathfrak{z}$. (Note that $\mathfrak{z}_s = \mathfrak{g}_{y,s} \cap \mathfrak{z}$ for any $y \in \mathcal{B}$.) For $Y \in \mathfrak{g} \smallsetminus (\mathfrak{z} + \mathcal{N})$, let $n(Y)$ be the optimal number defined by $Y \in (\mathfrak{z} + V_{n(Y)}) \smallsetminus (\mathfrak{z} + V_{n(Y)+})$. For $Y \in \mathfrak{z} + \mathcal{N}$, we define $n(Y) = \infty$. Recall that $Y \in V_{m(Y)} \smallsetminus V_{m(Y)+}$, so $n(Y) = \max_{X \in \mathfrak{z}} m(Y + X)$.

Let $\mathfrak{h}$ be an elliptic Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ is isomorphic to an extension of F of degree ℓ which is either unramified or totally ramified. Let e denote the ramification degree of $\mathfrak{h}/F$. $\mathfrak{h}$ is G-conjugate to a Cartan subalgebra $\mathfrak{g}'$ where $\mathfrak{g}' = C_{\mathfrak{g}}(b)$ for some x_e-generic element b. (Since $p > \ell$, this is easy to see since $\mathcal{B}(\mathfrak{h}^\times)$, which is a single point (modulo the center), embeds in $\mathcal{B}(G)$.) So, without loss of generality, we will work exclusively with an elliptic Cartan subalgebra $\mathfrak{g}'$ which is obtained from an x_e-generic element b. Fix such a $\mathfrak{g}'$. We will call $\mathfrak{g}'$ (un)ramified if $\mathfrak{g}'$ is an (un)ramified extension of F. Finally, define G'_t, $x = x_e$, $\mathfrak{g}'_t$, etc. as in §3.2. Since ℓ is prime, we have either $\mathfrak{g}_{x,s} = \mathfrak{k}_{\lceil s \rceil}$ or $\mathfrak{g}_{x,s} = \mathfrak{b}_{\lceil \ell \cdot s \rceil}$.

Let $N = N_G(\mathfrak{g}')$. We have $NG_{x,0} = G'G_{x,0}$. Since $p > \ell$, we have that G' is a torus. Let $\mathcal{W} = \mathcal{W}(G', G') \cong N/G'$ denote the Weyl group of G'.

Let dg be the Haar measure on G normalized so that $\mathrm{meas}(G_x) = 1$. We assume that both dk and dk_1 denote the normalized Haar measure on G_x. Let dg^* be the G-invariant measure on G/Z normalized so that $\mathrm{meas}(ZG_x/Z) = 1$. Let dt be the Haar measure on G' normalized so that $\mathrm{meas}(G'_0) = 1/e$.

Let dW be a Haar measure on $\mathfrak{g}$. We will assume that dW is normalized so that $\mathrm{meas}(\mathfrak{g}_{x,t}) = \mathrm{meas}(G_{x,t})$ for all $t > 0$. Let dT be the additive Haar measure on $\mathfrak{g}'$ so that $\mathrm{meas}(\mathfrak{g}'_t) = \mathrm{meas}(G'_t)$ for all $t > 0$. Let $c_\Lambda(\mathfrak{g}) \in \mathbb{Q}$ be the constant so that $\hat{\hat{f}}(v) = f(-v)$ with respect to the measure $c_\Lambda(\mathfrak{g}) \cdot dW$. Define $c_\Lambda(\mathfrak{g}')$ similarly.

Since $p > \ell$, we can refine Lemma 3.2.2. We have the direct sum decomposition $\mathfrak{g} = \mathfrak{g}' + \mathfrak{g}^\perp$ where the perpendicular is taken with respect to the trace form. If $\mathfrak{g}^\perp_s = \mathfrak{g}_{x,s} \cap \mathfrak{g}^\perp$, then it follows from [1] that $\mathfrak{g}_{x,s} = \mathfrak{g}'_s + \mathfrak{g}^\perp_s$.

Lemma 5.1.1 (Adler). *Suppose* $m < t$ *and* $Y \in \big(\mathfrak{g}'_m \smallsetminus (\mathfrak{z}_m + \mathfrak{g}'_{m^+})\big) + \mathfrak{g}_{x,t}$*. Then*

1. *for any* $s \in \mathbb{R}$*, the map* $\mathrm{ad}(Y) \colon \mathfrak{g}^\perp_s \to \mathfrak{g}^\perp_{(s+m)}$ *is a bijection and*
2. *there exists a* $k \in G_{x,t-m}$ *such that* ${}^kY \in \mathfrak{g}'_m$*.*

Proof. This is [1, Lemma 2.3.1 and Lemma 2.3.2]. □

Suppose $X \in \mathfrak{g}'_{-t} \smallsetminus (\mathfrak{z}_{-t} + \mathfrak{g}'_{(-t)^+})$. If $m < t$, then for $Y \in \mathfrak{g}'_m \smallsetminus (\mathfrak{z}_m + \mathfrak{g}'_{m^+})$ and $\bar{\omega} \in \mathcal{W}$ we write

$$H(\omega, Y) = \sum_{\bar{W} \in \mathfrak{g}^\perp_s / \mathfrak{g}^\perp_{s^+}} \Lambda\big(\mathrm{tr}(X \cdot [{}^\omega Y, W] \cdot W)\big)$$

where $s = (t-m)/2$. Here $\omega \in N$ is a representative of $\bar{\omega}$. For elements W and V of $\mathfrak{g}^\perp$ define

$$Q(V,W) = Q_{(X,{}^\omega Y)}(V,W) = \Lambda\big((\mathrm{tr}([X,W] \cdot [V, {}^\omega Y]))/2\big).$$

Q is a nondegenerate, symmetric, bilinear form on $\mathfrak{g}^\perp$. Let dV be a Haar measure on $\mathfrak{g}^\perp$. A short computation shows that

$$H(\omega, Y) = \mathrm{meas}(\mathfrak{g}^\perp_{s^+})^{-1} \cdot \int_{\mathfrak{g}^\perp_{s^+}} dV\, Q(V,V).$$

From [37, §27],

$$\left|\mathfrak{g}^\perp_s / \mathfrak{g}^\perp_{s^+}\right|^{-1/2} \cdot H(\omega, Y) = \mathrm{meas}(\mathfrak{g}^\perp_s)^{-1/2} \cdot \mathrm{meas}(\mathfrak{g}_{x,s^+})^{-1/2} \cdot \int_{\mathfrak{g}^\perp_{s^+}} dV\, Q(V,V)$$

is an eighth root of unity, which we will denote by $\gamma_{(X,{}^\omega Y)}$.

5.2 Explicit calculation of $\widehat{\mu_X}$ for regular elliptic X

Suppose that X and Y are regular elliptic elements of $\mathfrak{g}$ and $X = X' + Z_1$ while $Y = Y' + Z_2$ with $Z_i \in \mathfrak{z}$. Then

$$\widehat{\mu_X}(Y) = \Lambda\big(\mathrm{tr}(X \cdot Z_2)\big) \cdot \Lambda\big(\mathrm{tr}(Z_1 \cdot Y')\big) \cdot \widehat{\mu_{X'}}(Y').$$

So, we will restrict our attention to $X \in \mathfrak{g}'_{-r} \smallsetminus (\mathfrak{z}_{-r} + \mathfrak{g}'_{(-r)+})$ and $Y \in V_{n(Y)} \smallsetminus (\mathfrak{z}_{n(Y)} + V_{n(Y)+})$ throughout this section. In particular, this means that X is x-generic, $\mathfrak{g}' = C_{\mathfrak{g}}(X)$, and $n(Y) = m(Y)$.

In this section, we will prove the following theorem.

Theorem 5.2.1. *Suppose that $X \in \mathfrak{g}'_{-r} \smallsetminus (\mathfrak{z}_{-r} + \mathfrak{g}'_{(-r)+})$ and $Y \in V_{n(Y)} \smallsetminus (\mathfrak{z}_{n(Y)} + V_{n(Y)+})$ are regular elements of $\mathfrak{g}$. Let*

$$C = c_\Lambda(\mathfrak{g}') \cdot c_\Lambda^{-1}(\mathfrak{g}) \cdot |\eta(X)|^{-1/2} \cdot |\eta(Y)|^{-1/2}$$

and $C' = C \cdot \left|\mathfrak{g}_0^\perp / \mathfrak{g}_{0+}^\perp\right|^{-1/2}$. Then

$$\widehat{\mu_X}(Y) = \begin{cases} C \cdot \displaystyle\sum_{\bar{\omega} \in \mathcal{W}} \Lambda\big(\mathrm{tr}(X \cdot {}^{\omega}T)\big) \cdot \gamma_{(X, {}^{\omega}T)} & \text{if } n(Y) < r \text{ and } Y = {}^{g}T \text{ with} \\ & \quad g \in G \text{ and } T \in \mathfrak{g}'_{n(Y)} \smallsetminus \mathfrak{g}'_{n(Y)+}, \\ C' \cdot \displaystyle\sum_{\bar{k} \in G_x / M_0 G_{x,0+}} \Lambda\big(\mathrm{tr}(X \cdot {}^{k}T)\big) & \text{if } n(Y) = r \text{ and } Y = {}^{g}T \text{ with} \\ & \quad g \in G,\ T \text{ is } x\text{-generic}, \\ & \quad \text{and } T \in \mathfrak{g}_{x,r}, \\ \text{conjectured range of local expansion} & \text{if } n(Y) > r, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

where M is the centralizer of T in G and $M_0 = M \cap G_x$.

Remark 5.2.2. We know more than Theorem 5.2.1 suggests about the local expansion.

Suppose $\mathfrak{g}'$ is unramified. Then r is an integer and $n(Y) > r$ if and only if $Y \in {}^{G}(\varpi^r \mathfrak{b}_1)$. If $n(Y) > r$, it follows from [34] that $\widehat{\mu_X}(Y)$ is given by its local expansion. The constants $c_{\mathcal{O}}(X)$ can be found in [22].

Suppose $\mathfrak{g}'$ is ramified. If $r = k - 1/\ell$ for some integer k, then the local expansion holds for $n(Y) > r$ (see §4.3), and with the same r, the constants $c_{\mathcal{O}}(X)$ can be found in [22].

Remark 5.2.3. Suppose that X and Y are regular elements of $\mathfrak{g}$ as in the statement of Theorem 5.2.1. In [36] Waldspurger derives a formula for the function

$$\hat{i}^{\,G}_{G}(X,Y) \equiv c_\Lambda(\mathfrak{g}) \cdot |\eta(X)|^{1/2} \cdot |\eta(Y)|^{1/2} \cdot \widehat{\mu_X}(Y)$$

when Y is far from zero. In our situation, the result says that for Y far enough away from zero, $\hat{i}^{\,G}_{G}(X,Y)$ is zero unless the G-orbit of Y intersects $\mathfrak{g}'$; if $Y \in \mathfrak{g}'$, then

$$\hat{i}^{\,G}_{G}(X,Y) = c_\Lambda(\mathfrak{g}') \cdot \sum_{\bar{\omega} \in \mathcal{W}} \gamma_{(X,{}^{\omega}Y)} \cdot \Lambda\big(\operatorname{tr}(X \cdot {}^{\omega}Y)\big).$$

This formula is exactly the formula occurring in Theorem 5.2.1 for $Y \notin \mathfrak{z} + V_r$.

Vanishing results

This section is an adaptation of the material in [14].

Lemma 5.2.4. *Suppose that Y is regular and $Y \in V_{n(Y)} \smallsetminus (\mathfrak{z}_{n(Y)} + V_{n(Y)^+})$. If $n(Y) \leq r$, ${}^{G}Y \cap \mathfrak{g}_{x,r} = \emptyset$, and ${}^{G}Y \cap \mathfrak{g}' = \emptyset$, then $\widehat{\mu_X}(Y) = 0$.*

This lemma follows from the following two lemmas. Fix $g \in G$. We will assume that $W = {}^{g}Y \in \mathfrak{g}_{x,m} \smallsetminus (\mathfrak{z}_m + \mathfrak{g}_{x,m^+})$ for some $m < r$.

Lemma 5.2.5. *If ${}^{G_x}W \cap (\mathfrak{g}'_m + \mathfrak{g}_{x,(m+r)/2}) = \emptyset$, then*

$$\int_{G_x} dk\, \Lambda\big(\operatorname{tr}(X \cdot {}^{k}W)\big) = 0.$$

Proof. Let $s = (r-m)/2$. Then

$$\int_{G_x} dk\, \Lambda\big(\operatorname{tr}(X \cdot {}^{k}W)\big) = \text{const} \cdot \int_{G_x} dk \int_{G_{x,s^+}} dk_1\, \Lambda\big(\operatorname{tr}(X \cdot {}^{k_1 k}W)\big).$$

We will show that the inner integral is zero. Fix $k \in G_x$. Then we can write

$${}^{k}W = W' + W^{\perp}$$

with $W' \in \mathfrak{g}'_m$ and $W^\perp \in \mathfrak{g}^\perp_m \smallsetminus \mathfrak{g}^\perp_{(m+r)/2}$. As in the proof of Lemma 3.5.3, the inner integral becomes

$$\begin{aligned}\int_{\mathfrak{g}_{x,s^+}} dV\, \Lambda\big(\mathrm{tr}(X \cdot ({}^kW + [V, {}^kW]))\big)\\ &= \Lambda\big(\mathrm{tr}(X \cdot {}^kW)\big) \cdot \int_{\mathfrak{g}_{x,s^+}} dV\, \Lambda\big(\mathrm{tr}(X \cdot [V, {}^kW])\big)\\ &= \Lambda\big(\mathrm{tr}(X \cdot {}^kW)\big) \cdot \int_{\mathfrak{g}_{x,s^+}} dV\, \Lambda\big(\mathrm{tr}([W^\perp, X] \cdot V)\big).\end{aligned}$$

From Lemma 3.2.2, $[W^\perp, X] \in \mathfrak{g}_{x,m-r} \smallsetminus \mathfrak{g}_{x,-s}$ and so the final integral is zero. □

Lemma 5.2.6. *If* ${}^{G_x}W \cap \mathfrak{g}' = \emptyset$, *then*

$$\int_{G_x} dk\, \Lambda\big(\mathrm{tr}(X \cdot {}^kW)\big) = 0.$$

Proof. By Lemma 5.2.5, the integral under consideration vanishes if

$${}^{G_x}W \cap (\mathfrak{g}'_m + \mathfrak{g}_{x,(m+r)/2}) = \emptyset.$$

Since $\mathfrak{g}_{x,m} \smallsetminus (\mathfrak{g}_{x,m^+} + \mathfrak{z}_m)$ is G_x-stable, it will be enough to show that for $V \in \mathfrak{g}_{x,m} \smallsetminus (\mathfrak{g}_{x,m^+} + \mathfrak{z}_m)$ if

$$V \in \mathfrak{g}'_m + \mathfrak{g}_{x,(m+r)/2},$$

then

$${}^{G_x}V \cap \mathfrak{g}' \neq \emptyset.$$

If $V \in \mathfrak{g}'_m + \mathfrak{g}_{x,(m+r)/2}$, then

$$V \in (\mathfrak{g}'_m + \mathfrak{g}_{x,(m+r)/2}) \smallsetminus (\mathfrak{z}_m + \mathfrak{g}'_{m^+} + \mathfrak{g}_{x,(m+r)/2})$$

and so

$$V \in \big(\mathfrak{g}'_m \smallsetminus (\mathfrak{z}_m + \mathfrak{g}'_{m^+})\big) + \mathfrak{g}_{x,(m+r)/2}.$$

(Note that without any tameness restrictions, this implies that V must be an x-generic element of $\mathfrak{g}$ whose centralizer has the same ramification degree as $\mathfrak{g}'/F$.) With tameness restrictions, it follows from Lemma 5.1.1 that there is an element $k \in G_{x,(r-m)/2}$ such that ${}^kV \cap \mathfrak{g}' \neq \emptyset$. □

Lemma 5.2.4 follows immediately.

Lemma 5.2.7. *Suppose that* $\mathfrak{g}'$ *is an unramified extension of* F*,* $n(Y) = r$*, and* ${}^{G}Y \cap \mathfrak{g}_{x,r} \neq \emptyset$*. If* ${}^{G}Y \cap \mathfrak{g}' = \emptyset$*, then* $\widehat{\mu_X}(Y) = 0$*.*

Proof. From Lemma 5.2.6 it will be enough to assume that $Y \in \mathfrak{g}_{x,r}$ and show that

$$\int_{G_x} dk\, \Lambda\big(\mathrm{tr}(X \cdot {}^{k}Y)\big) = 0.$$

Since $n(Y) = r$, it follows that the coset $Y + \mathfrak{g}_{x,r+}$ contains no nilpotent elements. The lemma now follows from the proof of [14, Lemma 6.6]. $\square$

$\widehat{\mu_X}$ on $\mathfrak{g}' \smallsetminus \mathfrak{g}'_r$

Lemma 5.2.8. *Suppose that* $Y \in \mathfrak{g}'_m \smallsetminus (\mathfrak{z}_m + \mathfrak{g}'_{m+})$ *and* $m < r$*. Let* $s = (r - m)/2$*. Then*

$$\widehat{\mu_X}(Y) = C \cdot \sum_{\bar{\omega} \in \mathcal{W}} \Lambda\big(\mathrm{tr}(X \cdot^{\omega} Y)\big) \cdot H(\omega, Y)$$

where

$$C = e \cdot |G'_0/G'_{s+}| \cdot \mathrm{meas}(\mathfrak{g}_{x,s+}).$$

Proof.

$$\widehat{\mu_X}(Y) = \int_{G/Z} dg^* \int_{G_x} dk \int_{G_x} dk_1\, \Lambda\big(\mathrm{tr}(X \cdot {}^{k_1 g k}Y)\big).$$

Fix g and k. Since $n({}^{gk}Y) = n(Y) < r$, from Lemma 5.2.6 the inner integral is zero unless the G_x orbit of ${}^{gk}Y$ intersects $\mathfrak{g}'$ nontrivially. Suppose that ${}^{k_1 gk}Y \in \mathfrak{g}'$. Since Y is regular, all of its conjugates are also regular and so $k_1 gk$ must be an element of

$N = N_G(\mathfrak{g}')$. Consequently, g is an element of $G_x N G_x$. Since G_x is normal in NG_x, we have

$$\begin{aligned}
\widehat{\mu_X}(Y) &= \int_{NG_x/Z} dg^* \int_{G_x} dk \int_{G_x} dk_1\, \Lambda\big(\mathrm{tr}(X \cdot {}^{k_1 g k}Y)\big) \\
&= \sum_{\bar{g} \in NG_x/ZG_x} \int_{G_x} dk \int_{G_x} dk_1\, \Lambda\big(\mathrm{tr}(X \cdot {}^{k_1 g k}Y)\big) \\
&\qquad (\text{since } NG_x = G'G_x) \\
&= [G'G_x \colon ZG_x] \cdot \int_{G_x} dk \int_{G_x} dk_1\, \Lambda\big(\mathrm{tr}(X \cdot {}^{k_1 k}Y)\big) \\
&= e \cdot \int_{G_x} dk\, \Lambda\big(\mathrm{tr}(X \cdot {}^{k}Y)\big).
\end{aligned}$$

So we only need to compute

$$\int_{G_x} dk\, \Lambda\big(\mathrm{tr}(X \cdot {}^{k}Y)\big). \tag{5.2.9}$$

If $c_1 = \mathrm{meas}(\mathfrak{g}_{x,s^+})$, then (5.2.9) can be rewritten as

$$c_1^{-1} \cdot \int_{G_x} dk \int_{G_{x,s^+}} dk_1\, \Lambda\big(\mathrm{tr}(X \cdot {}^{k_1 k}Y)\big).$$

For fixed $k \in G_x$, the inner integral is zero unless ${}^{k}Y \in \mathfrak{g}'_m + \mathfrak{g}_{x,(m+r)/2}$ (see the proof of Lemma 5.2.5). So from Lemma 5.1.1, there exists a $b \in G_{x,s}$ such that ${}^{bk}Y \in \mathfrak{g}'_m$. This implies, as before, that k must be an element of $NG_{x,s} \cap G_x$. Consequently, since we can choose representatives of $\mathcal{W} \cong N/G'$ in G_x, we have

$$\begin{aligned}
(5.2.9) &= c_1^{-1} \cdot \sum_{\bar{\omega} \in \mathcal{W}} \int_{G'_0 G_{x,s}} dk \int_{G_{x,s^+}} dk_1\, \Lambda\big(\mathrm{tr}(X \cdot {}^{k_1 k \omega}Y)\big) \\
&= c_1^{-1} \cdot c_2 \cdot \sum_{\bar{\omega} \in \mathcal{W}} \int_{\mathfrak{g}_{x,s}} dW \int_{\mathfrak{g}_{x,s^+}} dV\, \Lambda\big(\mathrm{tr}(X \cdot {}^{(1+V)(1+W)\omega}Y)\big)
\end{aligned}$$

where $c_2 = |G'_0/G'_s|$.

Note that

$${}^{(1+V)(1+W)\omega}Y \equiv {}^{\omega}Y + [W, {}^{\omega}Y] + [{}^{\omega}Y, W] \cdot W + [V, {}^{\omega}Y] \mod \mathfrak{g}_{x,r^+}. \tag{5.2.10}$$

Therefore, since $[X, {}^{\omega}Y] = 0$, the inner integral above becomes

$$c_1 \cdot \Lambda\big(\mathrm{tr}(X \cdot {}^{\omega}Y)\big) \cdot \Lambda\big(\mathrm{tr}(X \cdot [{}^{\omega}Y, W] \cdot W)\big).$$

And so,

$$\begin{aligned}(5.2.9) &= c_2 \cdot c_1 \cdot \sum_{\bar{\omega} \in \mathcal{W}} \sum_{\bar{W} \in \mathfrak{g}_{x,s}/\mathfrak{g}_{x,s^+}} \Lambda\big(\mathrm{tr}(X \cdot {}^{\omega}Y)\big) \cdot \Lambda\big(\mathrm{tr}(X \cdot [{}^{\omega}Y, W] \cdot W)\big) \\ &= |G'_0/G'_{s^+}| \cdot c_1 \cdot \sum_{\bar{\omega} \in \mathcal{W}} \Lambda\big(\mathrm{tr}(X \cdot {}^{\omega}Y)\big) \sum_{\bar{W} \in \mathfrak{g}_s^{\perp}/\mathfrak{g}_{s^+}^{\perp}} \Lambda\big(\mathrm{tr}(X \cdot [{}^{\omega}Y, W] \cdot W)\big) \\ &= |G'_0/G'_{s^+}| \cdot c_1 \cdot \sum_{\bar{\omega} \in \mathcal{W}} \Lambda\big(\mathrm{tr}(X \cdot {}^{\omega}Y)\big) \cdot H(\omega, Y). \quad \square\end{aligned}$$

$\widehat{\mu_X}$ on $\mathfrak{g}_{x,r} \smallsetminus \mathfrak{g}_{x,r^+}$

Remark 5.2.11. Suppose that $Y \in \mathfrak{g}_{x,r} \smallsetminus \mathfrak{g}_{x,r^+}$ is x-generic. Then $M = F[Y]^{\times}$ normalizes $\mathfrak{g}_{x,t}$ and $G_{x,t}$ for all appropriate t and the natural filtration on M agrees with the filtration $G_{x,t}$, so it makes sense to define $M_0 = M \cap G_x$.

Lemma 5.2.12. *Suppose that $\mathfrak{g}'$ is a ramified extension of F and $Y \in \mathfrak{g}_{x,t} \smallsetminus (\mathfrak{z}_t + \mathfrak{g}_{x,t^+})$. Then $n(Y) = t$ if and only if Y is x-generic.*

Proof. If Y is x-generic, then from Example 3.2.3 the coset $Y + \mathfrak{g}_{x,t^+}$ can contain no nilpotent elements. Therefore, $Y \notin \mathfrak{z}_t + V_{t^+}$.

If $n(Y) = t$, then from Proposition 1.5.1 the coset $Y + \mathfrak{g}_{x,t^+}$ contains no nilpotent elements. It follows from Example 3.2.3 that Y is x-generic. $\square$

Lemma 5.2.13. *If $Y \in \mathfrak{g}_{x,r} \smallsetminus \mathfrak{g}_{x,r^+}$ and Y is x-generic, then*

$$\widehat{\mu_X}(Y) = C \cdot e \cdot \sum_{\bar{k} \in G_x/M_0 G_{x,0^+}} \Lambda\big(\mathrm{tr}(X \cdot {}^{k}Y)\big)$$

where M is the centralizer of Y in G and $C = \mathrm{meas}(G_{x,0^+}) \cdot \big|G'_0/G'_{0^+}\big|$.

Proof.

$$\widehat{\mu_X}(Y) = \int_{G/Z} dg^* \int_{G_x} dk \int_{G_x} dk_1 \, \Lambda\big(\mathrm{tr}(X \cdot {}^{k_1 gk}Y)\big).$$

If $W = {}^{gk}Y \notin \mathfrak{g}_{x,r}$, then $W \in \mathfrak{g}_{x,m} \smallsetminus \mathfrak{g}_{x,m^+}$ for some $m < r$. But Corollary 1.6.2 implies that $W \notin \mathfrak{g}'_m + \mathfrak{g}_{x,m^+}$. Therefore, from Lemma 5.2.5, the inner integral is zero unless ${}^{gk}Y \in \mathfrak{g}_{x,r}$.

If ${}^{gk}Y \in \mathfrak{g}_{x,r^+}$, then $Y \in \mathfrak{g}_{x,r^+} + \mathcal{N}$, which contradicts the fact that Y is x-generic.

For $\mathfrak{g}'$ unramified, if ${}^{gk}Y \in \mathfrak{g}_{x,r} \smallsetminus \mathfrak{g}_{x,r^+}$ is not x-generic, then Lemma 5.2.7 implies that the inner integral is zero. For $\mathfrak{g}'$ ramified, if ${}^{gk}Y \in \mathfrak{g}_{x,r} \smallsetminus \mathfrak{g}_{x,r^+}$, then since $n({}^{gk}Y) = r$, it follows from Lemma 5.2.12 that Y is x-generic.

So the only pairs (g, k) which can contribute are those for which

$${}^{gk}Y \in \mathfrak{g}_{x,r} \smallsetminus \mathfrak{g}_{x,r^+}$$

and ${}^{gk}Y$ is x-generic. Therefore, $g \in NG_x$ from [2, Propositon 3.4]. Thus, as in previous proofs, we have

$$\begin{aligned}\widehat{\mu_X}(Y) &= \int_{NG_x/Z} dg^* \int_{G_x} dk \int_{G_x} dk_1 \, \Lambda\big(\mathrm{tr}(X \cdot {}^{k_1 gk}Y)\big)\\ &= e \cdot \int_{G_x} dk \, \Lambda\big(\mathrm{tr}(X \cdot {}^{k}Y)\big).\end{aligned}$$

Since $\left|M_0/M_{0^+}\right| = \left|G'_0/G'_{0^+}\right|$, the lemma follows. ☐

Proof of Theorem 5.2.1

Lemma 5.2.14. *Suppose X and Y are as in the statement of Theorem 5.2.1. Let $s = \big(r - n(Y)\big)/2$ and $C = e \cdot \big|G'_0/G'_{s^+}\big| \cdot \operatorname{meas}(\mathfrak{g}_{x,s^+})$.*

$$\widehat{\mu_X}(Y) = \begin{cases} C \cdot \displaystyle\sum_{\bar\omega \in \mathcal{W}} \Lambda\big(\operatorname{tr}(X \cdot {}^{\omega}T)\big) \cdot H(\omega, T) & \text{if } n(Y) < r \text{ and } Y = {}^{g}T \text{ with} \\ & \quad g \in G \text{ and } T \in \mathfrak{g}'_{n(Y)} \smallsetminus \mathfrak{g}'_{n(Y)^+}, \\ C \cdot \displaystyle\sum_{\bar k \in G_x/M_0 G_{x,0^+}} \Lambda\big(\operatorname{tr}(X \cdot {}^{k}T)\big) & \text{if } n(Y) = r \text{ and } Y = {}^{g}T \text{ with} \\ & \quad g \in G,\ T \text{ is } x\text{-generic}, \\ & \quad \text{and } T \in \mathfrak{g}_{x,r}, \\ \text{conjectured range of local expansion} & \text{if } n(Y) > r, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

where M is the centralizer of T in G.

Proof. To complete the proof, we need to check that if

1. $n(Y) = r$ and ${}^{G}Y$ does not intersect the x-generic elements of $\mathfrak{g}_{x,r} \smallsetminus \mathfrak{g}_{x,r^+}$ or
2. $n(Y) < r$ and ${}^{G}Y \cap \mathfrak{g}' = \emptyset$,

then $\widehat{\mu_X}(Y) = 0$.

Lemma 5.2.4 covers the second situation.

Let us now consider the first case. If $\mathfrak{g}'$ is unramified, then Lemma 5.2.7 implies that $\widehat{\mu_X}(Y) = 0$ unless ${}^{G}Y \cap \mathfrak{g}_{x,r} = \emptyset$. However, if ${}^{G}Y \cap \mathfrak{g}_{x,r} = \emptyset$, then ${}^{G}Y \cap \mathfrak{g}' = \emptyset$ because $n(Y) = r$. If $\mathfrak{g}'$ is ramified, then Lemma 5.2.12 implies that ${}^{G}Y \cap \mathfrak{g}_{x,r} = \emptyset$. This implies that ${}^{G}Y \cap \mathfrak{g}' = \emptyset$. In both situations, we must now show that if $n(Y) = r$ and ${}^{G}Y \cap \mathfrak{g}' = \emptyset$, then $\widehat{\mu_X}(Y) = 0$. This is covered by Lemma 5.2.4. □

We now finish this section with the proof of Theorem 5.2.1.

Proof of Theorem 5.2.1. We begin by collecting some facts. It is not too difficult to show that

$$|\eta(X)|^{-1/2} \cdot |\eta(Y)|^{-1/2} = \frac{\left|\mathfrak{g}_0^\perp \,/\mathfrak{g}_{0+}^\perp\,\right|^{1/2} \cdot \left|\mathfrak{g}_s^\perp \,/\mathfrak{g}_{s+}^\perp\,\right|^{1/2}}{\left|\mathfrak{g}_0^\perp \,/\mathfrak{g}_{s+}^\perp\,\right|}.$$

We also have

$$c_\Lambda^{-1}(\mathfrak{g}) = \operatorname{meas}(\mathfrak{g}_{x,0+}) \cdot |\mathfrak{g}_{x,0}/\mathfrak{g}_{x,0+}|^{1/2}$$

and

$$c_\Lambda^{-1}(\mathfrak{g}') = \operatorname{meas}(\mathfrak{g}'_{0+}) \cdot |\mathfrak{g}'_0/\mathfrak{g}'_{0+}|^{1/2}.$$

We will verify that when $n(Y) = r$ and $\widehat{\mu_X}(Y) \neq 0$, the constant C' occurring in Lemma 5.2.14 is

$$c_\Lambda(\mathfrak{g}') \cdot c_\Lambda^{-1}(\mathfrak{g}) \cdot \left|\eta(X)\right|^{-1/2} \cdot \left|\eta(Y)\right|^{-1/2} \cdot \left|\mathfrak{g}_0^\perp/\mathfrak{g}_{0+}^\perp\right|^{-1/2}.$$

The remaining cases are similar; however, one must be careful when converting from $H(\omega, Y)$ to $\gamma_{(X,{}^\omega Y)}$.

Since $n(Y) = r$, we may ignore the term $|\eta(X)|^{-1/2} \cdot |\eta(Y)|^{-1/2}$. Therefore,

$$\begin{aligned} C' &= e \cdot |G'_0/G'_{0+}| \cdot \operatorname{meas}(\mathfrak{g}_{x,0+}) \\ &= \frac{\operatorname{meas}(\mathfrak{g}_{x,0+})}{\operatorname{meas}(\mathfrak{g}'_{0+})} \\ &= c_\Lambda(\mathfrak{g}') \cdot c_\Lambda^{-1}(\mathfrak{g}) \cdot \left|\mathfrak{g}_0^\perp \,/\mathfrak{g}_{0+}^\perp\,\right|^{-1/2}. \quad \square \end{aligned}$$

5.3 Explicit positive depth character values

Let U denote the set of unipotent elements in G and recall that Z denotes the center of G. For $s > 0$, define $U_s = 1 + V_s$. Note that $U = \bigcap_{s>0} U_s$. Define $U_{s+} = \bigcup_{t>s} U_t$.

Let γ be an element of G. If $\gamma \in ZU_{0+} \smallsetminus ZU$, then define the optimal number $n(\gamma) > 0$ by $\gamma \in ZU_{n(\gamma)} \smallsetminus ZU_{n(\gamma)+}$. If $\gamma \in ZU$, let $n(\gamma) = \infty$. If $\gamma \notin ZU_{0+}$, let $n(\gamma) = 0$.

Before presenting a character table for the representations constructed in §3.2, we need some information about the character of the inducing representation σ. In the tame case ($p > \ell$) the character of the inducing representation σ has been calculated.

If $\mathfrak{g}_{x,r/2} \neq \mathfrak{g}_{x,(r/2)^+}$, the representation σ is obtained by extending a representation r_χ of $ZG'_{0+}G_{x,(r/2)^+}$ and these extensions are indexed by characters of $G'G_{x,r/2}/ZG'_{0+}G_{x,r/2}$ (see §3.2). In this situation we describe the character of only one such extension; the characters of the remaining extensions are easy to determine from this.

Lemma 5.3.1. *The character of σ has the following form. Suppose that $\gamma \in G'G_{x,r/2}$, then*

$$\chi_\sigma(\gamma) = \begin{cases} \lambda(\sigma) \cdot \phi(\gamma) & \textit{if } \gamma \in G'G_{x,(r/2)^+} \smallsetminus ZG'_{0+}G_{x,(r/2)^+}, \\ \deg(\sigma) \cdot \phi(\gamma) & \textit{if } \gamma \in ZG'_{0+}G_{x,(r/2)^+}, \textit{ and} \\ 0 & \textit{if } \gamma \textit{ is not } G'G_{x,r/2}\textit{-conjugate to an element of } G'G_{x,(r/2)^+} \end{cases}$$

where $\lambda(\sigma) = 1$ if $\mathfrak{g}_{x,r/2} = \mathfrak{g}_{x,r/2^+}$. If $\mathfrak{g}_{x,r/2} \neq \mathfrak{g}_{x,r/2^+}$, then

$$\lambda(\sigma) = \begin{cases} (-1)^{\ell-1} & \textit{if } \mathfrak{g}' \textit{ is an unramified extension of } F, \textit{ and} \\ (\frac{q}{\ell}) & \textit{if } \mathfrak{g}' \textit{ is a ramified extension of } F. \end{cases}$$

($(\frac{q}{\ell})$ is the Legendre symbol, so $(\frac{q}{\ell})$ is 1 if q is a square mod ℓ and -1 otherwise.)

Proof. If $\mathfrak{g}_{x,r/2} = \mathfrak{g}_{x,r/2^+}$, there is nothing to prove. Otherwise, this follows from [15, Lemma 3.5.36]. ☐

Theorem 5.3.2. *Suppose that $\pi = \pi_\sigma$ is a supercuspidal representation of $\mathrm{GL}_\ell(F)$ of depth $\rho(\pi) > 0$ as described in §3.2. Suppose that $\gamma \in G^{\mathrm{reg}}$. Let*

$$C = c_\Lambda(\mathfrak{g}') \cdot c_\Lambda^{-1}(\mathfrak{g}) \cdot |D(\gamma)|^{-1/2} \cdot |\eta(X_\pi)|^{-1/2}.$$

Then

$$\frac{\Theta_\pi(\gamma)}{\deg(\pi)} = \begin{cases} C \cdot \lambda(\sigma) \cdot \sum_{\bar\omega \in \mathcal{W}} \phi({}^{\omega}t) & \textit{if } n(\gamma) = 0 \textit{ and } \gamma = {}^{g}t \textit{ with} \\ & \qquad g \in G \textit{ and } t \in G', \\ C \cdot \sum_{\bar\omega \in \mathcal{W}} \phi({}^{\omega}t) \cdot \gamma_{(X_\pi, {}^{\omega}Y)} & \textit{if } 0 < n(\gamma) \le r/2 \textit{ and } \gamma = {}^{g}t \textit{ where} \\ & \qquad t = z \cdot (1+Y) \textit{ with } z \in Z,\ g \in G, \\ & \qquad \textit{and } Y \in \mathfrak{g}'_{n(\gamma)} \smallsetminus (\mathfrak{z}_{n(\gamma)} + \mathfrak{g}'_{n(\gamma)^+}), \\ \phi(z) \cdot \widehat{\mu_{X_\pi}}(Y) & \textit{if } n(\gamma) > r/2 \textit{ and } \gamma = z \cdot (1 + {}^{g}Y) \\ & \qquad \textit{with } g \in G,\ z \in Z, \\ & \qquad \textit{and } Y \in V_{n(\gamma)}, \textit{ and} \\ 0 & \textit{otherwise.} \end{cases}$$

We break the proof into several pieces.

Some vanishing results for Θ_π

If $\mathfrak{g}'$ is a ramified extension of F, something nearly identical to the lemmas in this section can be found in [14]. Our tameness restrictions allow for a more precise vanishing result than the one found there.

Lemma 5.3.3. *Suppose that*

1. $0 < m < r$,
2. $\gamma \in G_{x,m} \smallsetminus G_{x,m^+}$, *and*
3. ${}^{G_x}\gamma \cap G'_m G_{x,(m+r)/2} = \emptyset$.

Then

$$\int_{G_x} dk\, \dot\chi_\sigma({}^{k}\gamma) = 0.$$

Proof. Let $s = (r-m)/2$. Then

$$\int_{G_x} dk\, \dot{\chi}_\sigma({}^k\gamma) = \text{const} \cdot \int_{G_x} dk \int_{G_{x,s^+}} dk_1\, \dot{\chi}_\sigma({}^{k_1k}\gamma).$$

We want to show that the inner integral is zero. In order to do this, we first analyze ${}^{k_1k}\gamma$. We then reduce the integral to something familiar.

Fix $k \in G_x$ and write ${}^k\gamma$ as th where $t = 1+T$ with $T \in \mathfrak{g}'_u \smallsetminus \mathfrak{g}'_{u^+}$ and $h = 1+H$ with $H \in \mathfrak{g}_v^\perp \smallsetminus \mathfrak{g}_{v^+}^\perp$. Note that $u \geq m$ and $v \geq m$ and at least one of u or v must equal m.

We first claim that we may assume that $v \geq r/2$. Suppose that $v < r/2$. Since $th \notin G_{x,m^+}$, we have $m < r/2$. For $b \in G_{x,s^+}$, we have ${}^b(th) \equiv th \bmod G_{x,(r/2)^+}$ and so $\dot{\chi}_\sigma({}^b(th)) = 0$ since ${}^b(th) \notin G'G_{x,r/2}$.

Without loss of generality we may assume that $T \in \mathfrak{g}'_m$ and $H \in \mathfrak{g}^\perp_{(r/2)^+} \cap \mathfrak{g}^\perp_m$. We then have that

$$\begin{aligned}\chi_\sigma({}^{k_1}(th)) &= \dim(\sigma) \cdot \phi({}^{k_1}t\,{}^{k_1}h) \\ &= \dim(\sigma) \cdot \phi(th) \cdot \phi(k_1tk_1^{-1}t^{-1}) \cdot \phi(k_1hk_1^{-1}h^{-1}).\end{aligned}$$

We write $k_1 = 1+K$, $k_1^{-1} = 1+K'$, and $t^{-1} = 1+T'$ with $K, K' \in \mathfrak{g}_{x,s^+}$ and $T' \in \mathfrak{g}'_m$. Then it is easy to check that $K+K'+KK' = 0$ and $T+T'+TT' = 0$. Since $tk_1t^{-1}k_1^{-1} \in G_{x,(r/2)^+}$, one can show that

$$\phi(k_1tk_1^{-1}t^{-1}) = \Lambda\big(\text{tr}(X_\pi \cdot (k_1tk_1^{-1}t^{-1} - 1))\big) = 1.$$

Therefore

$$\begin{aligned}\chi_\sigma({}^{k_1}(th)) &= \text{const} \cdot \Lambda\big(\text{tr}(X_\pi \cdot (k_1hk_1^{-1}h^{-1} - 1))\big) \\ &= \text{const} \cdot \Lambda\big(\text{tr}(X_\pi \cdot [K,H])\big).\end{aligned}$$

Arguing as at the end of the proof of Lemma 5.2.5 produces the desired result. ☐

Lemma 5.3.4. *Suppose that*

1. $0 < m < r$,

2. $\gamma \in G_{x,m} \smallsetminus (Z \cap G_{x,m})G_{x,m^+}$, *and*

3. ${}^{G_x}\gamma \cap G' = \emptyset$.

Then

$$\int_{G_x} dk\, \dot\chi_\sigma({}^k\gamma) = 0.$$

Proof. From Lemma 5.3.3 we need only show that if γ is an element of

$$(G'_m \smallsetminus (Z \cap G'_m)G'_{m+})\, G_{x,(m+r)2}$$

then

$${}^{G_x}\gamma \cap G' \neq \emptyset.$$

The lemma follows from the proof of Lemma 5.2.6, and the fact that the bijective map $X \mapsto 1 + X$ from $\mathfrak{g}_{x,0+}$ to $G_{x,0+}$ preserves levels and takes elements of $\mathfrak{g}'$ to elements of G'. $\square$

Lemma 5.3.5. *Suppose that γ is regular and ${}^{G_x}\gamma \subset G \smallsetminus (U_{0^+}Z)$. If the G_x-orbit of γ does not intersect G', then*

$$\int_{G_x} dk\, \dot\chi_\sigma({}^k\gamma) = 0.$$

Proof. It is enough to show that if

$$\gamma \in G'G_{x,r/2} \cap \big(G \smallsetminus (U_{0^+}Z)\big),$$

then there exists a $b \in G_x$ such that ${}^b\gamma \in G'$.

Write $\gamma = th$ with $t \in G'$ and $h \in G_{x,r/2}$.

First suppose that $t \in G'_0$. Since $th \in G'G_{x,r/2}$ but $th \notin G_{x,0^+}Z$, we must have $t \in G'_0$ yet $t \notin ZG'_{0^+}$. If $\mathfrak{g}'$ is ramified over F, no such t exists. So assume that $\mathfrak{g}'$ is an unramified extension of F. Then t is also an x-generic element of $\mathfrak{g}'_0 \smallsetminus \mathfrak{g}'_{0+}$ and $th \in t + \mathfrak{g}_{x,r/2}$. From Lemma 5.1.1, there exists a $b \in G_{x,r/2}$ such that ${}^b(th) \in G'$.

Now suppose that $t \in G' \smallsetminus G'_0Z$. If $\mathfrak{g}'$ is an unramified extension of F, then no such t exists. So assume that $\mathfrak{g}'$ is a ramified extension of F. Recall that $\mathfrak{g}_{x,r} = \mathfrak{b}_{\lceil r\cdot\ell\rceil}$. Let Π be a uniformizer for $\mathfrak{g}'$ such that $NB_0 = \langle\Pi\rangle B_0$ and $\Pi \cdot \mathfrak{b}_k = \mathfrak{b}_{k+1}$ for all integers k.

Modulo the center, we can write $t \in \Pi^j \cdot G'_0 \smallsetminus \Pi^{(j+1)} \cdot G'_0$ for some $0 < j < \ell$. Thus t is an x-generic element of $\mathfrak{g}'_{j/\ell} \smallsetminus \mathfrak{g}'_{(j+1)/\ell}$ and $th \in \mathfrak{g}'_{j/\ell} + \mathfrak{g}_{x,(r/2+j/\ell)}$. From Lemma 5.1.1 there exists a b in $G_{x,r/2}$ such that ${}^b(th) \in G'$. ☐

Character calculations on $G'_{0+} \smallsetminus G'_r$

Lemma 5.3.6. *Suppose that*

1. $0 < m < r$ *and*

2. $\gamma \in \left(G'_m \smallsetminus (Z \cap G_m)G'_{m+}\right) \cap G^{\mathrm{reg}}$

Write $Y = \gamma - 1$. *Let* $s = (r-m)/2$, *and*

$$C = e \cdot |G'_0/G'_{s+}| \cdot \mathrm{meas}(\mathfrak{g}_{x,s+}).$$

Then

$$\Theta_\pi(\gamma) = \deg(\pi) \cdot C \cdot \sum_{\bar{\omega} \in \mathcal{W}} \phi({}^\omega\gamma) \cdot H(\omega, Y).$$

Proof. We have

$$\Theta_\pi(\gamma) = \frac{\deg(\pi)}{\deg(\sigma)} \cdot \int_{G/Z} dg^* \int_{G_x} dk \int_{G_x} dk_1\, \dot{\chi}_\sigma({}^{k_1 g k}\gamma).$$

Fix $\gamma \in G$. We know from Lemma 5.3.4 and Lemma 5.3.5 that the inner integral is zero unless either ${}^{gk}\gamma \in G_{x,r}$ or ${}^{G_x}({}^{gk}\gamma) \cap G' \neq \emptyset$. Since $n(\gamma - 1) < r$, the G-orbit of γ does not intersect $G_{x,r}$. If

$${}^{G_x}({}^{gk}\gamma) \cap G' \neq \emptyset,$$

then there exists a $b \in G_x$ such that

$${}^{bgk}\gamma \in G'.$$

This implies that $g \in G_x N_G(G') G_x = G' G_x$. Therefore, the integral formula for $\Theta_\pi(\gamma)$ becomes

$$\frac{\deg(\pi)}{\deg(\sigma)} \cdot e \cdot \int_{G_x} dk\, \dot{\chi}_\sigma({}^k\gamma).$$

If $c_1^{-1} = \text{meas}(G_{x,s+})$, then this last integral can be written as

$$\frac{\deg(\pi)}{\deg(\sigma)} \cdot e \cdot c_1 \cdot \int_{G_x} dk \int_{G_{x,s+}} dk_1\, \dot\chi_\sigma({}^{k_1 k}\gamma).$$

From Lemma 5.3.3, the inner integral is zero unless ${}^k\gamma \in G'_m G_{x,(r+m)/2}$, so assume that this is true. From Lemma 5.1.1, there exists a $b \in G_{x,s}$ such that ${}^{bk}\gamma \in G'$. This implies that k is an element of $NG_{x,s} \cap G_x$. Consequently, since we can choose representatives of $\mathcal{W}$ in G_x, the above integral becomes

$$\frac{\deg(\pi)}{\deg(\sigma)} \cdot e \cdot c_1 \cdot \sum_{\bar\omega \in \mathcal{W}} \int_{G'_0 G_{x,s}} dk \int_{G_{x,s+}} dk_1\, \dot\chi_\sigma({}^{k_1 k \omega}\gamma)$$

$$= \frac{\deg(\pi)}{\deg(\sigma)} \cdot e \cdot c_1 \cdot |G'_0/G'_s| \cdot \sum_{\bar\omega \in \mathcal{W}} \int_{G_{x,s}} dk \int_{G_{x,s+}} dk_1\, \dot\chi_\sigma({}^{k_1 k \omega}\gamma).$$

If we write $\gamma = 1 + Y$, then the integrals become (see equation (5.2.10))

$$\phi({}^\omega\gamma) \cdot \int_{\mathfrak{g}_{x,s}} dW \int_{\mathfrak{g}_{x,s+}} dV\, \Lambda\big(\text{tr}(X_\pi \cdot ([W, {}^\omega Y] + [{}^\omega Y, W] \cdot W + [V, Y]))\big).$$

Arguing as in the proof of Lemma 5.2.8 completes the proof. □

Character values on $G' \setminus G'_{0+}$

Lemma 5.3.7. *If* $\gamma \in (G' \setminus G'_{0+} Z) \cap G^{\text{reg}}$, *then*

$$\Theta_\pi(\gamma) = \frac{\deg(\pi)}{\deg(\sigma)} \cdot C \cdot \lambda(\sigma) \cdot \sum_{\omega \in \mathcal{W}} \phi({}^\omega\gamma)$$

where

$$C = \left|G'_0/G'_{(r/2)}\right| \cdot e \cdot \text{meas}(\mathfrak{g}_{x,(r/2)}).$$

Proof. In the usual way (but relying on Lemma 5.3.5 this time) we can write

$$\Theta_\pi(\gamma) = \frac{\deg\,(\pi)}{\deg(\sigma)} \cdot e \cdot \int_{G_x} dk\, \dot\chi_\sigma({}^k\gamma).$$

The function being integrated will be zero unless ${}^k\gamma \in G' G_{x,r/2}$. It follows from the proof of Lemma 5.3.5 that k must be an element of $NG_{x,r/2}$. Therefore, as usual,

$$\Theta_\pi(\gamma) = \frac{\deg(\pi)}{\deg(\sigma)} \cdot \left|G'_0/G'_{r/2}\right| \cdot e \cdot \sum_{\bar\omega \in \mathcal{W}} \int_{G_{x,r/2}} dk\, \dot\chi_\sigma({}^{k\omega}\gamma).$$

The lemma now follows from Lemma 5.3.1 □

Proof of Theorem 5.3.2

Lemma 5.3.8. *Suppose that $\pi = \pi_\sigma$ is a positive depth supercuspidal representation of $\mathrm{GL}_\ell(F)$ as described in §3.2. Suppose that $\gamma \in G^{\mathrm{reg}}$. If $s = \big(r - n(\gamma)\big)/2$, $C_0 = e \cdot \left|G'_0/G'_{r/2}\right| \cdot \mathrm{meas}(\mathfrak{g}_{x,r/2})$, and $C = e \cdot \left|G'_0/G'_{s+}\right| \cdot \mathrm{meas}(\mathfrak{g}_{x,s+})$, then*

$$\frac{\Theta_\pi(\gamma)}{\deg(\pi)} = \begin{cases} \dfrac{C_0 \cdot \lambda(\sigma)}{\deg(\sigma)} \cdot \displaystyle\sum_{\bar\omega \in \mathcal{W}} \phi({}^{\omega}t) & \text{if } n(\gamma) = 0 \text{ and } \gamma = {}^{g}t \text{ with} \\ & \qquad g \in G \text{ and } t \in G', \\ C \cdot \displaystyle\sum_{\bar\omega \in \mathcal{W}} \phi({}^{\omega}t) \cdot H(\omega, Y) & \text{if } 0 < n(\gamma) \le r/2 \text{ and } \gamma = {}^{g}t \text{ where} \\ & \qquad t = z \cdot (1 + Y) \text{ with } z \in Z,\ g \in G, \\ & \qquad \text{and } Y \in \mathfrak{g}'_{n(\gamma)} \smallsetminus (\mathfrak{z}_{n(\gamma)} + \mathfrak{g}'_{n(\gamma)+}), \\ \phi(z) \cdot \widehat{\mu_{X_\pi}}(Y) & \text{if } n(\gamma) > r/2 \text{ and } \gamma = z \cdot (1 + {}^{g}Y) \\ & \qquad \text{with } g \in G,\ z \in Z, \\ & \qquad \text{and } Y \in V_{n(\gamma)}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We need to check that if $n(\gamma) \le r/2$ and ${}^{G}\gamma \cap G' = \emptyset$, then $\Theta_\pi(\gamma) = 0$. If $n(\gamma) = 0$, this is Lemma 5.3.5. So suppose that $0 < n(\gamma) < r/2$ and fix $g \in G$ such that ${}^{g}\gamma \in G'G_{x,r/2}$. Write ${}^{g}\gamma = th$ with $t \in G'$ and $h \in G_{x,r/2}$. If $t \notin ZG'_{0+}$, then, as in the proof of Lemma 5.3.5, ${}^{G}\gamma \cap G' \ne \emptyset$. Therefore, ${}^{g}\gamma \in ZG_{x,0+}$ and the result follows from Lemma 5.3.4. ☐

Proof of Theorem 5.3.2. We will verify that when $n(\gamma) = 0$ and $\gamma \in G'$, the ratio

$$\frac{C_0 \cdot \lambda(\sigma)}{\deg(\sigma)}$$

occurring in Lemma 5.3.8 is

$$c_\Lambda(\mathfrak{g}') \cdot c_\Lambda^{-1}(\mathfrak{g}) \cdot |D(\gamma)|^{-1/2} \cdot |\eta(X_\pi)|^{-1/2}.$$

The remaining cases are handled exactly as in the proof of Theorem 5.2.1.

Since $n(\gamma) = 0$, we have that $|D(\gamma)| = 1$, so we will ignore this term. Since the dimension of σ is equal to $\left| \mathfrak{g}^{\perp}_{r/2}/\mathfrak{g}^{\perp}_{(r/2)+} \right|^{1/2}$, we have

$$\begin{aligned}
\frac{C_0 \cdot \lambda(\sigma)}{\deg(\sigma)} &= e \cdot \operatorname{meas}(\mathfrak{g}_{x,r/2}) \cdot \left|G'_0/G'_{r/2}\right| \cdot \left| \mathfrak{g}^{\perp}_{r/2}/\mathfrak{g}^{\perp}_{(r/2)+} \right|^{-1/2} \\
&= \frac{\operatorname{meas}(\mathfrak{g}_{x,r/2}) \cdot \left|\mathfrak{g}'_{0+}/\mathfrak{g}'_{r/2}\right|}{\operatorname{meas}(\mathfrak{g}'_{0+}) \cdot \left| \mathfrak{g}^{\perp}_{r/2}/\mathfrak{g}^{\perp}_{(r/2)+} \right|^{1/2}} \\
&= \frac{\operatorname{meas}(\mathfrak{g}_{x,0+}) \cdot \left| \mathfrak{g}^{\perp}_{r/2}/\mathfrak{g}^{\perp}_{(r/2)+} \right|^{1/2}}{\operatorname{meas}(\mathfrak{g}'_{0+}) \cdot \left| \mathfrak{g}^{\perp}_{0+}/\mathfrak{g}^{\perp}_{(r/2)+} \right|} \\
&= \frac{\operatorname{meas}(\mathfrak{g}_{x,0+}) \cdot \left|\mathfrak{g}^{\perp}_{0}/\mathfrak{g}^{\perp}_{0+} \right|^{1/2}}{\operatorname{meas}(\mathfrak{g}'_{0+}) \cdot |\eta(X_\pi)|^{1/2}} \\
&= c_\Lambda(\mathfrak{g}') \cdot c_\Lambda^{-1}(\mathfrak{g}) \cdot |D(\gamma)|^{-1/2} \cdot |\eta(X_\pi)|^{-1/2}. \quad \square
\end{aligned}$$

5.4 Explicit depth zero character values

Suppose that π_σ is a depth zero representation of $\mathrm{GL}_\ell(F)$ as discussed in section 3.2.

Theorem 5.4.1. *Suppose that $\gamma \in K_0^{\mathrm{reg}}$.*

$$\frac{\Theta_\pi(\gamma)}{\deg(\pi)} = \begin{cases} \dfrac{\chi_\sigma(\gamma)}{\deg(\sigma)} & \text{if } \gamma \text{ is unramified elliptic and} \\ & \qquad \gamma \notin K_1, \\ \chi_\pi(z) \cdot \displaystyle\sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}}(\pi) \widehat{\mu_{\mathcal{O}}}(X) & \text{if } \gamma = z \cdot (1 + {}^{g}X) \text{ with} \\ & \qquad X \in \mathfrak{b}_1,\ g \in G, \text{ and } z \in Z, \\ 0 & \text{otherwise.} \end{cases}$$

Here χ_π is the central character of π.

Proof. The first line is [14, Proposition 6.11(1)]. The second line is Lemma 3.4.1. The final line is [14, Lemma 6.4]. $\square$

Note that the functions $\widehat{\mu_{\mathcal{O}}}$ are known [22, Lemma 5.1], as are the $c_{\mathcal{O}}$'s for these representations [22, Proposition 5.3].

5.5 A few comments on GL_3

In section 3.2, we discussed three "series" of supercuspidal representations of $\mathrm{GL}_3(F)$ — the unramified series and two ramified series. The unramified series and one of the ramified series are completely described above. We need only consider the ramified series of depth $\rho(\pi) = k + 1/3$ for k a nonnegative integer. For these representations we must verify that the local character expansion is valid for $\gamma \in 1 + V^{\mathrm{reg}}_{\rho(\pi)+}$ and compute the $c_{\mathcal{O}}$'s for $\mathcal{O} \in \mathcal{O}(0)$.

Fix a representation π as in §3.2 having depth $k + 1/3$. By analyzing the Moy-Prasad filtrations $\mathfrak{g}_{x,r}$ for $r = k + 1/3$, one sees that

$$\Theta_\pi(1 + X) = \deg(\pi) \cdot \widehat{\mu_{X_\pi}}(X) = \sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}} \widehat{\mu_{\mathcal{O}}}(X)$$

for all $X \in V^{\mathrm{reg}}_{(k+1/3)+}$ if the following lemma is true.

Lemma 5.5.1.

$$j_{\mathfrak{q}_2} J(\mathfrak{b}_2) = j_{\mathfrak{q}_2} J(\mathcal{N}).$$

Here $\mathfrak{q}_2$ is the set of matrices of the form

$$\begin{pmatrix} \wp & \wp & \wp \\ \wp & \wp & \wp \\ \wp^2 & \wp^2 & \wp \end{pmatrix}$$

in $M_3(F)$. Since the proof of Lemma 5.5.1 is extremely tedious and not particularly enlightening, we shall omit it.

We now need to compute the constants $c_{\mathcal{O}}$ occurring in the Harish-Chandra-Howe local expansion. I have computed these using equation (0.0.1). We need some notation before explicitly writing them down.

In Table 2 we classify the three nilpotent orbits of $M_3(F)$ by the size of the blocks in their Jordan decomposition. Table 3 lists the $c_{\mathcal{O}}$'s for both ramified series of representations subject to the normalizations of measures given in [22, §5]. Note that these values are independent of F.

One can compute the $c_{\mathcal{O}}$'s for $\mathrm{GL}_3(F)$ in other ways. For example, with certain tameness restrictions, by using Theorem 4.4 of [22] and the results of Repka [25,

partition	orbit representative
(3)	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$(2,1)$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$(1,1,1)$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Table 2: $\mathcal{O}(0)$ for $G = \mathrm{GL}_3(F)$

$\rho(\pi)$	$k + 1/3$	$k + 2/3$
$c_{(3)}(\pi)$	$\dfrac{q^{(3k+2)} \cdot (q^2 + q + 1)}{q^2}$	$\dfrac{q^{(3k+3)} \cdot (q^2 + q + 1)}{q^2}$
$c_{(2,1)}(\pi)$	$-q^k \cdot (2 + q)$	$-q^k \cdot (2q + 1)$
$c_{(1,1,1)}(\pi)$	1	1

Table 3: $c_{\mathcal{O}}(\pi)$'s for ramified supercuspidal representations of $\mathrm{GL}_3(F)$

26], one can compute $c_{(1,1,1)}$ and $c_{(2,1)}$. Calculating the constants in this way yields the same results.

5.6 Comparison with previous results

In Remark 5.2.3 we noted that the results obtained here agree with the relevant results of [36].

Now suppose that $\ell = 1$. Then $G = F^\times$ and $\mathfrak{g} = F$. We first consider our results on $\mathfrak{g}$. Fix $X \in F$. For $f \in C_c^\infty(F)$, we have

$$\widehat{\mu_X}(f) = \hat{f}(X) = \int_F dY\, f(Y) \cdot \Lambda(X \cdot Y),$$

and so we have $\widehat{\mu_X}(Y) = \Lambda(X \cdot Y)$. This is exactly what Theorem 5.2.1 provides. Now let σ be an irreducible supercuspidal representation of G. Any character of G qualifies, so suppose that σ is a character. Then $\Theta_\sigma = \sigma$. This agrees with Theorem 5.3.2.

Suppose that $G = \mathrm{GL}_2(F)$. After much hard work, one can show that the character values obtained in [29] agree with those listed in Theorem 5.3.2 except at level $\rho(\pi)$. At this level, it can be shown, by looking at specific examples, that the two sets of results are incompatible.

Now suppose that ℓ is odd, $G = \mathrm{GL}_\ell(F)$, and π is a ramified supercuspidal representation as discussed in §3.2 such that $\mathfrak{g}_{x,\rho(\pi)/2} \neq \mathfrak{g}_{x,(\rho(\pi)/2)+}$. In this case, the character values obtained in [3] agree (after correcting for a typographical error in equation (c) of Theorem 4.2) with those listed in Theorem 5.3.2 with the following possible exception. Since Corwin, Moy, and Sally use the matching theorem to obtain the character's values, their results often involve sums over certain sets related to a division algebra. Most of the time, these sums can easily be related to a sum over a finite field and then compared favorably with the values obtained in Theorem 5.3.2. However, at level $\rho(\pi)$ I have been unable to show that their results agree with those obtained here.

This difficulty at level $\rho(\pi)$ is quite vexing and will be the subject of future work.

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