The Hales-Moy-Prasad conjecture for Sp_4

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1 Introduction

This paper is really about SL_2. In fact, in principle, it is about things we have known about SL_2 since before I was born. Namely, where the Harish-Chandra-Howe local expansion holds for an irreducible admissible representation. So, in this sense, there is nothing new here. However, I hope to convince you that there is hope that not only is the conjecture of Hales, Moy, and Prasad true, but we might actually be able to prove it.

The results discussed below owe their existence to many people. I thank the organizers of the conference, Professors Sally and Vignéras, for inviting me to speak. I draw heavily on some work of J.-L. Waldspurger. Robert Kottwitz, in his usual way, spurred me to make what I consider to be a key realization that makes this all work. I thank Jeff Adler for his comments on an earlier version of this note. Of course, none of this would have been possible without the constant encouragement, mathematical and otherwise, of Professor Sally.

Finally, because I was unable to incorporate some of my "visual aids" into this note, the discussion here about what motivated the various ideas is (in my opinion) less satisfactory than in the original talk.

2 Notation

2.1 General notation

Let F denote a nonarchimedian local field[1]. Let R be its ring of integers and $\wp = \varpi R$ the prime ideal. We assume that $R/\wp \cong \mathbb{F}_q$ and q is odd. We fix an additive character Λ of F with conductor $\wp$.

G will denote $\mathrm{SL}_2(F)$ realized as the set of 2×2 matrices having determinant one. We denote by $\mathfrak{g}$ the Lie algebra of G. We will take $\mathfrak{g}$ to be $\mathfrak{sl}_2(F)$, the set of 2×2 matrices having trace zero with the usual bracket operation. φ will denote the Cayley transform from the set of topologically nilpotent elements in $\mathfrak{g}$ into G. (The set of topologically nilpotent elements will be defined later.) $\mathcal{N}$

[1] Additionally, I will assume that F has characteristic zero. Since I will assume that q is odd, I do not think that this assumption is needed in this note. However, since the status of convergence of orbital integrals in positive characteristic is unresolved, I will err on the side of caution. On the other hand, I have chosen to state results in a characteristic-free way.

denotes the set of nilpotent elements in $\mathfrak{g}$, i.e., the set

$$\{X \in \mathfrak{g}\colon X^2 = 0\}.$$

Regular will always mean regular semisimple and G^{reg} will denote the set of regular elements in G. (Similar notation applies to $\mathfrak{g}$.)

For $g \in G$ and $X \in \mathfrak{g}$, we denote by ${}^{g}X$ the adjoint action of g on X. If S is a subset of $\mathfrak{g}$, then

$${}^{G}S = \{{}^{g}s\colon s \in S, g \in G\}$$

denotes the G-orbit of S.

Example 1.

$${}^{G}\mathcal{N} = \mathcal{N}.$$

We recall that $\mathcal{N}$ is the union of a finite number of nilpotent orbits

$$\mathcal{N} = \bigcup_{\mathcal{O} \in \mathcal{O}(0)} \mathcal{O}$$

and the cardinality of $\mathcal{O}(0)$ is five.

2.2 Some filtrations for $\mathrm{SL}_2(F)$

2.2.1 Standard congruence filtration

The congruence filtration lattices: Let

$$\mathfrak{k}_0 = M_2(R) \cap \mathfrak{sl}_2(F)$$

$$\mathfrak{k}_i = \varpi^i \cdot \mathfrak{k}_0$$

The corresponding parahoric subgroup is $K_0 = \mathrm{SL}_2(R)$.

2.2.2 Standard Iwahori filtration

The standard Iwahori filtration lattices: Let $\mathfrak{b}_0$ denote the set of matrices in $\mathfrak{g}$ of the form

$$\begin{pmatrix} R & R \\ \wp & R \end{pmatrix}.$$

That is, $\mathfrak{b}_0$ is the set of matrices in $\mathfrak{sl}_2$ with upper triangular entries in R and strictly lower triangular entries in $\wp$.

$$\mathfrak{b}_0 = \begin{pmatrix} R & R \\ \wp & R \end{pmatrix} \supset \mathfrak{b}_1 = \begin{pmatrix} \wp & R \\ \wp & \wp \end{pmatrix} \supset \mathfrak{b}_2 = \varpi \cdot \mathfrak{b}_0.$$

The corresponding Iwahori subgroup is

$$B_0 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0 : c \in \wp\}.$$

2.2.3 Nonstandard congruence filtration

The nonstandard congruence filtration lattices: Let

$$\Pi = \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right) \in \mathrm{GL}_2(F).$$

Note that Π normalizes each $\mathfrak{b}_i$ and B_0.

$$\mathfrak{q}_0 = {}^{\Pi}\mathfrak{k}_0 = \left(\begin{smallmatrix} R & \wp^{-1} \\ \wp & R \end{smallmatrix}\right)$$

$$\mathfrak{q}_i = \varpi^i \cdot \mathfrak{q}_0$$

The corresponding parahoric is $Q_0 = {}^{\Pi}K_0$.

2.3 Moy-Prasad filtrations on $\mathfrak{g}$

Moy and Prasad in [3, 4] have defined a way to associate a lattice $\mathfrak{g}_{x,r}$ in $\mathfrak{g}$ to a pair (x, r) in $\mathcal{B}(G) \times \mathbb{R}$ (where $\mathcal{B}(G)$ denotes the Bruhat-Tits building of G). These lattices have the property that for $g \in G$,

$${}^{g}\mathfrak{g}_{x,r} = \mathfrak{g}_{gx,r}.$$

Fix an alcove (affine chamber) C in $\mathcal{B}(G)$. We know that any point in $\mathcal{B}(G)$ is conjugate to a point in $\bar{C}$, the closure of C. Therefore, in order to understand the lattices $\mathfrak{g}_{x,r}$ up to conjugation, it is enough to understand the $\mathfrak{g}_{x,r}$ for $(x, r) \in \bar{C} \times \mathbb{R}$.

Rather than repeat the definition of $\mathfrak{g}_{x,r}$, we offer a few illustrations for our situation. Let us choose the apartment $\mathcal{A}$ in $\mathcal{B} = \mathcal{B}(\mathrm{SL}_2(F))$ corresponding to the diagonal torus in $G = \mathrm{SL}_2(F)$. Let C be the alcove in $\mathcal{A}$ with stabilizer B_0. In Figure 1, we illustrate how the lattice $\mathfrak{g}_{x,r}$ varies as x and r vary for x near C (C is the open line segment with endpoints x' and z'). Since the apartment $\mathcal{A}$ can be thought of as a copy of $\mathbb{R}$, this figure may be thought of as a picture of $\mathbb{R}^2$. Each diagonal dotted line is the graph of $r = \psi(x)$ for some affine root ψ. The horizontal dotted lines correspond to the natural filtration of the diagonal Cartan subalgebra in $\mathfrak{sl}_2(F)$.

The dotted lines divide the plane into convex polygons. Fix one such polygon D. If (x_1, r_1) and (x_2, r_2) are two points in the interior of D, then $\mathfrak{g}_{x_1,r_1} = \mathfrak{g}_{x_2,r_2}$. Because of the conditions which define $\mathfrak{g}_{x,r}$ for arbitrary x and r, a point (y, s) on the boundary of D corresponds to the same filtration lattice as every other point in the interior of D if and only if there exists a point (y, r) in the interior of D with $r < s$.

Unfortunately, the Moy-Prasad filtration lattices which occur for $\mathfrak{sl}_2$ are all familiar. Less familiar lattices do occur in, for example, $\mathfrak{sl}_3$.

2.4 Important comments on Moy-Prasad filtrations

As a notational convenience, we define $\mathfrak{g}_{x,r^+} = \cup_{s>r}\mathfrak{g}_{x,s}$. Note that in figure 1, we have $\mathfrak{g}_{x,r} = \mathfrak{g}_{x,r^+}$ unless (x, r) lies on a dotted line.

Example 2. $\mathfrak{g}_{y',0} = \mathfrak{b}_0$ while $\mathfrak{g}_{y',0^+} = \mathfrak{b}_1$

If we restrict r to the nonnegative real numbers, then we can define filtration subgroups, $G_{x,r}$, of the parahoric subgroup G_x attached to x.

Example 3.

- $G_{x',0} = K_0$
- $G_{y',0} = B_0$
- $G_{z',0} = Q_0$

We define the Fourier transform of $f \in C_c^\infty(\mathfrak{g})$ by

$$\hat{f}(X) = \int_{\mathfrak{g}} dY f(Y) \cdot \Lambda\big(\mathrm{tr}(X \cdot Y)\big).$$

Here dY is a G-invariant measure on $\mathfrak{g}$. For $\mathrm{SL}_2(F)$ in odd residual characteristic, the Fourier transform behaves well with respect to the Moy-Prasad filtrations (in other situations, the Fourier transform still behaves well if you realize $\hat{f}$ as a function on the dual space of $\mathfrak{g}$). In particular, we have

$$f \in C(\mathfrak{g}_{x,r}/\mathfrak{g}_{y,s}) \text{ if and only if } \hat{f} \in C(\mathfrak{g}_{y,(-s)^+}/\mathfrak{g}_{x,(-r)^+})$$

for x and y in $\mathcal{B}(G)$. Finally, we will normalize the measure dY so that $\hat{\hat{f}}(X) = f(-X)$.

3 The conjecture of Hales, Moy, and Prasad

3.1 Background

3.1.1 Characters

Suppose that (π, V) is an admissible, irreducible representation of G. It is a deep result of Howe and Harish-Chandra that the character distribution

$$\Theta_\pi : C_c^\infty(G) \to \mathbb{C}$$

which sends f to $\mathrm{tr}\big(\pi(f)\big)$ is represented by a locally constant function F_π on G^{reg}. That is, if f is a locally constant, compactly supported function on G^{reg}, then we have

$$\Theta_\pi(f) = \int_G dg\, f(g) \cdot F_\pi(g) \tag{4}$$

where dg is a Haar measure on G. The only characteristic-free proof of this result of which I am aware may be found in [1].

Suppose that $\gamma \in G^{\mathrm{reg}}$. Let U be a neighborhood of γ such that F_π is constant on U. Let $[U]$ denote the characteristic function of U and let

$$f_U = \frac{[U]}{\mathrm{meas}\,(U)}.$$

Then equation (4) becomes

$$\Theta_\pi(f_U) = F_\pi(\gamma).$$

In other words, by taking the limit as U shrinks to γ we see that F_π really is the character of π and for this reason we will abuse notation and allow Θ_π to denote both the character distribution and the function which represents this distribution.

3.1.2 Depth

Moy and Prasad define the *depth* $\rho(\pi)$ of π. A property of $\rho(\pi)$ is that it is the smallest number for which there exists an $x \in \mathcal{B}(G)$ such that

$$V^{G_{x,\rho(\pi)+}} \neq \{0\}.$$

Note that if $\rho(\pi)$ is positive, then

$$V^{G_{x,\rho(\pi)}} = \{0\}.$$

Example 5. If π has Iwahori (B_0) fixed vectors, then $\rho(\pi) = 0$.

3.1.3 Fourier transform of a nilpotent orbital integral

For $f \in C_c^\infty(\mathfrak{g})$ and $\mathcal{O} \in \mathcal{O}(0)$, we define the Fourier transform of the nilpotent orbital integral $\mu_{\mathcal{O}}$ by

$$\widehat{\mu_{\mathcal{O}}}(f) = \mu_{\mathcal{O}}(\hat{f}).$$

In characteristic zero, Harish-Chandra showed that this distribution

$$\widehat{\mu_{\mathcal{O}}} : C_c^\infty(\mathfrak{g}) \to \mathbb{C}$$

is represented by a locally constant function on $\mathfrak{g}^{\mathrm{reg}}$ which we also denote by $\widehat{\mu_{\mathcal{O}}}$. Following Harish-Chandra, R. Huntsinger [2] derived an integral formula for this function and P.J. Sally, Jr. has used this formula to explicitly calculate $\widehat{\mu_{\mathcal{O}}}$ for our situation.

3.1.4 The Harish-Chandra-Howe local expansion

The Harish-Chandra-Howe local expansion says that for all regular X sufficiently near zero in $\mathfrak{g}$ we can write

$$\Theta_\pi(\varphi(X)) = \sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}}(\pi) \cdot \widehat{\mu_{\mathcal{O}}}(X)$$

for some constants $c_{\mathcal{O}}(\pi) \in \mathbb{C}$. In other words, sufficiently near the identity, the character of an admissible irreducible representation of G can be expressed as a finite linear combination of functions on $\mathfrak{g}$. Notice that there is no mention about where this equality should hold. Howe originally established this result for $\mathrm{GL}_n(F)$ in any characteristic and Harish-Chandra extended it to arbitrary reductive groups in characteristic zero. Of course, in their proofs neither used the Cayley transform, but it works in our situation.

3.2 The Hales-Moy-Prasad conjecture

The conjecture of Hales, Moy, and Prasad was first formulated by T. Hales for depth zero representations. Their conjecture identifies a neighborhood on which the Harish-Chandra-Howe expansion ought to hold.

Conjecture 6. *There exist constants $c_{\mathcal{O}}(\pi) \in \mathbb{C}$ indexed by $\mathcal{O} \in \mathcal{O}(0)$ such that*

$$\Theta_\pi(\varphi(X)) = \sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}}(\pi) \cdot \widehat{\mu_{\mathcal{O}}}(X)$$

for all $X \in \mathfrak{g}_{\rho(\pi)+} \cap \mathfrak{g}^{\mathrm{reg}}$.

Here we set $\mathfrak{g}_r = \cup_{x \in \mathcal{B}(G)} \mathfrak{g}_{x,r}$. We know that $\mathfrak{g}_r$ is an open, closed, G-invariant subset of $\mathfrak{g}$, i.e., it is a G-domain.

Example 7.

- $\mathfrak{g}_{0+} = \mathfrak{g}_{1/2} = {}^G\mathfrak{b}_1$. This is sometimes referred to as the set of topologically nilpotent elements in $\mathfrak{g}$.
- $\mathfrak{g}_{(1/2)+} = \mathfrak{g}_1 = {}^G\mathfrak{k}_1 \cup {}^G\mathfrak{q}_1$.

3.2.1 Evidence for the conjecture

If you are not dealing with a classical group, in the following list you will have to replace the Cayley transform with a suitable map from $\mathfrak{g}$ to G.

- If $\rho(\pi)$ is an integer, then J.-L. Waldspurger [5] has verified Conjecture 6 for a large class of groups (with some restrictions on q).
- If q is odd, then the conjecture can be verified for $\mathrm{SL}_2(F)$ by examining the work of P.J. Sally, Jr. and J. Shalika.
- For $\mathrm{GL}_2(F)$ the conjecture was verified by A. Moy.
- For $\mathrm{GL}_3(F)$ the conjecture is true without restrictions.
- For $\mathrm{Sp}_4(F)$ and $\mathrm{GSp}_4(F)$ the conjecture is true in odd residual characteristic.

3.3 How to prove the conjecture

The "traditional" approach to proving that there exists a local expansion proceeds as follows. Let $\tilde{J}$ denote a certain set of G-invariant distributions on $\mathfrak{g}$ and let $J(\mathcal{N})$ denote the linear span of the nilpotent orbital integrals. If $\mathcal{L}$ denotes a particularly well-chosen lattice in $\mathfrak{g}$, then we want to show that

$$\tilde{J}|_{C_c(\mathfrak{g}/\mathcal{L})} = J(\mathcal{N})|_{C_c(\mathfrak{g}/\mathcal{L})}.$$

We then want to invoke some version of Kirillov theory to relate Θ_π to the Fourier transform of a distribution in $\tilde{J}$. This will give us our desired result. If we want to prove the sharpened version of the local character expansion expressed in the Hales-Moy-Prasad conjecture, then we must carefully specify $\tilde{J}$ and $\mathcal{L}$.

A difficulty with this approach is that the space of functions $C_c(\mathfrak{g}/\mathcal{L})$ is small and therefore we have to work hard to make sure that we always operate within that space. Since $\tilde{J}$ is a G-invariant set, it might be nice to replace $C_c(\mathfrak{g}/\mathcal{L})$ with a G-invariant set of functions. This idea arose from a comment of R. Kottwitz, and it was used successfully in my proofs of the Hales-Moy-Prasad conjecture for $\mathrm{SL}_2(F)$, $\mathrm{Sp}_4(F)$, and $\mathrm{GSp}_4(F)$.

The natural candidate to replace the $\mathcal{L}$ invariant functions is the space

$$\begin{aligned} D_v &= \{f \in C_c^\infty(\mathfrak{g})\colon \operatorname{supp}(\hat{f}) \subset \mathfrak{g}_{(-v)^+}\} \\ &= \mathrm{F.T.}\big(C_c^\infty(\mathfrak{g}_{(-v)^+})\big) \\ &= \sum_{x \in \mathcal{B}(G)} C_c(\mathfrak{g}/\mathfrak{g}_{x,v}) \end{aligned}$$

for $v \in \mathbb{R}$. (Note, $f \in \sum_{x\in\mathcal{B}(G)} C_c(\mathfrak{g}/\mathfrak{g}_{x,v})$ if and only if $f = \sum f_i$, the sum being over a finite collection of $f_i \in C_c(\mathfrak{g}/\mathfrak{g}_{x_i,v})$ for some finite set of $x_i \in \mathcal{B}(G)$.)

Remark 8. Because $\mathfrak{g}_{(-v)^+} \cap \mathfrak{m} = \mathfrak{m}_{(-v)^+}$, we have that the constant term map $f \mapsto f_P$ takes

$$D_v \to D_v^{\mathfrak{m}}.$$

Remark 9. Similar objects can be defined for G and they have analogous properties.

3.4 The proof

Fix $v \in \mathbb{R}$. For a subset U of $\mathfrak{g}$ we let $[U]$ denote the characteristic function of the set U. There are essentially three steps in the proof. First, we define a set of distributions $\tilde{J}_v$ and show that

Lemma 10.

$$\tilde{J}_v|_{D_v} = J(\mathcal{N})|_{D_v}.$$

We then define a distribution

$$\widehat{\Theta}_\pi\colon C_c^\infty(\mathfrak{g}) \to \mathbb{C}$$

which takes f to
$$\Theta_\pi(\hat{f} \cdot [\mathfrak{g}_{\rho(\pi)^+}] \circ \varphi^{-1}).$$
Here and in what follows I am implicitly assuming that
$$h \circ \varphi^{-1}(g) = \begin{cases} h(\varphi^{-1}(g)) & \text{if } g \in \varphi(\mathfrak{g}_{\rho(\pi)^+}) \\ 0 & \text{otherwise} \end{cases}$$
for $h \in C_c^\infty(\mathfrak{g}_{\rho(\pi)^+})$. The second step is to show that

Lemma 11.
$$\widehat{\Theta}_\pi \in \tilde{J}_{-\rho(\pi)}.$$

The final step is to use the above two lemmas to verify the Hales-Moy-Prasad conjecture. We need to show that for all $f \in C_c^\infty(\mathfrak{g}^{\mathrm{reg}}_{\rho(\pi)^+})$,
$$\Theta_\pi(f \circ \varphi^{-1}) = \sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}}(\pi) \cdot \widehat{\mu_{\mathcal{O}}}(f).$$

Fix $f \in C_c^\infty(\mathfrak{g}^{\mathrm{reg}}_{\rho(\pi)^+})$ and let $f' = \hat{\hat{f}}$. Note that $\widehat{f'} \in D_{-\rho(\pi)}$. From Lemma 10 and Lemma 11 we have
$$\begin{aligned} \Theta_\pi(f \circ \varphi^{-1}) &= \widehat{\Theta}_\pi(\widehat{f'}) \\ &= \sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}} \cdot \mu_{\mathcal{O}}(\widehat{f'}) \\ &= \sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}} \cdot \widehat{\mu_{\mathcal{O}}}(f') \\ &= \sum_{\mathcal{O} \in \mathcal{O}(0)} c'_{\mathcal{O}} \cdot \widehat{\mu_{\mathcal{O}}}(f). \end{aligned}$$

So it only remains to verify the two lemmas above. Fix an $x \in \mathcal{B}(G)$. Define
$$\begin{aligned} \tilde{J}_{v,x} = \{T \in J(\mathfrak{g})\colon & \text{ for all } w \in \mathbb{R} \text{ and for all } f \in C\big((\mathfrak{g}_{x,w} \smallsetminus \mathfrak{g}_{x,w^+})/\mathfrak{g}_{x,v}\big), \\ & \text{if } \operatorname{supp}(f) \cap (\mathcal{N} + \mathfrak{g}_{x,w^+}) = \emptyset \text{ then } T(f) = 0\} \end{aligned}$$
and
$$\tilde{J}_v = \bigcap_{x \in \mathcal{B}(G)} \tilde{J}_{v,x}.$$
The proof of Lemma 11 is nearly identical to the proof given in [5, §6.7]. In fact, it is easier here. So we will concentrate on the proof of Lemma 10. This will be a proof by example.

Suppose that $v = 1/2$ and $T \in \tilde{J}_{1/2}$.

The idea is to show that $T|_{D_{1/2}}$ depends only on
$$T|_{C(\mathfrak{k}_0/\mathfrak{k}_1) \cup C(\mathfrak{b}_0/\mathfrak{b}_1) \cup C(\mathfrak{q}_0/\mathfrak{q}_1)}.$$

Suppose for the time being that we can show this.

Recall that $\mathfrak{k}_i \leftrightarrow x'$, $\mathfrak{b}_i \leftrightarrow y'$, $\mathfrak{q}_i \leftrightarrow z'$. Consider first the filtration associated to x'. From the definition of $\tilde{J}_{1/2}$, we know that $T|_{C(\mathfrak{k}_0/\mathfrak{k}_1)}$ is completely determined by functions of the form $[n+\mathfrak{k}_1]$ with $n \in \mathcal{N} \cap \mathfrak{k}_0$. Up to K_0-conjugacy, there are only three such functions. Similar statements apply to the filtrations determined by y' and z'. For the point y' we get one function and for the point z' we get three functions. So, T is completely determined by its values on seven functions. However, there is an obvious equation relating $T([\mathfrak{b}_1])$ with T evaluated at the three functions corresponding to x'. Similarly for the point z'. Therefore, T is completely determined by its values on five functions. Since T was arbitrary, we have

$$\dim_{\mathbb{C}}(\tilde{J}_{1/2}|_{D_{1/2}}) \leq 5.$$

Of course, $J(\mathcal{N}) \subset \tilde{J}_{1/2}$ and the dimension of $J(\mathcal{N})|_{D_{1/2}}$ is five. Therefore, we have

$$\tilde{J}_{1/2}|_{D_{1/2}} = J(\mathcal{N})|_{D_{1/2}}.$$

We now wish to show that T depends only on

$$T|_{C(\mathfrak{k}_0/\mathfrak{k}_1) \cup C(\mathfrak{b}_0/\mathfrak{b}_1) \cup C(\mathfrak{q}_0/\mathfrak{q}_1)}.$$

Because of the G-invariance of everything in sight, we will know $T|_{D_{1/2}}$ if we know

$$T|_{C_c(\mathfrak{g}/\mathfrak{k}_1) \cup C_c(\mathfrak{g}/\mathfrak{b}_1) \cup C_c(\mathfrak{g}/\mathfrak{q}_1)}.$$

Let us start with a function $f \in C_c(\mathfrak{g}/\mathfrak{k}_1)$. We suppose that there is some work to do, i.e., there exists some $m \geq 1$ such that $\operatorname{supp}(f) \subset \mathfrak{k}_{-m} \setminus \mathfrak{k}_{1-m}$. We wish to show that $T(f) = T(f')$ where the support of f' is "closer" to the origin than that of f. According to the definition of $\tilde{J}_{1/2}$, we may assume that $f = [n+Y+\mathfrak{k}_1]$ with $n \in \mathcal{N} \cap (\mathfrak{k}_{-m} \setminus \mathfrak{k}_{1-m})$ and $Y \in \mathfrak{k}_{1-m} = \mathfrak{g}_{x',(-m)^+}$. Since $\mathfrak{k}_1$ is K_0-invariant, we may assume that

$$n = \varpi^{-m} \begin{pmatrix} 0 & \theta \\ 0 & 0 \end{pmatrix}$$

with $\theta \in R^\times$.

Note that

$$n + Y + \mathfrak{k}_1 \subset \varpi^{-m}\mathfrak{b}_1 = \mathfrak{b}_{1-m} = \mathfrak{g}_{y',(-m)^+} = \mathfrak{g}_{y',(1/2-m)}$$

so that the support of f is closer to the origin with respect to the y' filtration. (This is a special case of [3, Proposition 6.3].) Of course, we cannot continue this process inductively unless we can somehow transform f into a function in $C_c(\mathfrak{g}/\mathfrak{b}_1)$. Let

$$t_\alpha = \begin{pmatrix} (1+\alpha) & 0 \\ 0 & (1+\alpha)^{-1} \end{pmatrix} \in \mathrm{SL}_2(F)$$

for $\alpha \in \wp$. Then

$$\begin{aligned} T\big([n+Y+\mathfrak{k}_1]\big) &= \frac{1}{q} \cdot \sum_{\bar{\alpha} \in \wp^m/\wp^{(m+1)}} \big(T^{t_\alpha}\big)\big([n+Y+\mathfrak{k}_1]\big) \\ &= \frac{1}{q} \cdot T\big([n+Y+\mathfrak{b}_1]\big). \end{aligned}$$

Now the function $[n + Y + \mathfrak{b}_1]$ is certainly in $C_c(\mathfrak{g}/\mathfrak{b}_1)$, and its support is closer to the origin with respect to the y' filtration.

This program can be carried out for the points y' and z' as well.

References

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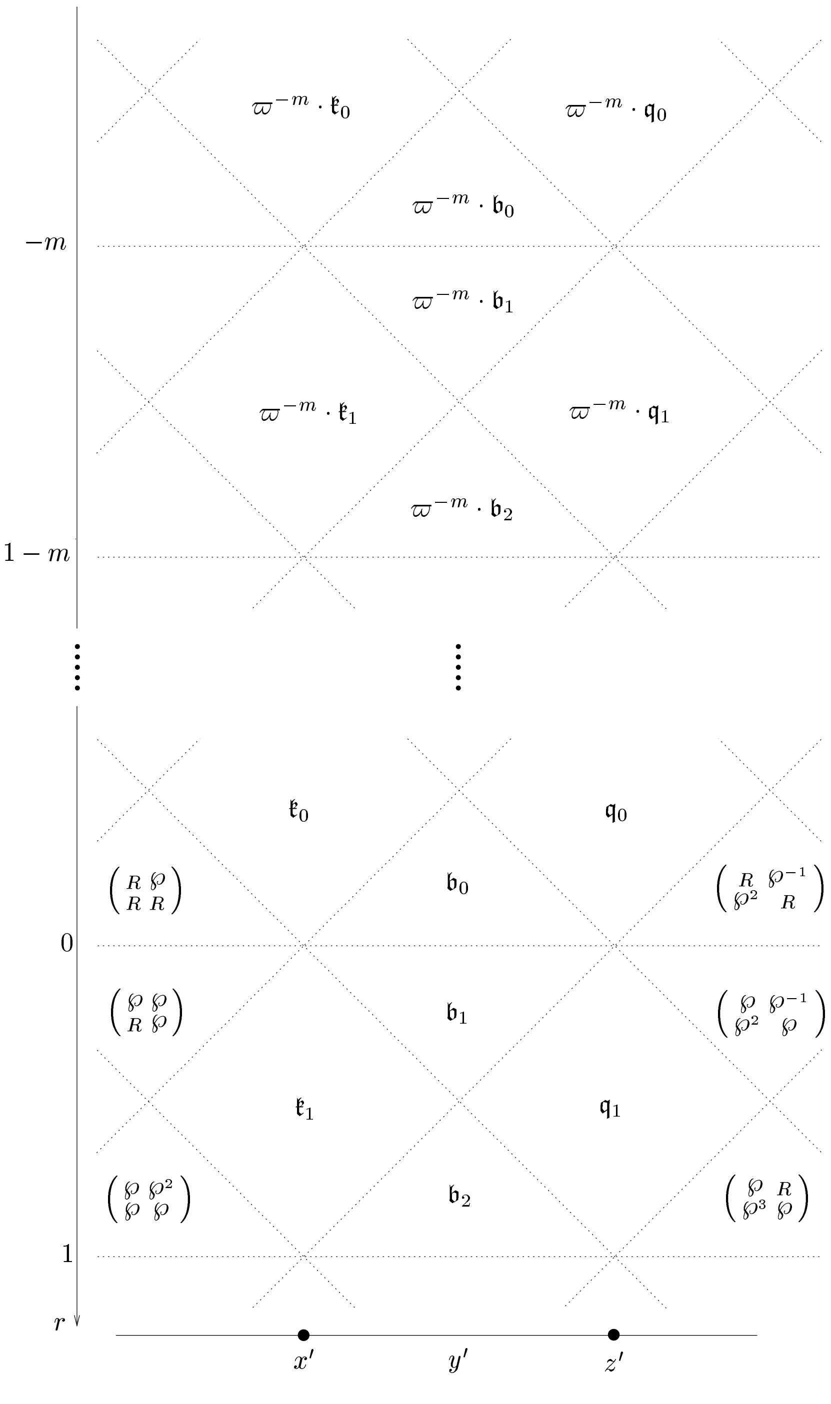

Figure 1: $(\mathfrak{sl}_2)_{x,r}$, x near C