Homogeneity for reductive p-adic groups: an introduction

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ABSTRACT. We discuss, in a fairly conversational manner, homogeneity results for reductive p-adic groups. We provide some motivation for why we expect such results to be true, and we discuss why they are important. We also discuss most of the mathematics required to prove homogeneity statements.

1. Introduction

The goal of these notes is to introduce the idea of homogeneity for reductive p-adic groups. Except in trivial cases, we are not in any position to verify homogeneity statements; rather, we shall try to motivate both why such results are important and why we should believe that they are true. To this end, we will also discuss many of the important mathematical ideas surrounding these statements. Finally, while I think that they are mathematically accurate, these notes are intended as an introduction, not as a reference.

I thank Joseph Rabinoff for producing the computer graphics for Figure 8. I learned nearly all that I know about harmonic analysis while under the excellent guidance of Bob Kottwitz and Paul Sally, Jr.. Although they are not directly referenced here, my understanding of Bruhat-Tits theory has been deeply influenced by the beautiful papers of Allen Moy and Gopal Prasad. Finally, I thank Jeff Adler for his excellent proofreading of these notes.

2. An introduction to homogeneity

We begin with some motivations for considering homogeneity questions and try to illustrate why their answers look the way that they do.

2.1. The case GL_1**.** We begin with the completely trivial yet illuminating case of $G = k^\times = \mathrm{GL}_1(k)$ where k is a p-adic field.

We first consider homogeneity statements on $k^\times$ and then turn our attention to its Lie algebra k. Let $C_c^\infty(k^\times)$ denote the space of compactly supported, locally constant functions on $k^\times$ (similar notation applies to k).

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Suppose $\chi \in \widehat{k^\times}$, that is, χ is a complex-valued continuous multiplicative character of $k^\times$. We may define a distribution $\Theta_\chi \colon C_c^\infty(k^\times) \to \mathbb{C}$ by setting

$$\Theta_\chi(f) = \int_{k^\times} \chi(x) \cdot f(x)\, dx$$

for $f \in C_c^\infty(k^\times)$. Here dx denotes a (fixed) Haar measure on $k^\times$.

Let R denote the ring of integers of k and let $\wp$ denote the prime ideal. Fix a uniformizer ϖ (that is, $\wp = \varpi \cdot R$). To avoid complications, we suppose χ has depth $(m-1)$ with $m > 1$, that is, the restriction of χ to the filtration subgroup $1 + \wp^m$ is trivial and the restriction of χ to the filtration subgroup $1 + \wp^{(m-1)}$ is nontrivial.

Note that if the support of f is contained in $1 + \wp^m$, then

$$\Theta_\chi(f) = \Theta_1(f)$$

where Θ_1 denotes the distribution associated to the trivial character on $k^\times$. Therefore, we may write

$$\mathrm{res}_{C_c^\infty(1+\wp^m)} \Theta_\chi = \mathrm{res}_{C_c^\infty(1+\wp^m)} \Theta_1.$$

This is a homogeneity[1] statement: the distributions Θ_χ and Θ_1 agree on $C_c^\infty(1+\wp^m)$.

We now focus on the Lie algebra k of $k^\times$. We let μ_0 denote the distribution on $C_c^\infty(k)$ which sends f to $f(0)$. Suppose T is a distribution on k, that is, a linear map from $C_c^\infty(k)$ to $\mathbb{C}$. Suppose m is an integer such that T belongs to $J(\wp^m)$, the space of distributions on k having support in $\wp^m$. If f belongs to $C_c(k/\wp^m)$, the space of compactly supported functions on k which are translation invariant with respect to the lattice[2] $\wp^m$, then we can write

$$f = \sum_{\bar{X} \in k/\wp^m} f(X) \cdot [X + \wp^m]$$

where $[X + \wp^m]$ denotes the characteristic function of the coset $X + \wp^m$. For such a function we have

$$\begin{aligned} T(f) &= T\big(\sum_{\bar{X} \in k/\wp^m} f(X) \cdot [X + \wp^m]\big) = f(0) \cdot T([\wp^m]) \\ &= T([\wp^m]) \cdot \mu_0(f). \end{aligned}$$

That is, we have the homogeneity statement

$$\mathrm{res}_{C_c(k/\wp^m)} J(\wp^m) = \mathrm{res}_{C_c(k/\wp^m)} \mathbb{C} \cdot \mu_0.$$

Since $\mathrm{GL}_1(k)$ is abelian, we have not yet said anything nontrivial. The main idea you should keep in mind is that, by restricting to a subspace of a larger function space, we'd like to be able to express fairly arbitrary distributions in terms of well-understood distributions:

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[1] According to the Oxford English Dictionary [**11**], the word *homogeneity* means "identity of kind with something else," and according to Webster's Dictionary [**15**] it means "the state of having identical distribution functions or values."

[2] A compact, open R-submodule of a p-adic vector space is called a *lattice*.

STATEMENT 2.1.1.

$$\operatorname{res}_{\substack{\textit{Function}\\ \textit{space}}} \left\{ \begin{array}{c} \textit{Fairly arbitrary} \\ \textit{distributions} \end{array} \right\} = \operatorname{res}_{\substack{\textit{Function}\\ \textit{space}}} \left\{ \begin{array}{c} \textit{Well-understood} \\ \textit{distributions} \end{array} \right\}.$$

Moreover, we'd like this statement to be optimal in some sense. For example, the following exercise shows that the homogeneity statements we made above are optimal.

EXERCISE 2.1.2. Suppose $\ell \leq m < n$. Show that

$$\operatorname{res}_{C_c(k/\wp^\ell)} J(\wp^m) = \operatorname{res}_{C_c(k/\wp^\ell)} \mathbb{C} \cdot \mu_0$$

and

$$\operatorname{res}_{C_c(k/\wp^n)} J(\wp^m) \neq \operatorname{res}_{C_c(k/\wp^n)} \mathbb{C} \cdot \mu_0.$$

Formulate and prove a similar statement for distributions on $k^\times$.

2.2. Some history and an application. If we do not wish to make an optimal homogeneity statement, then the type of results we seek have been known for a long time — we shall call these "prehomogeneity" results. However, it has become clear that a great many of the interesting problems in representation theory and harmonic analysis require more precision than these prehomogeneity results provide.

Let G be a reductive p-adic group and let $\mathfrak{g}$ be the Lie algebra of G. So, for example, we could take G to be $\mathrm{SL}_n(k)$ or $\mathrm{Sp}_{2n}(k)$ and then $\mathfrak{g}$ would be $\mathfrak{sl}_n(k)$ or $\mathfrak{sp}_{2n}(k)$. If $S \subset \mathfrak{g}$, then we set

$${}^G S := \{{}^g s := \mathrm{Ad}(g)s \mid g \in G \text{ and } s \in S\}.$$

The first result we discuss is a conjecture of Howe which was proved by Howe [**8**] for the general linear group and by Harish-Chandra [**7**] in a general context.

THEOREM 2.2.1 (Howe's conjecture for the Lie algebra). *If L is a lattice in $\mathfrak{g}$ and $\omega \subset \mathfrak{g}$ is compact, then*

$$\dim_{\mathbb{C}}\bigl(\operatorname{res}_{C_c(\mathfrak{g}/L)} J(\omega)\bigr) < \infty.$$

In the statement of Howe's conjecture, the notation $J(\omega)$ denotes the space of invariant distributions[3] supported on the closure of the set ${}^G\omega$. So, for example, if $X \in \omega$, then the orbital integral μ_X belongs to $J(\omega)$. (Since $\mathrm{GL}_1(k)$ is abelian, this agrees with our earlier use of the notation J.) Note that since Howe's conjecture is not equating two sets of distributions, it is not really a homogeneity result — or even a prehomogeneity statement. However, for fixed ω and expanding L, the dimension of the left-hand side will stabilize. Thus, for sufficiently large L, we might expect to find a basis for the left hand side consisting of well-understood distributions on $\mathfrak{g}$ (see §2.3). In this section, we use the above result to prove a useful harmonic analysis result (which will later be improved using homogeneity results).

[3]A distribution T is said to be *invariant* provided that $T(f^g) = T(f)$ for all $g \in G$ and $f \in C_c^\infty(\mathfrak{g})$. Here $f^g(X) = f({}^gX)$.

Suppose that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}^{\text{r.s.s}}$. (Here $\mathfrak{g}^{\text{r.s.s}}$ denotes the set of regular semisimple elements in $\mathfrak{g}$, that is, those elements of $\mathfrak{g}$ whose centralizer in G is a torus.) We consider the map $\mathfrak{h}' \times C_c^\infty(\mathfrak{g}) \to \mathbb{C}$ defined by

$$(H, f) \mapsto \widehat{\mu}_H(f) := \mu_H(\hat{f}). \tag{*}$$

Here, we realize the Fourier transform as a map from $C_c^\infty(\mathfrak{g})$ to itself by setting

$$\hat{f}(X) = \int_{\mathfrak{g}} f(Y) \cdot \Lambda(B(Y, X))\, dY$$

where dY is a Haar measure on $\mathfrak{g}$, B is a nondegenerate, symmetric, invariant, bilinear form on $\mathfrak{g}$, and Λ is a continuous additive character of k that is trivial on the lattice $\wp$ and nontrivial on the lattice R.

There are two ways to think about the map defined by Equation (*):

(1) If we fix H and vary f, then we are looking at a distribution on $\mathfrak{g}$. It is a result of Harish-Chandra that this distribution is represented by a locally integrable function on $\mathfrak{g}$ which we also call $\widehat{\mu}_H$. This means that for all $f \in C_c^\infty(\mathfrak{g})$ we have

$$\widehat{\mu}_H(f) = \int_{\mathfrak{g}} f(Y) \cdot \widehat{\mu}_H(Y)\, dY.$$

(2) If we fix f and vary H, then we are looking at a locally constant function on $\mathfrak{h}'$.

We can combine these two ways of thinking about the map defined in Equation (*) by formulating a statement about the local constancy of the function $\widehat{\mu}_H$. Namely,

THEOREM 2.2.2 ([**7**]). *For all $H \in \mathfrak{h}'$ and for all compact open $\omega \subset \mathfrak{g}$, there exists a compact open $\omega_H \subset \mathfrak{h}'$ such that*

(1) *$H \in \omega_H$ and*
(2) *$\widehat{\mu}_{H'}(Y) = \widehat{\mu}_H(Y)$ for all $H' \in \omega_H$ and all $Y \in \omega$.*

To illustrate the usefulness of Howe's conjecture, we present here Harish-Chandra's proof of this result. In the proof, Howe's conjecture reduces a seemingly intractable problem to a simple linear algebra problem.

PROOF. Fix $H \in \mathfrak{h}'$ and $\omega \subset \mathfrak{g}$ compact and open. We begin by reformulating statement (2) of the theorem:

$$\widehat{\mu}_{H'}(Y) = \widehat{\mu}_H(Y) \text{ for all } H' \in \omega_H \text{ and all } Y \in \omega.$$

This statement is equivalent to the statement

$$\widehat{\mu}_{H'}(f) = \widehat{\mu}_H(f) \text{ for all } H' \in \omega_H \text{ and all } f \in C_c^\infty(\omega),$$

which, in turn, is equivalent to the statement

$$\mu_{H'}(\hat{f}) = \mu_H(\hat{f}) \text{ for all } H' \in \omega_H \text{ and all } f \in C_c^\infty(\omega).$$

By choosing a lattice L in $\mathfrak{g}$ so that $f \in C_c^\infty(\omega)$ implies that $\hat{f} \in C_c(\mathfrak{g}/L)$, we see that this last formulation of the statement would be true if we knew that

$$\mu_{H'}(\varphi) = \mu_H(\varphi) \text{ for all } H' \in \omega_H \text{ and all } \varphi \in C_c(\mathfrak{g}/L).$$

We will establish this last statement (which, in itself, is a type of prehomogeneity statement).

Let ω'_H be any compact open neighborhood of H in $\mathfrak{h}'$. Note that $\mu_{H'}$ belongs to $J(\omega'_H)$ for all $H' \in \omega'_H$. From Howe's conjecture for the Lie algebra, we have

$$\dim_{\mathbb{C}}\left(\operatorname{res}_{C_c(\mathfrak{g}/L)} J(\omega'_H)\right) < \infty.$$

Hence, we can choose $H_1, H_2, \ldots, H_m \in \omega'_H$ such that for every $H' \in \omega'_H$ the distribution $\operatorname{res}_{C_c(\mathfrak{g}/L)} \mu_{H'}$ belongs to the span of the linearly independent distributions

$$\operatorname{res}_{C_c(\mathfrak{g}/L)} \mu_{H_i}.$$

Fix $f_1,\ f_2,\ \ldots,\ f_m \in C_c(\mathfrak{g}/L)$ such that

$$\mu_{H_i}(f_j) = \delta_{ij}.$$

So, for all $H' \in \omega'_H$ we have

$$\mu_{H'}(f) = \sum_i \mu_{H'}(f_i) \cdot \mu_{H_i}(f)$$

for all $f \in C_c(\mathfrak{g}/L)$.

Fix a neighborhood ω_H of H for which

(1) $\omega_H \subset \omega'_H$ and
(2) $\mu_{H'}(f_i) = \mu_H(f_i)$ for all $1 \le i \le m$ and for all $H' \in \omega_H$.

We then have that

$$\begin{aligned}\mu_{H'}(f) &= \sum_i \mu_{H'}(f_i) \cdot \mu_{H_i}(f)\\ &= \sum_i \mu_H(f_i) \cdot \mu_{H_i}(f)\\ &= \mu_H(f)\end{aligned}$$

for all $f \in C_c(\mathfrak{g}/L)$ and all $H' \in \omega_H$. □

2.3. The nilpotent cone in $\mathrm{SL}_2(\mathbb{R})$. In this section we look at the $\mathrm{SL}_2(\mathbb{R})$-orbits in $\mathfrak{sl}_2(\mathbb{R})$. We do this for two reasons: First, it gives us a way to visualize[4] the problems we are discussing. Second, it will help to clear up many of the common misunderstandings the reader may harbor about how things work over non-algebraically closed fields.

As vector spaces, we have $\mathbb{R}^3 \cong \mathfrak{sl}_2(\mathbb{R})$ via the map

$$(x,y,z) \mapsto \mathrm{M}(x,y,z) := \begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix}.$$

The characteristic polynomial of $\mathrm{M}(x,y,z)$ is

$$t^2 - \left((x^2+y^2) - z^2\right),$$

and so we have three distinct types of elements depending on the eigenvalues of $\mathrm{M}(x,y,z)$ (see Table 1).

[4]It is hard to draw pictures of p-adic vector spaces; to paraphrase Paul Sally, Jr.: "We all have our own picture of the p-adics, but we dare not discuss it with others."

Type of element	$(x^2+y^2)-z^2$
nilpotent	0
split	>0
elliptic	<0

TABLE 1. Types of elements in $\mathfrak{sl}_2(\mathbb{R})$

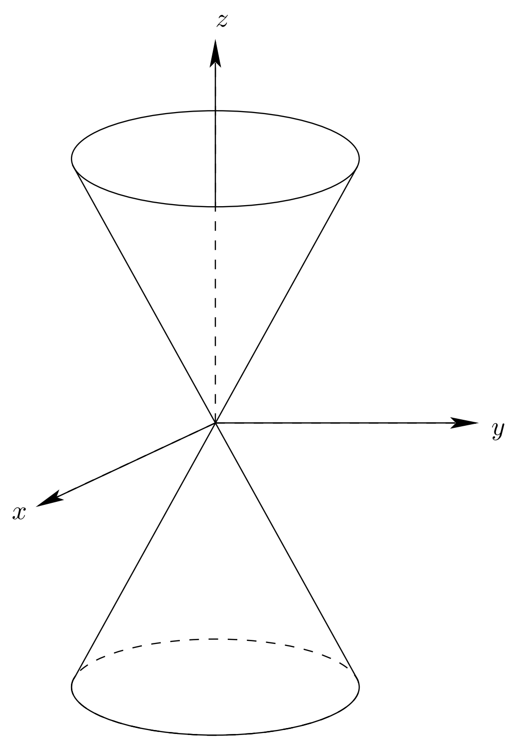

FIGURE 1. The nilpotent cone for $\mathrm{SL}_2(\mathbb{R})$

2.3.1. *Nilpotent elements.* In this case, we have $z^2 = x^2+y^2$, and so $\mathcal{N}$, the nilpotent elements, is a cone in $\mathbb{R}^3$ (see Figure 1).

We let $\mathcal{O}(0)$ denote the set of nilpotent orbits. To decompose $\mathcal{N}$ into orbits, we notice that the unit circle S^1 embeds into $\mathrm{SL}_2(\mathbb{R})$ under the map

$$\theta \mapsto s(\theta) := \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

and

$${}^{s(\theta)}M(x,y,z) = M(x\cdot\cos(2\theta)+y\cdot\sin(2\theta), y\cdot\cos(2\theta)-x\cdot\sin(2\theta), z).$$

Consequently, the set of nilpotent elements in $\mathfrak{sl}_2(\mathbb{R})$ having a fixed z value are all conjugate. From the Jacobson-Morozov theorem [**3**, §5.3], for all $X \in \mathcal{N}$ we can produce a one-parameter subgroup

$$\lambda\colon\ \mathrm{GL}_1 \to \mathrm{SL}_2$$

such that

$${}^{\lambda(t)}X = t^2 X$$

for all $t \in \mathbb{R}^\times$.

EXERCISE 2.3.1. Prove the above assertion.

Combining the action of S^1 with the above consequence of Jacobson-Morozov, we conclude that $\mathcal{O}(0)$ has at most three elements. In fact, there are exactly three nilpotent orbits.

REMARK 2.3.2. It is important to note that, except for the trivial orbit, it is not true that there is a single $g \in \mathrm{SL}_2(\mathbb{R})$ that acts by dilation on every element of a nilpotent orbit. More precisely, we know that if $\mathcal{O} \in \mathcal{O}(0)$, then

(1) $t^2\mathcal{O} = \mathcal{O}$ for all $t \in \mathbb{R}^\times$ and
(2) for each $X \in \mathcal{O}$, there is a $g_X \in \mathrm{SL}_2(\mathbb{R})$ such that ${}^{g_X}X = t^2 X$.

Consequently, if $\mu_\mathcal{O}$ denotes an invariant measure (it is unique up to a constant) on $\mathcal{O}$ and f is a nice function on $\mathcal{O}$, then

(1) for all $t \in \mathbb{R}^\times$
$$\mu_\mathcal{O}(f_{t^2}) = |t|^{-\dim(\mathcal{O})}\,\mu_\mathcal{O}(f)$$
where $f_{t^2}(Y) = f(t^2Y)$ for $Y \in \mathfrak{sl}_2(\mathbb{R})$ and
(2) for all $g \in \mathrm{SL}_2(\mathbb{R})$,
$$\mu_\mathcal{O}(f^g) = \mu_\mathcal{O}(f).$$

2.3.2. *Split and elliptic elements.* We now consider the two remaining cases. In both cases, the characteristic polynomial has distinct eigenvalues: real in the split case and complex in the elliptic case. Fix $\alpha > 0$.

We first consider the split case. The set of $\mathrm{M}(x, y, z)$ for which $\alpha^2 = z^2 - (x^2 + y^2)$ form a single orbit all of whose elements are conjugate to

$$\mathrm{M}(\alpha, 0, 0) = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}.$$

The orbit is a one sheeted hyperboloid which is asymptotic to (and outside of) the nilpotent cone.

For the elliptic case the elements $M(x, y, z)$ for which $-\alpha^2 = z^2 - (x^2 + y^2)$ form two orbits all of whose elements are conjugate to either

$$\mathrm{M}(0, 0, \alpha) = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$$

or

$$\mathrm{M}(0, -\alpha, 0) = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}.$$

Note that these two matrices are conjugate by an element of $\mathrm{SL}_2(\mathbb{C})$. These orbits form a two sheeted hyperboloid which is asymptotic to (and inside of) the nilpotent cone.

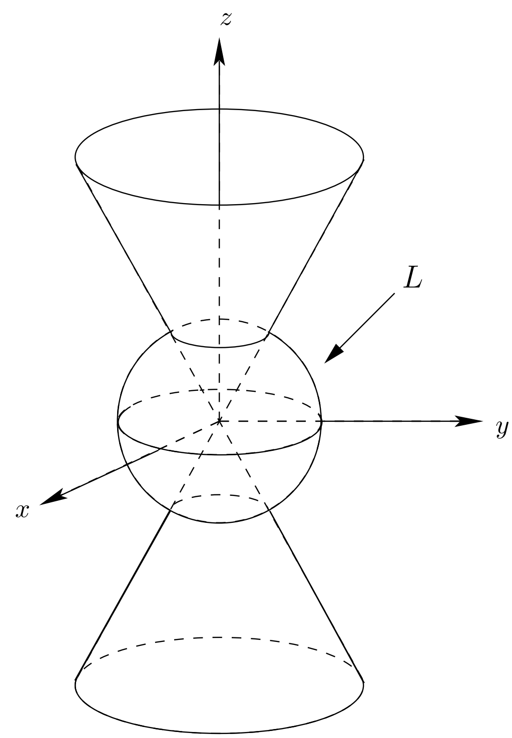

FIGURE 2. A "picture" of the lattice L

To complete our discussion of split and elliptic elements, we recall that a Cartan subalgebra (CSA) is a maximal subalgebra consisting of commuting semisimple elements. (If you prefer, you may think of a CSA as the Lie algebra of a maximal $\mathbb{R}$-torus of SL_2.) For $\mathfrak{sl}_2(\mathbb{R})$, the CSAs are one-dimensional, given by lines through the origin of the form

$$\{M(\lambda a, \lambda b, \lambda c) \mid \lambda \in \mathbb{R}\}$$

with $a^2 + b^2 \neq c^2$. We therefore recover the "standard" split CSA

$$\{M(\lambda, 0, 0) \mid \lambda \in \mathbb{R}\} = \{\left(\begin{smallmatrix} x & 0 \\ 0 & -x \end{smallmatrix}\right) \mid x \in \mathbb{R}\}$$

and the "standard" elliptic CSA

$$\{M(0, 0, \lambda) \mid \lambda \in \mathbb{R}\} = \{\left(\begin{smallmatrix} 0 & z \\ -z & 0 \end{smallmatrix}\right) \mid z \in \mathbb{R}\}.$$

2.3.3. *A return to homogeneity.* We again consider Statement 2.1.1. From the preceding discussion, it is clear (at least for $\mathfrak{sl}_2(\mathbb{R})$) that every orbit is asymptotic to the nilpotent cone. Thus, it is believable that the right-hand side of Statement 2.1.1 should, ideally, consist of nilpotent orbital integrals.

If we pretend that we can draw pictures of what the nilpotent cone looks like p-adically, then we can even visualize Statement 2.1.1 . For simplicity, let us assume that we we are interested in invariant distributions supported on the closure of ${}^{\mathrm{SL}_2(k)}L$ for the lattice L "drawn" in Figure 2

From our discussion above, we know that the closure of ${}^{\mathrm{SL}_2(k)}L$ is asymptotic to the nilpotent cone, and we "see" that, in fact,

$${}^{\mathrm{SL}_2(k)}L \subset \mathcal{N} + L.$$

(Compare this with Lemma 5.1.1.) Consequently, it is not much of a stretch to think that our homogeneity statements should look like

$$\mathrm{res}_{C_c(\mathfrak{g}/L)} J(L) = \mathrm{res}_{C_c(\mathfrak{g}/L)} J(\mathcal{N})$$

where $J(\mathcal{N})$ denotes the space of invariant distributions spanned by the nilpotent orbital integrals.

3. An introduction to some aspects of Bruhat-Tits theory

We now have a guess as to what belongs on the right-hand side of Statement 2.1.1. The purpose of this section is to introduce, via examples, enough Bruhat-Tits theory to help us refine our understanding of what to place on the left-hand side.

A good introduction to Bruhat-Tits theory may be found in Joe Rabinoff's Harvard senior thesis [**12**].

3.1. Apartments. Our immediate goal is to understand a bit of the mathematics behind the "Coxeter paper" that Bill Casselman has posted on his web page[5].

Recall that G is a p-adic group, that is, G is the group of k-rational points of a connected reductive linear algebraic k-group $\mathbf{G}$. For simplicity, we shall assume that $\mathbf{G}$ is a semisimple, k-split group which is defined over $\mathbb{Z}$. Thus, the notations $\mathbf{G}(R)$ and $\mathfrak{g}(R)$ make sense. So, for example, $\mathbf{G}$ could be Sp_{2n}, realized in the usual way.

Following earlier lecturers, we fix a maximal k-split torus $\mathbf{A}$ in $\mathbf{G}$ which is defined over $\mathbb{Z}$. We let A denote the group of k-rational points of $\mathbf{A}$. So, for example, A could be the set of diagonal matrices in $\mathrm{Sp}_{2n}(k)$.

We let $\mathcal{A} = \mathbf{X}_*(\mathbf{A}) \otimes \mathbb{R}$ and call $\mathcal{A}$ the *apartment*[6] attached to A. For the group $\mathrm{Sp}_{2n}(k)$, the apartment is isomorphic to $\mathbb{R}^n$.

An apartment carries a natural polysimplicial decomposition; we now describe how this arises. We let $\Phi = \Phi(\mathbf{G}, \mathbf{A})$ denote the set of nontrivial eigencharacters for the action of A on $\mathfrak{g}$. We assume that the valuation map $\nu\colon k^\times \to \mathbb{Z}$ is surjective, and we let $\Psi = \Psi(\mathbf{G}, \mathbf{A}, \nu)$ denote the corresponding set of affine roots, that is

$$\Psi = \{\gamma + n \,|\, \gamma \in \Phi\,,\, n \in \mathbb{Z}\}.$$

Each $\psi = \gamma + n \in \Psi$ defines an affine function on $\mathcal{A}$ by

$$(\gamma + n)(\lambda \otimes r) := r \cdot \langle \lambda, \gamma \rangle + n$$

where $\langle\ ,\ \rangle$ denotes the natural perfect pairing $\mathbf{X}_*(\mathbf{A}) \times \mathbf{X}^*(\mathbf{A}) \to \mathbb{Z}$. (Here, $\mathbf{X}^*(\mathbf{A})$ denotes the group of characters of $\mathbf{A}$.) Consequently, for each $\psi \in \Psi$, we can define the

[5] Look under Frivolities at `http://www.math.ubc.ca/people/faculty/cass/`

[6] Generally speaking, one does not want to fix (as we have) an origin.

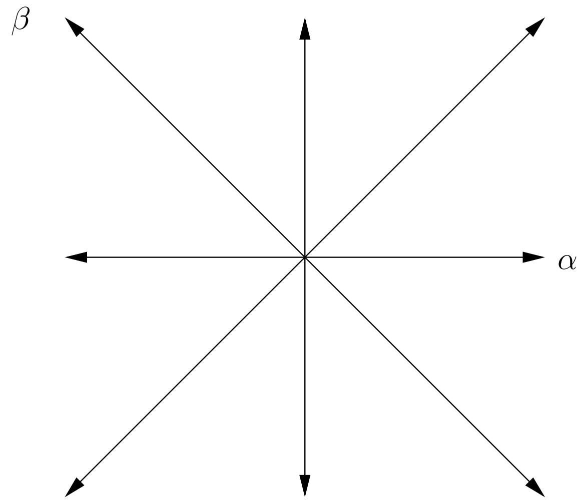

FIGURE 3. The C_2 root system

hyperplane $H_\psi := \{x \in \mathcal{A} \,|\, \psi(x) = 0\} \subset \mathcal{A}$. These hyperplanes give us the familiar polysimplicial decomposition of $\mathcal{A}$. We usually call a polysimplex occurring in this decomposition a *facet* and the maximal facets are called *alcoves*.

Finally, just as the Weyl group $W = N_G(A)/A$ acts transitively on (spherical) chambers, the extended affine Weyl group $\tilde{W} = N_G(A)/\mathbf{A}(R)$ acts transitively on alcoves (but not, in general, simply transitively — think about the image of $\left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right)$ in $\mathrm{PGL}_2(k)$ and how it acts on the standard apartment of $\mathrm{PGL}_2(k)$).

3.1.1. $\mathrm{Sp}_4(k)$ *in detail.* For this subsection only, we let $G = \mathrm{Sp}_4(k)$ realized as the subgroup of the group of 4×4 matrices of nonzero determinant which preserve

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

We take A to be the set of matrices $\{a(x,y) \,|\, x, y \in k^\times\}$ where

$$a(x,y) := \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & y^{-1} & 0 \\ 0 & 0 & 0 & x^{-1} \end{pmatrix}.$$

If we define $\alpha, \beta \in \mathbf{X}^*(\mathbf{A})$ by $\alpha(a(x,y)) = xy^{-1}$ and $\beta(a(x,y)) = y^2$, then

$$\Phi = \{\pm\alpha, \pm\beta, \pm(\beta+\alpha), \pm(\beta+2\alpha)\}$$

and the root system has the familiar diagram given in Figure 3.

The $\mathbb{Z}$-lattice of cocharacters $\mathbf{X}_*(\mathbf{A})$ is the $\mathbb{Z}$-linear span of λ_1 and λ_2 where $\lambda_1(t) = a(t,1)$ and $\lambda_2(t) = a(1,t)$ for $t \in k^\times$. In Figure 4 we have begun a sketch of the simplicial decomposition of $\mathcal{A}$ arising from the above data. The reader is encouraged to spend some time thinking about how we arrived at Figure 4.

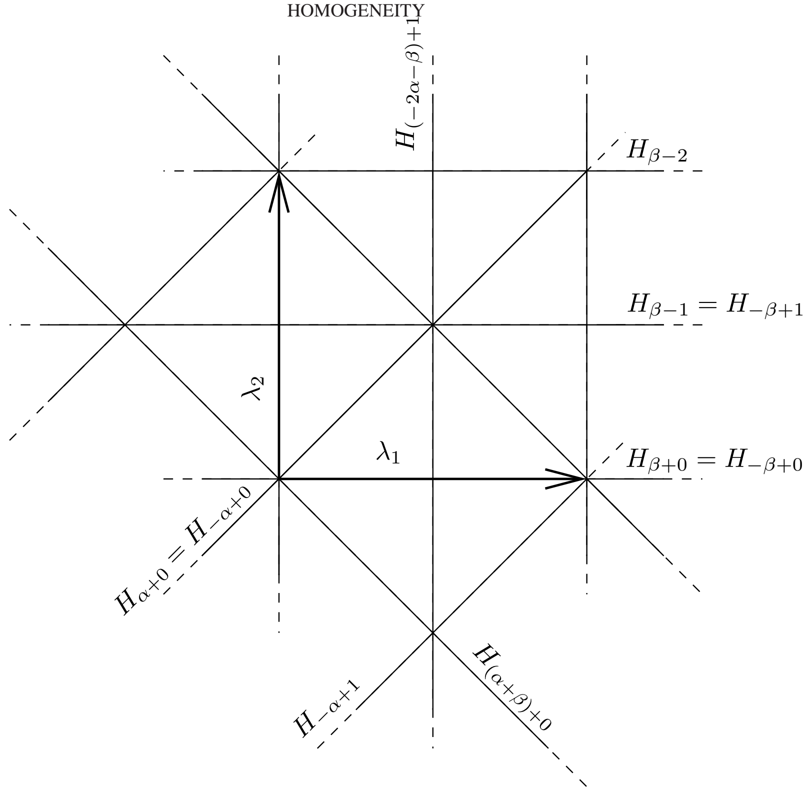

FIGURE 4. A sketch of an apartment for $\mathrm{Sp}_4(k)$

REMARK 3.1.1. For those familiar with coroots, we note that $\check{\alpha} = \lambda_1 - \lambda_2$ while $\check{\beta} = \lambda_2$.

3.2. Objects associated to facets. To each facet in $\mathcal{A}$ we can attach many types of objects. Some of these live in G, others in $\mathfrak{g}$, and still others are properly thought of as objects over $\mathfrak{f} := R/\wp$, the residue field of k. In this section, we introduce these items.

For each $\gamma \in \Phi$ we have a root group, denoted U_γ, in G and a root space, denoted $\mathfrak{g}_\gamma$, in $\mathfrak{g}$. In each case, these groups are isomorphic to k.

EXAMPLE 3.2.1. In the example of $\mathrm{Sp}_4(k)$ introduced above, we have that U_α consists of matrices of the form

$$\begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\mathfrak{g}_\alpha$ consists of 4×4 matrices of the form

$$\begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The field k carries a natural filtration, indexed by $\mathbb{Z}$, consisting of compact open subgroups:

$$k \supset \cdots \supset \wp^{-2} \supset \wp^{-1} \supset R \supset \wp \supset \wp^2 \supset \cdots \supset \{0\}.$$

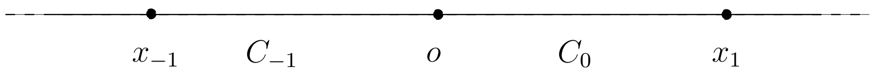

FIGURE 5. A sketch of an apartment for $\mathrm{SL}_2(k)$

We'd like to use the set $\{\gamma + n \mid n \in \mathbb{Z}\}$ to index the corresponding natural filtration in U_γ (resp. $\mathfrak{g}_\gamma$). To fix this indexing, we make the following choices:

$$U_{\gamma+1} \subsetneq U_{\gamma+0} := \mathbf{G}(R) \cap U_\gamma$$

and

$$\mathfrak{g}_{\gamma+1} \subsetneq \mathfrak{g}_{\gamma+0} := \mathfrak{g}(R) \cap \mathfrak{g}_\gamma.$$

We can now define some of the objects we are interested in. For $x \in \mathcal{A}$, we define G_x, the parahoric subgroup attached to x, by

$$G_x := \langle \mathbf{A}(R), U_\psi \rangle_{\psi \in \Psi; \psi(x) \geq 0}.$$

That is, G_x is the group generated by $\mathbf{A}(R)$ and the subgroups U_ψ for $\psi \in \Psi$ with $\psi(x) \geq 0$. Since a facet F in $\mathcal{A}$ is determined by the intersection of hyperplanes, we have $G_x = G_y$ for $x, y \in F$. Consequently, the notation G_F makes sense. If o is the origin in $\mathcal{A}$, then $G_o = \mathbf{G}(R)$.

EXAMPLE 3.2.2. We consider the case of $\mathrm{SL}_2(k)$ with A realized as the set of diagonal matrices. In Figure 5 we have sketched and labeled part of the corresponding apartment. After fixing an orientation, the parahoric subgroups associated to each facet are given[7] in the second column of Table 2.

F	G_F	G_F^+	G_F/G_F^+
x_{-1}	$\begin{pmatrix} R & \wp \\ \wp^{-1} & R \end{pmatrix}$	$1 + \begin{pmatrix} \wp & \wp^2 \\ R & \wp \end{pmatrix}$	$\mathrm{SL}_2(\mathfrak{f})$
C_{-1}	$\begin{pmatrix} R & \wp \\ R & R \end{pmatrix}$	$1 + \begin{pmatrix} \wp & \wp \\ R & \wp \end{pmatrix}$	$\mathrm{GL}_1(\mathfrak{f})$
o	$\mathrm{SL}_2(R)$	$1 + \begin{pmatrix} \wp & \wp \\ \wp & \wp \end{pmatrix}$	$\mathrm{SL}_2(\mathfrak{f})$
C_0	$\begin{pmatrix} R & R \\ \wp & R \end{pmatrix}$	$1 + \begin{pmatrix} \wp & R \\ \wp & \wp \end{pmatrix}$	$\mathrm{GL}_1(\mathfrak{f})$
x_1	$\begin{pmatrix} R & \wp^{-1} \\ \wp & R \end{pmatrix}$	$1 + \begin{pmatrix} \wp & R \\ \wp^2 & \wp \end{pmatrix}$	$\mathrm{SL}_2(\mathfrak{f})$

TABLE 2. Various groups associated to facets in $\mathrm{SL}_2(k)$

[7]For example, the notation $\begin{pmatrix} R & \wp \\ \wp^{-1} & R \end{pmatrix}$ means the group of matrices in $\mathrm{SL}_2(k)$ having entries in the indicated rings.

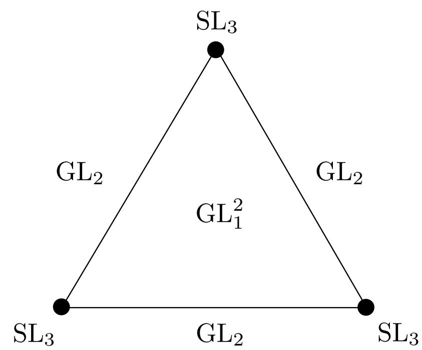

FIGURE 6. An alcove for $\mathrm{SL}_3(k)$

The parahoric G_F always has a normal subgroup G_F^+, called the pro-unipotent radical, with the property that the quotient G_F/G_F^+ is the group of $\mathfrak{f}$-rational points of a connected reductive $\mathfrak{f}$-group G_F. To define G_F^+, we must first consider the torus A. We set

$$\mathbf{A}(R)^+ := \{a \in \mathbf{A}(R) \mid \nu(\chi(a) - 1) > 0 \text{ for all } \chi \in \mathbf{X}^*(\mathbf{A})\}.$$

EXAMPLE 3.2.3. In $\mathrm{SL}_2(k)$, $\mathbf{A}(R)^+$ consists of the matrices

$$\begin{pmatrix} 1+\wp & 0 \\ 0 & 1+\wp \end{pmatrix}.$$

and in $\mathrm{Sp}_4(k)$, we have $\mathbf{A}(R)^+ := \{a(x,y) \mid x, y \in 1 + \wp\}$.

For $x \in \mathcal{A}$ we define G_x^+ by

$$G_x^+ := \langle \mathbf{A}(R)^+, U_\psi \rangle_{\psi \in \Psi; \psi(x) > 0}.$$

As before, for a facet F in $\mathcal{A}$, the notation G_F^+ makes sense. The various subgroups associated to each facet in $\mathcal{A}$ for $\mathrm{SL}_2(k)$ are given in Table 2.

It is a general fact, which is clearly exhibited in the example of $\mathrm{SL}_2(k)$, that if F_1 and F_2 are two facets for which F_1 belongs to the closure of F_2, then

$$G_{F_1}^+ < G_{F_2}^+ < G_{F_2} < G_{F_1}$$

and $G_{F_2}/G_{F_1}^+$ is a parabolic subgroup of $\mathsf{G}_{F_1}(\mathfrak{f}) = G_{F_1}/G_{F_1}^+$ with unipotent radical isomorphic to $G_{F_2}^+/G_{F_1}^+$ and Levi factor isomorphic to $\mathsf{G}_{F_2}(\mathfrak{f})$. In particular, if F_2 is an alcove, then $G_{F_2}/G_{F_1}^+$ may be identified with a Borel subgroup of $\mathsf{G}_{F_1}(\mathfrak{f})$.

We end this section with a few examples.

EXAMPLE 3.2.4. In Figure 6 we label each of the facets in a fixed alcove of an apartment for $\mathrm{SL}_3(k)$ with the name of the corresponding $\mathfrak{f}$-group.

EXAMPLE 3.2.5. In Figure 7 we label each of the facets in a fixed alcove of an apartment for $\mathrm{Sp}_4(k)$ with the name of the corresponding $\mathfrak{f}$-group.

EXAMPLE 3.2.6. In Figure 8 there is a model, produced by Joseph Rabinoff, for an alcove of $\mathrm{Sp}_6(k)$. Each of the facets has been labeled with the name of the corresponding $\mathfrak{f}$-group. This model can be quite instructive. For example, after assembling the model, one

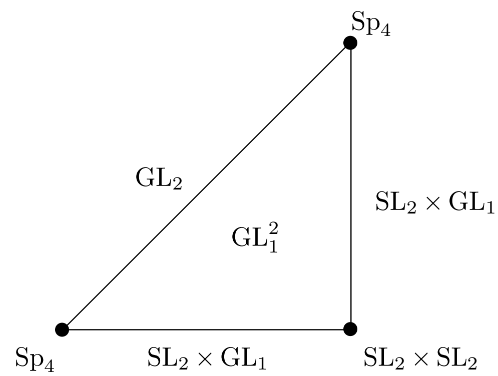

FIGURE 7. An alcove for $\mathrm{Sp}_4(k)$

sees that it can be realized as that part of a cube cut out by placing vertices at a vertex of the cube, the midpoint of an adjacent edge, the center of an adjacent face, and the center of the cube. The cube decomposes into forty-eight such solids, and the Weyl group of $\mathrm{Sp}_6(k)$ acts simply transitively on them (take the origin of $\mathcal{A}$ as the center of the cube).

All of the above can be carried out for the Lie algebra. In particular, for a facet F there is a lattice $\mathfrak{g}_F^+$ so that $\mathfrak{g}_F/\mathfrak{g}_F^+$ is $\mathsf{L}_F(\mathfrak{f}) := \mathrm{Lie}(\mathsf{G}_F)(\mathfrak{f})$, the Lie algebra of $\mathsf{G}_F(\mathfrak{f})$.

4. Parameterizations via Bruhat-Tits theory: nilpotent orbits

The main idea of this section is to relate certain aspects of the structure theory of G to the structure theory of the various finite groups of Lie type that arise naturally via Bruhat-Tits theory. We shall treat the structure theory of finite groups of Lie type as a black box. These results will play a key role in our understanding and use of the homogeneity statements to come.

4.1. A parameterization of nilpotent orbits: examples. None of the material in this section works unless p, the residual characteristic of k, is sufficiently large (as a function of the root datum of $\mathbf{G}$). We begin with an example.

EXAMPLE 4.1.1. When $p \neq 2$ the group $\mathrm{SL}_2(k)$ has five nilpotent orbits. These are represented by the elements of the set

$$\left\{ \left(\begin{smallmatrix} 0 & \theta \\ 0 & 0 \end{smallmatrix}\right) \mid \theta \in \{0, 1, \varepsilon, \varpi^{-1}, \varpi^{-1}\varepsilon\} \right\}$$

where $\varepsilon \in R^\times \setminus (R^\times)^2$. On the other hand, we have that the group $\mathrm{SL}_2(\mathfrak{f})$ has two distinguished[8] orbits

$$^{\mathrm{SL}_2(\mathfrak{f})}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad ^{\mathrm{SL}_2(\mathfrak{f})}\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$$

where $\varepsilon \in \mathfrak{f}^\times \setminus (\mathfrak{f}^\times)^2$, and $\mathrm{GL}_1(\mathfrak{f})$ has one distinguished orbit — the trivial orbit. When we encode this information in our preferred chamber, we produce a picture like Figure 9. Note that in the diagram we've included the factor ϖ^{-1} to emphasize the obvious p-adic lift. For

[8]A nilpotent orbit which does not intersect a proper Levi subalgebra is called *distinguished*.

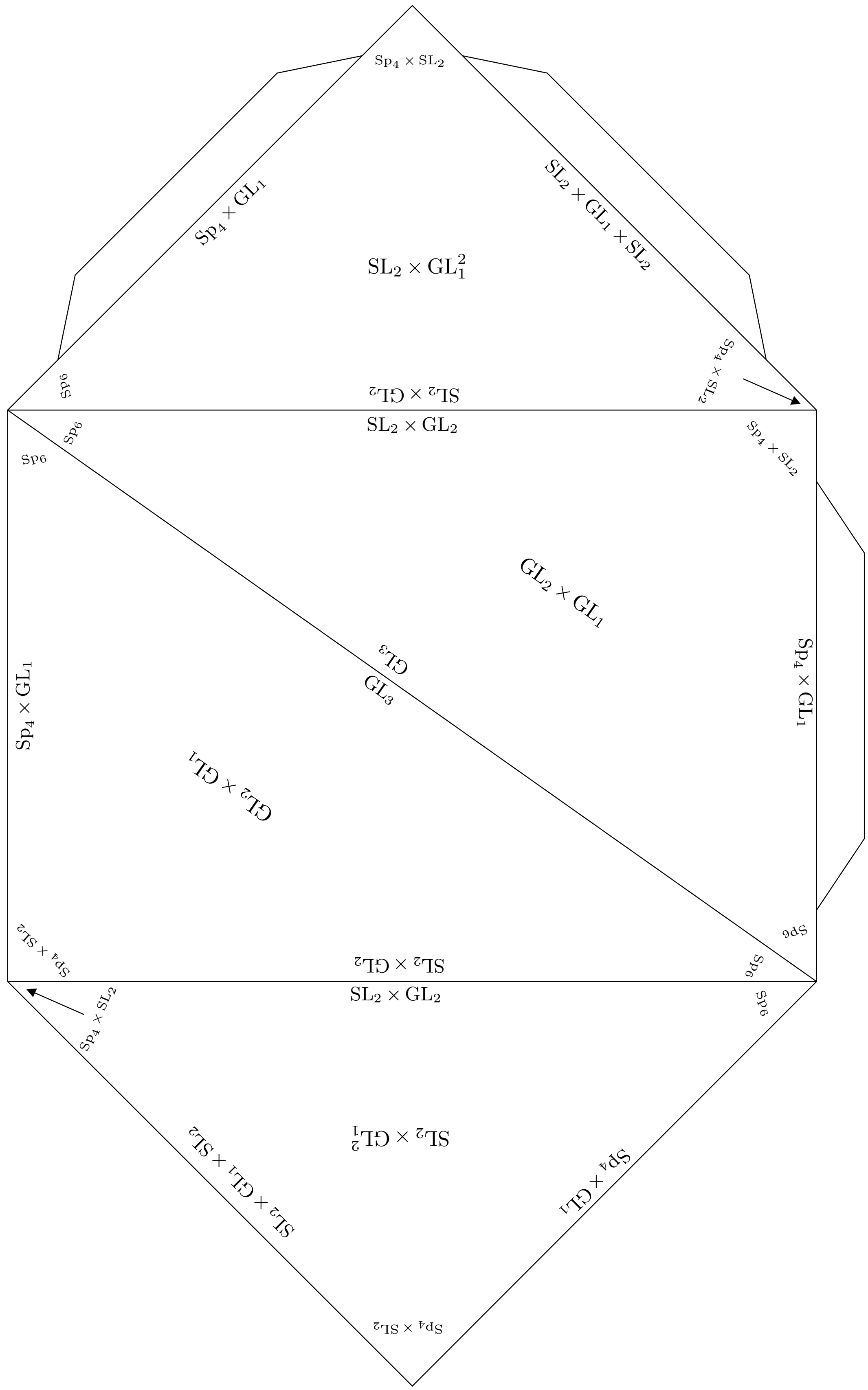

FIGURE 8. An alcove for $\mathrm{Sp}_6(k)$

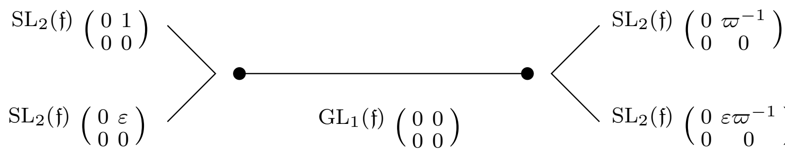

FIGURE 9. Distinguished nilpotent orbits associated to facets for $\mathrm{SL}_2(k)$.

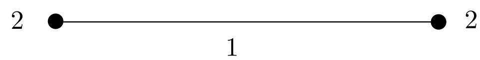

FIGURE 10. Enumeration of distinguished $\mathsf{G}_F(\mathfrak{f})$-orbits for $\mathrm{SL}_2(k)$

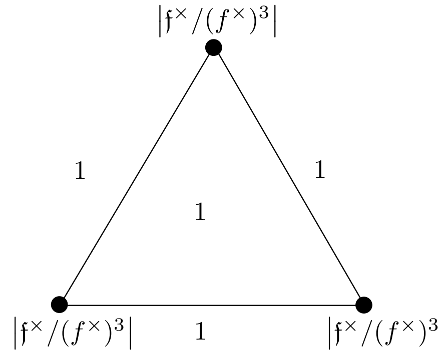

FIGURE 11. Enumeration of distinguished $\mathsf{G}_F(\mathfrak{f})$-orbits for $\mathrm{SL}_3(k)$

consistency with later examples, in Figure 10 we enumerate the distinguished $\mathsf{G}_F(\mathfrak{f})$-orbits attached to each facet in an alcove for $\mathrm{SL}_2(k)$. Note that there are five orbits enumerated in Figure 10.

The example of $\mathrm{SL}_2(k)$ indicates that there is a simple connection between $\mathcal{O}(0)$, the set of nilpotent orbits for a p-adic group, and the nilpotent orbits for Lie groups of finite type. We have the following result due to D. Barbasch and A. Moy [**2**].

FACT 4.1.2. If F is a facet and $\bar{\mathcal{O}} \subset \mathsf{L}_F(\mathfrak{f}) = \mathfrak{g}_F/\mathfrak{g}_F^+$ is a nilpotent orbit, then there exists a unique nilpotent orbit in $\mathfrak{g}$ of minimal dimension which intersects the preimage of $\bar{\mathcal{O}}$ nontrivially.

REMARK 4.1.3. The reader is urged to verify this fact for the group $\mathrm{SL}_2(k)$.

EXAMPLE 4.1.4. Since the heuristics of $\mathrm{SL}_2(k)$ worked so well, let us now turn our attention to $\mathrm{SL}_3(k)$ with $p > 3$. It is easy to see that $\mathcal{O}(0)$ has $2 + 3 \cdot \left|\mathfrak{f}^\times/(\mathfrak{f}^\times)^3\right|$ elements. On the other hand, in Figure 11 we have enumerated the number of distinguished $\mathsf{G}_F(\mathfrak{f})$-orbits in $\mathsf{L}_F(\mathfrak{f})$ for each facet in an alcove of $\mathrm{SL}_3(k)$. When we proceed without thinking (that is, we sum), we find that our indexing set has $4 + 3 \cdot \left|\mathfrak{f}^\times/(\mathfrak{f}^\times)^3\right|$ elements — two too many! However, whenever two line segments in the closure of an alcove are incident (see

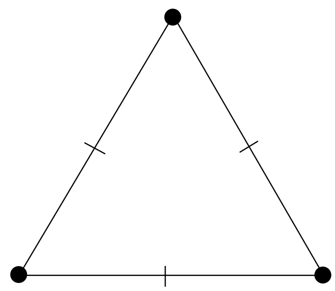

FIGURE 12. Equivalent edges in an alcove for $\mathrm{SL}_3(k)$

Figure 6), the associated general linear groups are conjugate in $\mathrm{SL}_3(\mathfrak{f})$. That is, in some real sense we are summing two too many things.

4.2. An equivalence relation on $\mathcal{A}$. We now introduce an equivalence relation on the set of facets of $\mathcal{A}$ that will account for the over counting encountered in the $\mathrm{SL}_3(k)$ example above.

DEFINITION 4.2.1. If F is a facet in $\mathcal{A}$, then we let $A(F)$ denote the smallest affine subspace of $\mathcal{A}$ containing F.

EXAMPLE 4.2.2. If F is a vertex, then $A(F)$ is the vertex itself. At the opposite extreme, if F is an alcove, then $A(F)$ is $\mathcal{A}$.

Recall that $\tilde{W} = N_G(A)/\mathbf{A}(R)$ acts transitively on the set of alcoves in $\mathcal{A}$

DEFINITION 4.2.3. Suppose F_1 and F_2 are two facets in $\mathcal{A}$. If there is a $w \in \tilde{W}$ such that

$$A(F_1) = A(wF_2),$$

then we write $F_1 \sim F_2$.

One easily verifies that the rule $\sim$ defines an equivalence relation on the set of facets in $\mathcal{A}$. Moreover, since $\tilde{W}$ acts transitively on alcoves, a set of representatives for the equivalence classes under $\sim$ can always be found among the facets occurring in the closure of a fixed alcove.

EXAMPLE 4.2.4. Here are some examples that the reader is encouraged to verify.

- Two vertices are equivalent if and only if they belong to the same $\tilde{W}$-orbit.
- If C_1 and C_2 are two alcoves in $\mathcal{A}$, then $C_1 \sim C_2$.
- For $\mathrm{SL}_2(k)$ and $\mathrm{Sp}_4(k)$, the set of facets occurring in the closure of a fixed alcove forms a complete set of representatives for the relation $\sim$.
- The only equivalent facets occurring in the closure of an alcove for $\mathrm{Sp}_6(k)$ are the two faces for which G_F is $\mathrm{GL}_2 \times \mathrm{GL}_1$.
- The only equivalent facets occurring in the closure of an alcove for $\mathrm{SL}_3(k)$ are the three edges. That is, the facets with hatch marks in Figure 12.

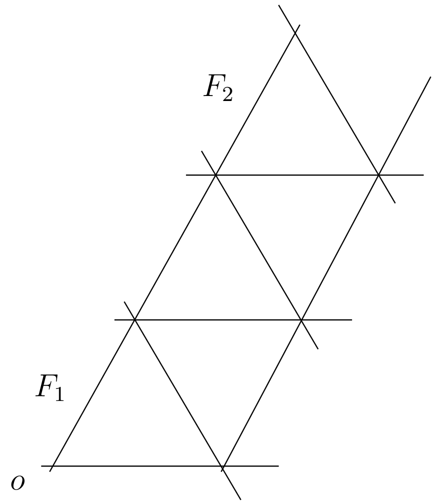

FIGURE 13. Part of the apartment for $\mathrm{SL}_3(k)$

4.3. The key idea. We now present the key ingredient that makes everything work. If F_1 and F_2 are two facets in $\mathcal{A}$ such that $A(F_1) = A(F_2)$, then the natural map

$$G_{F_1} \cap G_{F_2} \to \mathsf{G}_{F_i}(\mathfrak{f})$$

is surjective with kernel $G^+_{F_1} \cap G^+_{F_2}$. In fact, this leads to an $\mathfrak{f}$-isomorphism between G_{F_1} and G_{F_2} which we write as $\mathsf{G}_{F_1} \stackrel{i}{=} \mathsf{G}_{F_2}$ (or, for the Lie algebra, as $\mathsf{L}_{F_1} \stackrel{i}{=} \mathsf{L}_{F_2}$).

If you recall how the facets were created, then the above observation becomes less surprising. We now present an example to reinforce the idea.

EXAMPLE 4.3.1. Consider the facets F_1 and F_2 in the standard apartment for $\mathrm{SL}_3(k)$ as in Figure 13. In Table 3, we list the parahoric subgroup, its pro-unipotent radical, and

F	G_F	G_F^+	G_F
F_1	$\begin{pmatrix} R & R & R \\ R & R & R \\ \wp & \wp & R \end{pmatrix}$	$1 + \begin{pmatrix} \wp & \wp & R \\ \wp & \wp & R \\ \wp & \wp & R \end{pmatrix}$	GL_2
F_2	$\begin{pmatrix} R & R & \wp^{-2} \\ R & R & \wp^{-2} \\ \wp^3 & \wp^3 & R \end{pmatrix}$	$1 + \begin{pmatrix} \wp & \wp & \wp^{-2} \\ \wp & \wp & \wp^{-2} \\ \wp^3 & \wp^3 & \wp \end{pmatrix}$	GL_2

TABLE 3. Various groups associated to some facets for $\mathrm{SL}_3(k)$

the $\mathfrak{f}$-group associated to each of these facets. The reader may verify that this example works as advertised.

4.4. A parameterization of nilpotent orbits: the general case. We now present some definitions which allow us to extend the examples presented in §4.1.

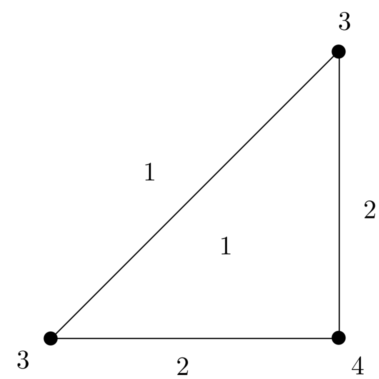

FIGURE 14. An enumeration of the distinguished $\mathsf{G}_F(\mathfrak{f})$-orbits for $\mathrm{Sp}_4(k)$.

DEFINITION 4.4.1. Let I^d denote the set of pairs $(F, \bar{\mathcal{O}})$ where F is a facet in $\mathcal{A}$ and $\bar{\mathcal{O}}$ is a distinguished $\mathsf{G}_F(\mathfrak{f})$-orbit in $\mathsf{L}_F(\mathfrak{f})$.

DEFINITION 4.4.2. Suppose $(F_1, \bar{\mathcal{O}}_1)$ and $(F_2, \bar{\mathcal{O}}_2)$ are two elements of I^d. We write $(F_1, \bar{\mathcal{O}}_1) \sim (F_2, \bar{\mathcal{O}}_2)$ provided that there exists $n \in N_G(A)$ such that

(1) $A(F_1) = A(nF_2)$ and
(2) $\bar{\mathcal{O}}_1 \stackrel{i}{=} {}^n\bar{\mathcal{O}}_2$ in $\mathsf{L}_{F_1}(\mathfrak{f}) \stackrel{i}{=} \mathsf{L}_{nF_2}(\mathfrak{f})$.

We can now state the main result for this section.

THEOREM 4.4.3 ([**6**]). *Suppose p is sufficiently large. The map that sends $(F, \bar{\mathcal{O}}) \in I^d$ to the unique nilpotent G-orbit of minimal dimension which intersects the preimage of $\bar{\mathcal{O}}$ nontrivially induces a bijective correspondence*

$$I^d/\!\sim\;\longleftrightarrow\;\mathcal{O}(0).$$

We remark that the theorem is false if p is not large enough. Consider, for example, $\mathrm{SL}_2(\mathbb{Q}_2)$.

We finish our discussion with some examples.

EXAMPLE 4.4.4. It is known that for $\mathrm{Sp}_4(k)$ and $p \neq 2$ the cardinality of $\mathcal{O}(0)$ is sixteen. We have already discussed the fact that none of the facets in the closure of a fixed alcove for $\mathrm{Sp}_4(k)$ are equivalent under $\sim$. In Figure 14 we enumerate the number of distinguished $\mathsf{G}_F(\mathfrak{f})$-orbits in $\mathsf{L}_F(\mathfrak{f})$ for each facet F in the closure of an alcove of $\mathrm{Sp}_4(k)$. As a warning to those who might wish to think further about these matters, we note that the three distinguished orbits found at each of the Sp_4 vertices arise in a somewhat surprising way: Over the algebraic closure, there is one regular nilpotent orbit and one subregular nilpotent orbit (which intersects the Lie algebra of the GL_2-Levi of Sp_4). Upon descent to the field $\mathfrak{f}$, the regular orbit breaks into two distinguished $\mathrm{Sp}_4(\mathfrak{f})$-orbits and the subregular orbit breaks into two $\mathrm{Sp}_4(\mathfrak{f})$-orbits. One of these orbits intersects the $\mathfrak{f}$-rational points of the Lie algebra of the GL_2-Levi; the other is distinguished.

EXAMPLE 4.4.5. It is known that for $\mathrm{Sp}_6(k)$ and $p \neq 2$ the cardinality of $\mathcal{O}(0)$ is forty-five. We have already discussed the fact that exactly two of the facets in the closure of a fixed alcove for $\mathrm{Sp}_6(k)$ are equivalent under $\sim$. In Table 4 we enumerate the number of

G	number of distinguished $\mathsf{G}(\mathfrak{f})$ -orbits
Sp_6	six
$\mathrm{Sp}_4 \times \mathrm{SL}_2$	six
$\mathrm{Sp}_4 \times \mathrm{GL}_1$	three
$\mathrm{SL}_2 \times \mathrm{GL}_1^2$	two
$\mathrm{SL}_2 \times \mathrm{GL}_1 \times \mathrm{SL}_2$	four
$\mathrm{SL}_2 \times \mathrm{GL}_2$	two
$\mathrm{GL}_2 \times \mathrm{GL}_1$	one
GL_3	one
GL_1^3	one

TABLE 4. An enumeration of distinguished $\mathsf{G}(\mathfrak{f})$-orbits

distinguished $\mathsf{G}_F(\mathfrak{f})$-orbits in $\mathsf{L}_F(\mathfrak{f})$ for each facet F in the closure of an alcove of $\mathrm{Sp}_6(k)$. The subsequent counting exercise is left to the reader.

Finally, we note that there does not exist a complete description of the distinguished orbits in the Lie algebra of a finite group of Lie type. But, although it seems that we have reduced one problem about which we know very little to another problem about which we also know very little, this reduction will be quite useful.

5. A precise homogeneity statement

Recall that our goal is to make Statement 2.1.1 into something reasonable and provable. In §2.3.3 we discussed the fact that the "G-orbit" of every compact set was asymptotic to the nilpotent cone. This motivated the idea that perhaps $J(\mathcal{N})$, the span of the nilpotent orbital integrals, was a reasonable candidate for the right-hand side of Statement 2.1.1. We are still searching for a candidate for the left-hand side; we begin with a very precise asymptotic result.

5.1. An asymptotic result.

LEMMA 5.1.1 ([**1**]). *For facets F_1, F_2 in $\mathcal{A}$ we have $\mathfrak{g}_{F_1} \subset \mathfrak{g}_{F_2} + \mathcal{N}$.*

EXAMPLE 5.1.2. In Figure 15 we have described the lattices $\mathfrak{g}_F$ for the standard apartment in $\mathrm{SL}_2(k)$. We observe that if F_2 lies to the left of F_1, then $\mathfrak{g}_{F_1} \subset \mathfrak{g}_{F_2} + \mathfrak{u}$ where $\mathfrak{u}$ is the set of strictly upper triangular two-by-two matrices.

PROOF. Choose $x \in F_1$ and $y \in F_2$. Let $\vec{v} = y - x$. Let Φ^+ denote the set of roots that pair nonnegatively against $\vec{v}$ and let $\Phi^- = \Phi \setminus \Phi^+$. We have

$$\sum_{\alpha \in \Phi^-} \mathfrak{g}_\alpha \subset \mathcal{N}$$

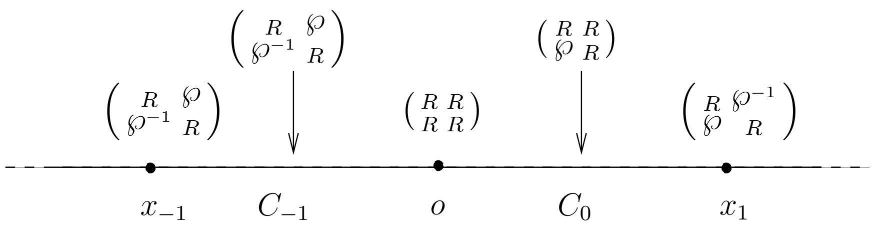

FIGURE 15. Some lattices in $\mathfrak{sl}_2(k)$

and

$$\begin{aligned}
\mathfrak{g}_{F_1} &= \mathrm{Lie}(\mathbf{A})(R) \oplus \sum_{\alpha\in\Phi;\, n\in\mathbb{Z};\, (\alpha+n)(x)>0} \mathfrak{g}_{\alpha+n} \\
&= \mathrm{Lie}(\mathbf{A})(R) \oplus \sum_{\alpha\in\Phi^+;\, n\in\mathbb{Z};\, (\alpha+n)(x)>0} \mathfrak{g}_{\alpha+n} \oplus \sum_{\alpha\in\Phi^-;\, n\in\mathbb{Z};\, (\alpha+n)(x)>0} \mathfrak{g}_{\alpha+n} \\
&\subset \mathfrak{g}_{F_2} + \sum_{\alpha\in\Phi^-;\, n\in\mathbb{Z};\, (\alpha+n)(x)>0} \mathfrak{g}_{\alpha+n} \\
&\subset \mathfrak{g}_{F_2} + \mathcal{N}.
\end{aligned}$$

The second to last line is true because if $\alpha \in \Phi^+$, then

$$\begin{aligned}
(\alpha+n)(y) = (\alpha+n)(x+\vec{v}) &= (\alpha+n)(x) + \langle \vec{v}, \alpha\rangle \\
&\geq (\alpha+n)(x).
\end{aligned}$$

□

To facilitate our discussion, we fix an alcove C in $\mathcal{A}$.

DEFINITION 5.1.3. We set

$$\mathfrak{g}_0 := \bigcup_{F\subset\bar{C}} {}^G(\mathfrak{g}_F)$$

where the union is over the facets occurring in the closure of a fixed alcove C.

The set $\mathfrak{g}_0$ is usually referred to as the set of compact elements in $\mathfrak{g}$; for $\mathrm{GL}_n(k)$ it is exactly the set of elements in $\mathrm{M}_n(k)$ for which each eigenvalue has nonnegative valuation.

COROLLARY 5.1.4. *We have* $\mathfrak{g}_0 \subset \mathfrak{g}_C + \mathcal{N}$.

PROOF. From Bruhat-Tits theory we can write

$$G = G_C \tilde{W} G_C.$$

The result follows. □

5.2. A homogeneity statement. The above asymptotic results, along with our previous discussions should, I hope, make the following homogeneity statement both natural and plausible.

THEOREM 5.2.1 ([**14**], [**4**]). *Suppose p is sufficiently large.*

(1)
$$\operatorname{res}_{C_c(\mathfrak{g}/\mathfrak{g}_C)} J(\mathfrak{g}_0) = \operatorname{res}_{C_c(\mathfrak{g}/\mathfrak{g}_C)} J(\mathcal{N}).$$

(2) *For $T \in J(\mathfrak{g}_0)$ we have*
$$\operatorname{res}_{C_c(\mathfrak{g}/\mathfrak{g}_C)} T = 0$$
if and only if
$$\operatorname{res}_{\sum_{F \subset \bar{C}} C(\mathfrak{g}_F/\mathfrak{g}_C)} T = 0.$$

The first proof of this result, for "unramified classical" groups, is due to Waldspurger [**14**]. We shall not attempt to prove this theorem, which is a special case of a much more general result. However, we do have enough tools on hand to sketch how statement (2) implies statement (1): We have
$$\operatorname{res}_{\sum_{F \subset \bar{C}} C(\mathfrak{g}_F/\mathfrak{g}_C)} T = 0$$
if and only if
$$\operatorname{res}_{\sum_{F \subset \bar{C}} C(\mathfrak{g}^+_C/\mathfrak{g}^+_F)} \widehat{T} = 0.$$
(Note, we are assuming in this statement that the form B introduced in §2.2 has certain properties — for example, that it descends to a nondegenerate, symmetric, nondegenerate, bilinear form on $\mathsf{L}_F(\mathfrak{f})$.) However, as discussed previously, $\mathfrak{g}^+_C/\mathfrak{g}^+_F$ is the nilradical of a Borel subgroup of $\mathsf{G}_F(\mathfrak{f})$. Thus
$$\operatorname{res}_{\sum_{F \subset \bar{C}} C(\mathfrak{g}_F/\mathfrak{g}_C)} T = 0$$
if and only if
$$\widehat{T}([(F, \bar{\mathcal{O}})]) = 0$$
for all $(F, \bar{\mathcal{O}}) \in I^d$ where $[(F, \bar{\mathcal{O}})]$ denotes the characteristic function of the preimage of $\bar{\mathcal{O}}$. It is then not difficult to see that this is equivalent to the statement
$$\widehat{T}([(F, \bar{\mathcal{O}})]) = 0$$
where $(F, \bar{\mathcal{O}}) \in I^d$ runs over a set of representatives for $I^d/\sim$. But from Theorem 4.4.3, this implies that the dimension of $\operatorname{res}_{C_c(\mathfrak{g}/\mathfrak{g}_C)} J(\mathfrak{g}_0)$ is less than or equal to the cardinality of $\mathcal{O}(0)$. On the other hand, $J(\mathcal{N}) \subset J(\mathfrak{g}_0)$ and from Harish-Chandra [**7**] we know that the dimension of
$$\operatorname{res}_{C_c(\mathfrak{g}/\mathfrak{g}_C)} J(\mathcal{N})$$
is equal to the number of nilpotent orbits. So (1) follows from (2).

5.3. Some applications. We present here two quick applications that are related to material presented elsewhere in this workshop. The final section of these notes is dedicated to giving a more thorough (yet still incomplete) treatment of an application.

First, the above homogeneity statement gives us a sharpened version of the Harish-Chandra–Howe local character expansion. Suppose, as usual, that p is large. Let (π, V) be an irreducible admissible representation of G. If there exists a facet F in $\mathcal{A}$ for which

$V^{G_F^+} \neq \{0\}$ (that is, (π, V) has depth zero), then there exist complex constants $c_{\mathcal{O}}(\pi)$ for which

$$\Theta_\pi(\exp(X)) = \sum_{\mathcal{O}\in\mathcal{O}(0)} c_{\mathcal{O}}(\pi) \cdot \widehat{\mu}_{\mathcal{O}}(X)$$

for all regular semisimple $X \in \mathfrak{g}_{0^+}$. Here $\mathfrak{g}_{0^+}$ denotes the set of topologically nilpotent elements, or, more precisely,

$$\mathfrak{g}_{0^+} := \bigcup_{F\subset\bar{C}} {}^G\mathfrak{g}_F^+.$$

For $\mathrm{GL}_n(k)$ the set of topologically nilpotent elements is exactly the set of elements in $\mathrm{M}_n(k)$ for which each eigenvalue has positive valuation. Note that we are also assuming that $\exp\colon \mathfrak{g}_{0^+} \to G_{0^+}$ is bijective.

Second, again assuming that p is sufficiently large, we can derive a sharpened Shalika-germ expansion. Namely, for all regular semisimple $X \in \mathfrak{g}_0$ we have

$$\widehat{\mu}_X(Y) = \sum_{\mathcal{O}\in\mathcal{O}(0)} \Gamma_{\mathcal{O}}(X) \cdot \widehat{\mu}_{\mathcal{O}}(Y)$$

for all regular semisimple $Y \in \mathfrak{g}_{0^+}$.

6. An application: stable distributions supported on the nilpotent cone

In this section, we sketch a final application of the homogeneity result stated above. This section should be thought of as an introduction to the techniques found in Waldspurger's tome [**13**].

6.1. Stability. For some purposes, the concept of stable invariance is more natural than the concept of invariance; however, the definition of stable invariance is far less natural. In order to motivate the definition of stability, we begin by recalling a result of Harish-Chandra.

We define $\mathcal{D}^{\mathrm{ann}}$ to be the space of functions that vanish on every regular semisimple orbital integral. That is

DEFINITION 6.1.1.

$$\mathcal{D}^{\mathrm{ann}} = \{f \in C_c^\infty(\mathfrak{g}) \mid \mu_X(f) = 0 \text{ for all regular semisimple } X \in \mathfrak{g}\}.$$

We then have

THEOREM 6.1.2 ([**7**]). *Suppose $T \in C_c^\infty(\mathfrak{g})^*$, that is, T is a distribution on $\mathfrak{g}$ (not necessarily invariant). We have*

$$T \text{ is invariant if and only if } \operatorname{res}_{\mathcal{D}^{\mathrm{ann}}} T = 0.$$

In other words, regular semisimple orbital integrals are dense in the space of invariant distributions. We remark that a key step in the proof is to show that $\operatorname{res}_{\mathcal{D}^{\mathrm{ann}}} \mu_{\mathcal{O}} = 0$ for each $\mathcal{O} \in \mathcal{O}(0)$.

Motivated by this result of Harish-Chandra, we can now define $J^{\mathrm{st}}(\mathfrak{g})$, the space of stably invariant distributions on $\mathfrak{g}$. We begin by introducing the idea of a stable orbital integral. Suppose $X \in \mathfrak{g}$ is regular semisimple. There is a finite set $\{X_\ell \,|\, 1 \leq \ell \leq n\}$ of regular semisimple elements in $\mathfrak{g}$ so that ${}^{\mathbf{G}(\bar{k})}X \cap \mathfrak{g}$ can be written as a disjoint union

$$^{\mathbf{G}(\bar{k})}X \cap \mathfrak{g} = {}^G X_1 \sqcup {}^G X_2 \sqcup \cdots \sqcup {}^G X_n.$$

After suitably normalizing measures, we set

$$S\mu_X = \sum_{\ell=1}^{n} \mu_{X_\ell}$$

and we call $S\mu_X$ a *stable orbital integral*.

The analogue of $\mathcal{D}^{\mathrm{ann}}$ becomes the space of functions that vanish on every stable orbital integral. That is,

DEFINITION 6.1.3.

$$\mathcal{D}^{\mathrm{stann}} := \{f \in C_c^\infty(\mathfrak{g}) \,|\, S\mu_X(f) = 0 \text{ for all regular semisimple } X \in \mathfrak{g}\}.$$

We then define

DEFINITION 6.1.4.

$$J^{\mathrm{st}}(\mathfrak{g}) := \{T \in C_c^\infty(\mathfrak{g})^* \mid \mathrm{res}_{D^{\mathrm{stann}}} T = 0\}.$$

Note that since $\mathcal{D}^{\mathrm{ann}} \subset \mathcal{D}^{\mathrm{stann}}$, every element of $J^{\mathrm{st}}(\mathfrak{g})$ is an invariant distribution on $\mathfrak{g}$.

EXAMPLE 6.1.5. Here are some examples of elements of $J^{\mathrm{st}}(\mathfrak{g})$.

- For all regular semisimple $X \in \mathfrak{g}$, the distribution $S\mu_X$ is stable.
- The distribution $\mu_{\{0\}}$ is stable.
- The distribution which sends $f \in C_c^\infty(\mathfrak{g})$ to $\int_{\mathfrak{g}} f(X)\, dX$ is stable.

Herein lies the basic problem: beyond the examples listed above, we have essentially no general understanding of $J^{\mathrm{st}}(\mathfrak{g})$. A natural first question to ask is: can we understand $J^{\mathrm{st}}(\mathcal{N}) := J(\mathcal{N}) \cap J^{\mathrm{st}}(\mathfrak{g})$? For certain unramified classical groups, Waldspurger has provided an affirmative answer to this question.

6.2. A first step towards understanding $J^{\mathrm{st}}(\mathcal{N})$. The following result, due to Waldspurger [**13**], gives us a way to tackle the problem of describing $J^{\mathrm{st}}(\mathcal{N})$. The argument is very similar to one that Harish-Chandra used to prove Theorem 6.1.2.

LEMMA 6.2.1 ([**13**]). *Suppose $T \in J(\mathfrak{g}_0)$. Let*

$$D = \sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}}(T) \cdot \mu_{\mathcal{O}}$$

(with $c_{\mathcal{O}}(T) \in \mathbb{C}$) denote the unique element in $J(\mathcal{N})$ for which

$$\mathrm{res}_{C_c(\mathfrak{g}/\mathfrak{g}_C)} T = \mathrm{res}_{C_c(\mathfrak{g}/\mathfrak{g}_C)} D.$$

If $T \in J^{st}(\mathfrak{g})$, then $D \in J^{st}(\mathcal{N})$.

PROOF. Fix $f \in \mathcal{D}^{\text{stann}}$. We need to show that $D(f) = 0$.

We note that if $t \in k^\times$, then $f_{t^2} \in \mathcal{D}^{\text{stann}}$. Choose $t \in k^\times \smallsetminus R^\times$ such that $f_{t^{2n}} \in C_c(\mathfrak{g}/\mathfrak{g}_C)$ for all $n \geq 1$. For all $n \geq 1$ we have

$$\begin{aligned} 0 &= T(f_{t^2 n}) = D(f_{t^2 n}) \\ &= \sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}}(T) \cdot \mu_{\mathcal{O}}(f_{t^{2n}}) \\ &= \sum_{i=0} |t|^{-in} \sum_{\mathcal{O} \in \mathcal{O}(0); \dim(\mathcal{O}) = i} c_{\mathcal{O}}(T) \cdot \mu_{\mathcal{O}}(f). \end{aligned}$$

Since the characters $n \mapsto |t|^{-in}$ are linearly independent, each of the terms

$$\sum_{\mathcal{O} \in \mathcal{O}(0); \dim(\mathcal{O}) = i} c_{\mathcal{O}}(T) \cdot \mu_{\mathcal{O}}(f)$$

must be zero. Consequently $D(f) = 0$. $\square$

Thus, one way to find a basis for $J^{\text{st}}(\mathcal{N})$ is to first produce a basis for $\operatorname{res}_{C_c(\mathfrak{g}/\mathfrak{g}_C)} J(\mathfrak{g}_0)$ with the properties

- the elements of the basis are of the form $\operatorname{res}_{C_c(\mathfrak{g}/\mathfrak{g}_C)} \mu_X$ with $X \in \mathfrak{g}_0$ regular semisimple, and
- we can easily describe which combinations of the μ_X are stable.

6.3. A dual basis. Fix a set of representatives $\{(F_i, \bar{\mathcal{O}}_i) \in I^d \mid 1 \leq i \leq |\mathcal{O}(0)|\}$ for $I^d/\sim$. Recall that for $T \in J(\mathfrak{g}_0)$ we have $\operatorname{res}_{C_c(\mathfrak{g}/\mathfrak{g}_C)} T = 0$ if and only if $\widehat{T}([(F_i, \bar{\mathcal{O}}_i)]) = 0$ for $1 \leq i \leq |\mathcal{O}(0)|$. Thus the Fourier transforms of the functions $[(F_i, \bar{\mathcal{O}}_i)]$ form a dual basis for $\operatorname{res}_{C_c(\mathfrak{g}/\mathfrak{g}_C)} J(\mathfrak{g}_0)$. (Note that the Fourier transform of the function $[(F, \bar{\mathcal{O}})]$ does not belong to $C_c(\mathfrak{g}/\mathfrak{g}_C)$, but, rather, it belongs to $\sum_{g \in G} C_c(\mathfrak{g}/{}^g\mathfrak{g}_C)$. However, since T is an invariant distribution, this will not cause us any difficulties.) So, the idea is to produce well-understood functions on $\mathsf{L}_F(\mathfrak{f})$ that separate distinguished nilpotent orbits and (might) have something to do with regular semisimple orbital integrals. Thanks to work of Deligne, Kazhdan, Lusztig, and others, such functions exist:

FACT 6.3.1 ([**10**]). There exist class functions on $\mathsf{L}_F(\mathfrak{f})$, called *generalized Green functions*, such that

- the functions span the set of class functions supported on the nilpotent elements in $\mathsf{L}_F(\mathfrak{f})$,
- the cuspidal[9] generalized Green functions separate distinguished orbits, and
- the functions are well understood.

[9]A function is called *cuspidal* provided that summing against the nilradical of any proper parabolic yields zero.

EXAMPLE 6.3.2. If $\mathsf{T} \leq \mathsf{G}_F$ is an $\mathfrak{f}$-minisotropic torus[10], then the usual Green function

$$Q_{\mathsf{T}}^{\mathsf{G}_F}(\bar{X}) = \begin{cases} 0 & \bar{X} \text{ is not nilpotent} \\ R_{\mathsf{T}}^{\mathsf{G}_F}(1)(\exp(\bar{X})) & \text{otherwise.} \end{cases}$$

is a cuspidal generalized Green function. Note that $\exp$ makes sense in this context because $\bar{X}$ is nilpotent, and we are assuming that p is not too small.

Note that not all cuspidal generalized Green functions occur as in this example; this is already the case for $\mathrm{SL}_2(\mathfrak{f})$.

We define I^G to be the set of pairs $(F, \mathcal{G})$ where F is a facet in $\mathcal{A}$ and $\mathcal{G}$ is a cuspidal generalized Green function on $\mathsf{L}_F(\mathfrak{f})$. As in the case of I^d, the set I^G carries a natural equivalence relation, which we also denote by $\sim$. Given the above discussion, it is not hard to believe that the following lemma is valid.

LEMMA 6.3.3 ([**13**]). *Suppose $T \in J(\mathfrak{g}_0)$. We have*

$$\operatorname{res}_{C_c(\mathfrak{g}/\mathfrak{g}_C)} T = 0 \quad \textit{if and only if } T(\hat{\mathcal{G}}_F) = 0$$

for all $(F, \mathcal{G}) \in I^G/\sim$. Here $\hat{\mathcal{G}}_F$ denotes the inflation of $\hat{\mathcal{G}}$ to a function on $\mathfrak{g}$.

6.4. A well-chosen basis for $\operatorname{res}_{C_c(\mathfrak{g}/\mathfrak{g}_C)} J(\mathfrak{g}_0)$. As discussed before, we want to find a basis for $\operatorname{res}_{C_c(\mathfrak{g}/\mathfrak{g}_C)} J(\mathfrak{g}_0)$ with several good properties. It would be even better if this basis were dual to $I^G/\sim$. As evidenced by the size of [**13**], this is quite a difficult problem. However, it is not too difficult to sketch how to carry out this program for the generalized Green functions of the form $Q_{\mathsf{T}}^{\mathsf{G}_F}$.

Fix an element of I^G of the form $(F, Q_{\mathsf{T}}^{\mathsf{G}_F})$. Choose absolutely any $X_{\mathsf{T}} \in \mathfrak{g}_F$ for which the centralizer in G_F of the image of X_{T} in L_F is T. Note that such an X_{T} is necessarily regular semisimple and $\mu_{X_{\mathsf{T}}} \in J(\mathfrak{g}_0)$. Using results of Kazhdan [**9**], Waldspurger proves

LEMMA 6.4.1 ([**13**]). *For X_{T} as above and $(F', \mathcal{G}') \in I^G$ we have*

$$\mu_{X_{\mathsf{T}}}(\hat{\mathcal{G}}_{F'}) = \begin{cases} 0 & (F', \mathcal{G}') \not\sim (F, Q_{\mathsf{T}}^{\mathsf{G}_F}) \\ N & \textit{otherwise.} \end{cases}$$

where N is a nice nonzero number which is independent of the choice of X_{T}.

As a consequence of this lemma, we have that $\operatorname{res}_{C_c^\infty(\mathfrak{g}_{0+})} \hat{\mu}_{X_{\mathsf{T}}}$ is independent of how X_{T} was chosen. This is a much stronger version of Lemma 2.2.2. To see why the elements $\operatorname{res}_{C_c^\infty(\mathfrak{g}_{0+})} \mu_{X_{\mathsf{T}}}$ are particularly nice to deal with, we must return to Bruhat-Tits theory.

[10] An $\mathfrak{f}$-torus is called $\mathfrak{f}$-*minisotropic* in G_F provided that its maximal $\mathfrak{f}$-split subtorus lies in the center of G_F.

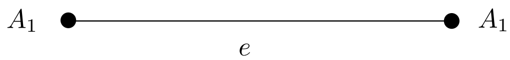

FIGURE 16. An enumeration of classes of $\mathfrak{f}$-minisotropic tori for SL_2

6.5. Parameterizing maximal unramified tori. A subgroup $T \leq G$ is called an *unramified torus* provided that it is the group of k-rational points of a torus which splits over an unramified extension of k.

EXAMPLE 6.5.1. We begin by considering some examples.

- The group A is always a maximal unramified torus.
- If $p \neq 2$ and $\varepsilon \in R^\times \smallsetminus (R^\times)^2$, then

$$\{ \left(\begin{smallmatrix} a & b \\ b\varepsilon & a \end{smallmatrix}\right) \mid a^2 - b^2\varepsilon = 1 \}$$

is a maximal unramified torus in $\mathrm{SL}_2(k)$, but the torus

$$\{ \left(\begin{smallmatrix} a & b \\ b\varpi & a \end{smallmatrix}\right) \mid a^2 - b^2\varpi = 1 \}$$

is not.

Just as we parameterized the elements of $\mathcal{O}(0)$ in terms of similar objects over the finite field, we would like to do the same for conjugacy classes of maximal unramified tori. This time, the objects over the finite field will be conjugacy classes of maximal $\mathfrak{f}$-minisotropic tori.

Suppose G is a connected $\mathfrak{f}$-split reductive group. From Carter [**3**] the $\mathsf{G}(\mathfrak{f})$-conjugacy classes of maximal $\mathfrak{f}$-tori in G are parameterized by the conjugacy classes in the Weyl group of G. We sketch how this parameterization works: Let S be a maximal $\mathfrak{f}$-split torus in G and let σ denote a topological generator for $\mathrm{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$. If T is any $\mathfrak{f}$-torus, then there is a $g \in \mathsf{G}(\bar{\mathfrak{f}})$ such that $\mathsf{T} = {}^g\mathsf{S}$. Since T and S are σ-stable, the element $\sigma(g)^{-1}g$ belongs to the normalizer of S in G and so determines a conjugacy class in the Weyl group.

The maximal $\mathfrak{f}$-minisotropic tori in G are parameterized by the anisotropic[11] conjugacy classes of the Weyl group. We shall use Carter's notation for the conjugacy classes in the Weyl group.

EXAMPLE 6.5.2. The group SL_2 has two $\mathrm{SL}_2(\mathfrak{f})$-conjugacy classes of maximal $\mathfrak{f}$-tori. One is $\mathfrak{f}$-minisotropic and corresponds to A_1 (see Figure 16), the nontrivial conjugacy class in the Weyl group, while the other is $\mathfrak{f}$-split and corresponds to the trivial conjugacy class in the Weyl group. The group GL_1 has a single $\mathrm{GL}_1(\mathfrak{f})$-conjugacy class of maximal $\mathfrak{f}$-tori, namely $\mathrm{GL}_1(\mathfrak{f})$ itself. In Figure 16 we enumerate the number of $\mathsf{G}_F(\mathfrak{f})$-conjugacy classes of $\mathfrak{f}$-minisotropic tori for each facet F in an alcove for $\mathrm{SL}_2(k)$. The sum of the enumerated classes is three, and the number of $\mathrm{SL}_2(k)$-conjugacy classes of maximal unramified tori is three. (Can you produce a representative for the third class?)

[11]A conjugacy class in a Weyl group is called *anisotropic* provided that it does not intersect a proper parabolic subgroup of the Weyl group.

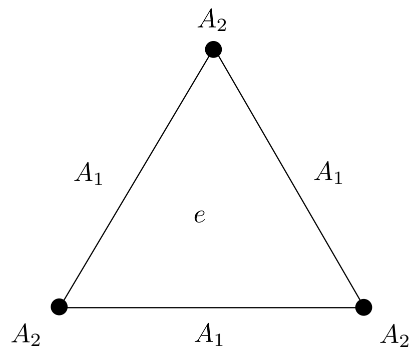

FIGURE 17. An enumeration of classes of $\mathfrak{f}$-minisotropic tori for SL_3

The map from tori over $\mathfrak{f}$ to tori over k is not as easy to describe as in the nilpotent case, but it has the advantage of working independent of the residual characteristic.

In general, we want to consider the set of pairs

$$I^m := \{(F, {}^{\mathsf{G}_F(\mathfrak{f})}\mathsf{T})\}$$

where F is a facet in $\mathcal{A}$ and ${}^{\mathsf{G}_F(\mathfrak{f})}\mathsf{T}$ is short-hand for the set of $\mathfrak{f}$-tori which are $\mathsf{G}_F(\mathfrak{f})$-conjugate to the $\mathfrak{f}$-minisotropic torus T. As with the sets I^d and I^G, the set I^m carries a natural equivalence relation, which we again denote by $\sim$.

THEOREM 6.5.3 ([**5**]). *We have a natural bijective correspondence between $I^m/\sim$ and the set of G-conjugacy classes of maximal unramified tori.*

EXAMPLE 6.5.4. The group $\mathrm{SL}_3(k)$ has five conjugacy classes of maximal unramified tori. In Figure 17 we use Carter's labeling for the conjugacy classes in the Weyl group to enumerate the $\mathsf{G}_F(\mathfrak{f})$-conjugacy classes of $\mathfrak{f}$-minisotropic tori for each facet F in an alcove for $\mathrm{SL}_3(k)$. (Recall that the line segments in the closure of the alcove are equivalent.)

EXAMPLE 6.5.5. The group $\mathrm{Sp}_4(k)$ has nine conjugacy classes of maximal unramified tori. In Figure 18 we again list the anisotropic Weyl group conjugacy classes to enumerate the $\mathsf{G}_F(\mathfrak{f})$-conjugacy classes of $\mathfrak{f}$-minisotropic tori for each facet F in an alcove for $\mathrm{Sp}_4(k)$.

6.6. The finish. To complete these notes, we remark that it is now nearly trivial to describe the number of distributions in $J^{\mathrm{st}}(\mathcal{N})$ arising from pairs of the form $(F, Q_{\mathsf{T}}^{\mathsf{G}_F})$.

In the preceding sections, we have discussed how to associate to the pair $(F, Q_{\mathsf{T}}^{\mathsf{G}_F}) \in I^G$ a regular semisimple orbital integral $\mu_{X_{\mathsf{T}}}$. On the other hand, $(F, Q_{\mathsf{T}}^{\mathsf{G}_F})$ is naturally associated to the pair (F, T) which is associated to a conjugacy class in the Weyl group of G_F. We can lift this conjugacy class to a $\tilde{W}$-conjugacy class in the extended affine Weyl group $\tilde{W}$ and then quotient by A to arrive at a conjugacy class, call it w_{T}, in W.

Suppose $(F', Q_{\mathsf{T}'}^{\mathsf{G}_{F'}})$ is another element of I^G with associated regular semisimple orbital integral $\mu_{X_{\mathsf{T}'}}$. From [**5**] the elements X_{T} and $X_{\mathsf{T}'}$ can be chosen to be stably conjugate if and only if $w_{\mathsf{T}} = w_{\mathsf{T}'}$. Consequently, to each W-conjugacy class in W we can associate

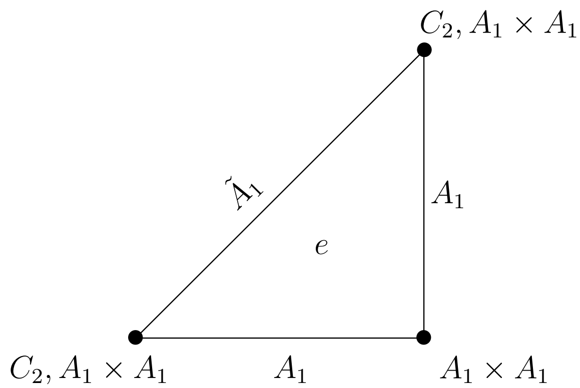

FIGURE 18. An enumeration of classes of $\mathfrak{f}$-minisotropic tori for Sp_4

one distribution in $J^{\mathrm{st}}(\mathcal{N})$. Thus, the dimension of $J^{\mathrm{st}}(\mathcal{N})$ is at least equal to the number of W-conjugacy classes in W.

EXAMPLE 6.6.1. From the above discussion, we can conclude the following. For $\mathrm{SL}_2(k)$, the dimension of $J^{\mathrm{st}}(\mathcal{N})$ is at least two (in fact, it is two). For $\mathrm{SL}_3(k)$, the dimension of $J^{\mathrm{st}}(\mathcal{N})$ is at least three (in fact, it is three). For $\mathrm{Sp}_4(k)$, the dimension of $J^{\mathrm{st}}(\mathcal{N})$ is at least five (in fact, it is six).

To describe the elements of $J^{\mathrm{st}}(\mathcal{N})$ is an entirely different and much more demanding problem. Such a description will rely on all that we have discussed here and more.

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