# Optimizing Infrastructure Design and Recovery Operations Under Stochastic Disruptions

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# Outline

# Critical Infrastructure Analysis: Literature Review

- Considered as networks with supply/demand/transshipment nodes, and service flows.
- Important to applications in energy, transportation, telecommunication, and many other areas.
- The literature includes
  - system survivability under malicious attacks, nature disasters, or component failures (e.g., Brown et al. 2006, Murray et al. 2007, San Martin 2007).
  - network design against deliberate attacks and the research of network interdiction (see, e.g., Cormican et al. 1998, Wood 1993).
  - network vulnerability (e.g., Pinar et al. 2010) and cascading failures (e.g., Crucitti et al. 2004, Nedic et al. 2006).
  - particular use in designing power grids (Faria Jr et al. 2005, Yao et al. 2007) and operations against blackouts (Alguacil et al. 2010).

# Our Problems

Combine phases of network design and operational planning, to minimize the expected costs of arc construction, flow operation, and service recovery under stochastic arc disruptions.

**Motivation:**

- The forms of service recovery vary depending on disruption severity, system interdependency, and service priority.
- For small-scale failures, local repairing can be done immediately for fully restoring service.
- During large-scale and severe damages, disconnection operations are used to avoid cascading failures.

**Two stochastic model variants:**

- Model 1 for repairing small-scale failures in a single network.
- Model 2 for avoiding large-scale cascading failures in multiple interdependent infrastructures.

# A Single Network: Notation I

Model 1 considers a single network with

- $G(\mathcal{N}, \mathcal{A}^0 \cup \mathcal{A})$: a directed connected graph with node set $\mathcal{N} = \mathcal{N}_+ \cup \mathcal{N}_= \cup \mathcal{N}_-$
- $\mathcal{N}_+$, $\mathcal{N}_=$, and $\mathcal{N}_-$: sets of supplies, intermediate transmissions, and demands.
- $\mathcal{A}^0$ and $\mathcal{A}$: the current existing arcs and potential arcs to be constructed ($\mathcal{A}^0 = \emptyset$ in this paper).

Parameters:

- $a_{ij}$, $c_{ij}$, and $d_{ij}$: flow capacity, construction cost, and unit flow cost of arc $(i, j)$, $\forall (i, j) \in \mathcal{A}$.
- $h_i$: unit generation cost of each supply node, $\forall i \in \mathcal{N}_+$.
- $S_i$: the maximum capacity of supply node $i \in \mathcal{N}_+$.
- $D_i$: consumer's demand at node $i \in \mathcal{N}_-$, with $\sum_{i \in \mathcal{N}_+} S_i \geq \sum_{i \in \mathcal{N}_-} D_i$.

# A Single Network: Notation II

- $Q$: a finite set of random disruption scenarios.
- $I_{ij}^q \in \{0, 1\}$: an effect of a disruptive event on arc $(i, j)$, $\forall (i, j) \in \mathcal{A},\ q \in Q$, where $I_{ij}^q = 0$ if arc $(i, j)$ fails, and 1 otherwise.
- $b_{ij}^q$: cost of repairing arc $(i, j)$, $\forall (i, j) \in \mathcal{A},\ q \in Q$ with $\max_{q \in Q} b_{ij}^q < c_{ij}$ by assumption, $\forall (i, j) \in \mathcal{A}$.

Decision Variables:

- $x_{ij} \in \{0, 1\}$: such that $x_{ij} = 1$ if we construct arc $(i, j)$, and 0 otherwise.
- $y_{ij}^q \in \{0, 1\}$, such that $y_{ij}^q = 1$ if arc $(i, j)$ is repaired in scenario $q$, and 0 otherwise.
- $f_{ij}^q \geq 0$: the amount of flow on arc $(i, j)$ in a repaired network, $\forall q \in Q$.

# Formulation of Model 1

$$\text{min:} \quad \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij} + \frac{1}{|Q|}\sum_{q\in Q}\left(\sum_{i\in\mathcal{N}_+} h_i g_i^q + \sum_{(i,j)\in\mathcal{A}} b_{ij}^q y_{ij}^q + \sum_{(i,j)\in\mathcal{A}} d_{ij} f_{ij}^q\right) \quad (1a)$$

$$\text{s.t.} \quad \sum_{j:(i,j)\in\mathcal{A}} f_{ij}^q - \sum_{j:(j,i)\in\mathcal{A}} f_{ji}^q - g_i^q = 0 \quad \forall i \in \mathcal{N}_+,\ q \in Q \quad (1b)$$

$$\sum_{j:(i,j)\in\mathcal{A}} f_{ij}^q - \sum_{j:(j,i)\in\mathcal{A}} f_{ji}^q = -D_i \quad \forall i \in \mathcal{N}_-,\ q \in Q \quad (1c)$$

$$\sum_{j:(i,j)\in\mathcal{A}} f_{ij}^q - \sum_{j:(j,i)\in\mathcal{A}} f_{ji}^q = 0 \quad \forall i \in \mathcal{N}_=,\ q \in Q \quad (1d)$$

$$y_{ij}^q \le x_{ij}(1 - I_{ij}^q) \quad \forall (i,j) \in \mathcal{A},\ q \in Q \quad (1e)$$

$$f_{ij}^q \le a_{ij}(I_{ij}^q x_{ij} + y_{ij}^q) \quad \forall (i,j) \in \mathcal{A},\ q \in Q \quad (1f)$$

$$0 \le g_i^q \le S_i \quad \forall i \in \mathcal{N}_+,\ q \in Q \quad (1g)$$

$$x_{ij} \in \{0,1\}\ \forall (i,j) \in \mathcal{A},\ y_{ij}^q \in \{0,1\},\ \text{and}\ \ f_{ij}^q \ge 0\ \forall (i,j) \in \mathcal{A},\ q \in Q \quad (1h)$$

where

- Variables $g_i^q$ in (1b) provide flow amount generated from supply nodes $i \in \mathcal{N}_+$.

# A Decomposition Framework

Decompose Model 1 into two stages with binary variables $x$ at the first stage, and $|Q|$ independent subproblems at the second stage.

- A relaxed master problem:

$$
\begin{aligned}
\text{min:} \quad & \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij} + \frac{1}{|Q|}\sum_{q\in Q}\eta_q \\
\text{s.t.} \quad & L_q(\eta_q, x) \geq 0 \quad \forall q \in Q \\
& x_{ij} \in \{0,1\} \quad \forall (i,j)\in\mathcal{A}, \quad \eta_q \geq \underline{\eta}^q \quad \forall q \in Q.
\end{aligned}
$$

- Given a solution $x$, subproblem $\mathbf{SP}^q(x)$-**Model 1** is

$$
\begin{aligned}
\eta_q = \text{min:} \quad & \sum_{i\in\mathcal{N}_+} h_i g_i^q + \sum_{(i,j)\in\mathcal{A}} b_{ij}^q y_{ij}^q + \sum_{(i,j)\in\mathcal{A}} d_{ij} f_{ij}^q \\
\text{s.t.} \quad & \text{(1b)–(1g)}, \\
& y_{ij}^q \in \{0,1\}, \text{ and } \ f_{ij}^q \geq 0 \quad \forall (i,j)\in\mathcal{A}.
\end{aligned}
$$

# Cutting Plane Generations I

Generate $L_q(\eta_q, x) \geq 0$ as LP-based Benders Cuts:

- Relax $y^q \geq 0$ in SP$^q(x)$-Model 1, and let $\tilde{\lambda}_i^q$, $\tilde{\alpha}_{ij}^q$, and $\tilde{\beta}_{ij}^q$ be optimal dual solutions associated with (1b)–(1d), (1e) and (1f), respectively.
- Given that SP$^q(x)$-Model 1 has a feasible solution,

$$\eta_q \geq -\sum_{(i,j)\in\mathcal{A}} \left((1 - I_{ij}^q)\tilde{\alpha}_{ij}^q + a_{ij} I_{ij}^q \tilde{\beta}_{ij}^q\right) x_{ij} - \sum_{i\in\mathcal{N}_+} \tilde{\lambda}_i^q S_i - \sum_{i\in\mathcal{N}_-} \tilde{\lambda}_i^q D_i \quad (2)$$

  is valid for all $q \in Q$.
- Proof: Weak duality theorem.

# Cutting Plane Generations II

Combine Benders cuts with Laporte-Louveaux (LL) inequalities to enforce convergence:

- Given $\hat{x}$, denote $\hat{X}^1$ as the set of arcs $\{(i,j) \in \mathcal{A} : \hat{x}_{ij} = 1\}$ and $\hat{X}^0$ as the set of arcs $\{(i,j) \in \mathcal{A} : \hat{x}_{ij} = 0\}$.
- Suppose that the current $\hat{x}$ is not optimal.
- Because at least one $x$ variable will change its current value in next iteration,

$$\sum_{(i,j)\in\hat{X}^1} (1 - x_{ij}) + \sum_{(i,j)\in\hat{X}^0} x_{ij} \geq 1. \tag{3}$$

is valid to MP.

# Modifying Model 1 for Power Systems

Apply Model 1 for optimizing design and service restoration in power transmission networks.

- Let $\theta_i$ and $\theta_j$ be voltages at locations $i$ and $j$, and $f_{ij}$ be the electricity flow between $i$ and $j$.
- The Kirchhoff's Voltage Law: $\theta_i - \theta_j = R_{ij} f_{ij}$, where $R_{ij}$ is the reactance between locations $i$ and $j$ (a DC flow model).
- Add two constraints to $\text{SP}^q(x)$-Model 1:

$$\theta_i^q - \theta_j^q \geq R_{ij} f_{ij}^q + M^+(I_{ij}^q x_{ij} + y_{ij}^q - 1) \; \forall (i,j) \in \mathcal{A} \quad (4)$$
$$\theta_i^q - \theta_j^q \leq R_{ij} f_{ij}^q - M^-(I_{ij}^q x_{ij} + y_{ij}^q - 1) \; \forall (i,j) \in \mathcal{A}, \quad (5)$$

where both $M^+$ and $M^-$ are sufficiently large numbers.

# A Penalty-based Subproblem Relaxation

- Develop valid cuts by allowing unsatisfied demands at nodes $i \in \mathcal{N}_-$.
- This variant refers to the "load shedding" operation in practice, in which the goal is to minimize costs of arc construction, repair, and the penalties incurred by unmet demands.
- $u_i^q \geq 0$: unsatisfied demands at nodes $i \in \mathcal{N}_-$.
- The new Model 1 imposes a penalty $p_i^q$ for each unit of unsatisfied demand, and formulate **R-SP**$^q(x)$**-Model 1**:

$$
\begin{aligned}
\text{min:} \quad & \sum_{i \in \mathcal{N}_+} h_i g_i^q + \sum_{i \in \mathcal{N}_-} p_i^q u_i^q + \sum_{(i,j) \in \mathcal{A}} b_{ij}^q y_{ij}^q + \sum_{(i,j) \in \mathcal{A}} d_{ij} f_{ij}^q \\
\text{s.t.} \quad & \text{(1b), (1d)–(1g), (4), (5)} \\
& -\sum_{j:(j,i) \in \mathcal{A}} f_{ji}^q - u_i^q = -D_i \quad \forall i \in \mathcal{N}_- \\
& u_i^q \geq 0, \ \forall i \in \mathcal{N}_-, \ y_{ij}^q \in \{0,1\}, \text{ and } f_{ij}^q \geq 0, \ \forall (i,j) \in \mathcal{A}.
\end{aligned}
$$

# Valid Inequalities Through Branch-and-Cut I

- Given $\hat{x}$, in subproblem $q$, branch on arc sets $\mathcal{A}^+ \subseteq \mathcal{A}$ and $\mathcal{A}^- \subseteq \mathcal{A} \setminus \mathcal{A}^+$, such that $y_{ij}^q = 1$, $\forall (i,j) \in \mathcal{A}^+$, and $y_{ij}^q = 0$, $\forall (i,j) \in \mathcal{A}^-$.
- To ensure binary $y_{ij}^q$-values for all arcs $(i,j) \in \mathcal{A}^+ \bigcup \mathcal{A}^-$ after branching, add

$$y_{ij}^q \geq 1 \quad \forall (i,j) \in \mathcal{A}^+ \tag{6}$$
$$-y_{ij}^q \geq 0 \quad \forall (i,j) \in \mathcal{A}^-. \tag{7}$$

to subproblems and compute $\eta_q$.

# Valid Inequalities Through Branch-and-Cut II

- Denote $\hat{\lambda}_i^q$, $\hat{\alpha}_{ij}^q$, $\hat{\beta}_{ij}^q$, $\hat{\pi}_{ij}^{q+}$, $\hat{\pi}_{ij}^{q-}$, $\hat{\omega}_{ij}^{q+}$, and $\hat{\omega}_{ij}^{q-}$ as optimal dual solutions to the corresponding subproblem.
- Let $M^+ = M^- = M$.
- For any $\mathcal{A}^+$ and $\mathcal{A}^-$, where $\mathcal{A}^+ \subseteq \mathcal{A}$, $\mathcal{A}^- \subseteq \mathcal{A} \setminus \mathcal{A}^+$,

$$\begin{aligned}\eta_q \geq & -\sum_{(i,j)\in\mathcal{A}} \left((1-I_{ij}^q)\hat{\alpha}_{ij}^q + a_{ij} I_{ij}^q \hat{\beta}_{ij}^q + M I_{ij}^q \left(\hat{\pi}_{ij}^{q+} + \hat{\pi}_{ij}^{q-}\right)\right) x_{ij} \\ & -\sum_{i\in\mathcal{N}_+} \hat{\lambda}_i^q S_i - \sum_{i\in\mathcal{N}_-} \hat{\lambda}_i^q D_i - M \sum_{(i,j)\in\mathcal{A}} \left(\hat{\pi}_{ij}^{q+} + \hat{\pi}_{ij}^{q-}\right) \\ & + \sum_{(i,j)\in\mathcal{A}^+} \hat{\omega}_{ij}^{q+} y_{ij}^q - \sum_{(i,j)\in\mathcal{A}^-} \hat{\omega}_{ij}^{q-} y_{ij}^q \end{aligned} \tag{8}$$

is valid to the relaxed MP of Model 1. (The proof is omitted.)

# Bounding the Big-$M$ in Cut (8)

- Any feasible solutions to (4) and (5) require that $|R_{ij}f_{ij} - M^+| = |R_{ij}f_{ij} + M^-|$ (values of both right-hand sides when $I_{ij}^q x_{ij} + y_{ij}^q - 1 = -1$) to be the maximum absolute difference between $\theta_i^q$ and $\theta_j^q$.
- Use

$$M = (|\mathcal{N}| - 1) \max_{(u,v)\in\mathcal{A}} \left\{ R_{uv} a_{uv} \right\}$$

for all node pairs, because any path between $i$ and $j$ contains no more than $|\mathcal{N}| - 1$ arcs, and the maximum voltage difference on any arc is bounded by $\max_{(u,v)\in\mathcal{A}} \left\{ R_{uv} a_{uv} \right\}$.

# Model 2: Multiple Interdependent Infrastructures

Model 2 analyzes multiple infrastructures, being *interdependent* and possessing risk of cascading failures.

- First stage: Network design and arc construction ($x$).
- Second stage: Consider two major responses after arcs are randomly destroyed:

  1. allowing load shedding at demand nodes.
  2. isolating failures by disconnecting pairs of interdependent nodes in different infrastructures.

# Notation of Model 2 I

- $K$: a set of all infrastructures.
- $\mathcal{N}^k$: contains sets of supply, transshipment, and demand nodes, denoted as $\mathcal{N}^k_+$, $\mathcal{N}^k_=$, and $\mathcal{N}^k_-$ with no common nodes.
- $\mathcal{A}^k$: a set of arcs to be constructed.
- $P(k_1, k_2)$: a set of node pairs carrying the interdependency between infrastructures $k_1$ and $k_2$, such that pair $(i, j) \in P(k_1, k_2)$ implies that node $j \in \mathcal{N}^{k_2}$ is dependent on demand node $i \in \mathcal{N}^{k_1}_-$.

# Notation of Model 2 II

Parameters:

- $a_{ij}^k$, $c_{ij}^k$, $d_{ij}^k$: the capacity, construction cost, and unit flow cost of arc $(i, j) \in \mathcal{A}^k$.
- $S_i^k$ and $D_j^k$: the maximum supply and required demand at nodes $i \in \mathcal{N}_+^k$ and $j \in \mathcal{N}_-^k$.
- $h_i^k$: unit generation cost varying at each supply node $i \in \mathcal{N}_+^k$.
- $p_i^k$: a penalty cost incurred for each unit of unsatisfied demand at node $i \in \mathcal{N}_-^k$.
- $s_{ij}^{k_1 k_2}$: fixed cost for disconnecting two interdependent nodes $i$ and $j$.
- $I_{ij}^{kq} \in \{0, 1\}$: the status of arc $(i, j) \in \mathcal{A}^k$ in scenarios $q \in Q$, where $I_{ij}^{kq} = 0$ if arc $(i, j)$ fails, and 1 otherwise.

# Notation of Model 2 III

Decision Variables:

- $x_{ij}^{k} \in \{0,1\}$: such that $x_{ij}^{k} = 1$ if we construct arc $(i,j)$ in infrastructure $k$, and 0 otherwise. (No arcs can be constructed between two nodes from different infrastructures.)
- $g_{i}^{kq}$: the amount of flow generated at node $i \in \mathcal{N}_{+}^{k}$.
- $u_{i}^{kq} \geq 0$: unsatisfied demand realized at node $i \in \mathcal{N}_{-}^{k}$.
- $z_{ijq}^{k_1 k_2} \in \{0,1\}$: such that $z_{ijq}^{k_1 k_2} = 1$ if we disconnect a parent node $i \in \mathcal{N}_{-}^{k_1}$ from its children node $j \in \mathcal{N}^{k_2}$, and 0 otherwise.
- $e_{ijq}^{k_1 k_2} \in \{0,1\}$: such that $e_{ijq}^{k_1 k_2} = 1$ means demand at node $i \in \mathcal{N}_{-}^{k_1}$ is not fully satisfied (i.e., $u_{i}^{kq} > 0$), and thus node $j \in \mathcal{N}^{k_2}$ becomes dysfunctional if it is still connected to $i$.
- $f_{ij}^{kq} \geq 0$: an amount of flow on arc $(i,j) \in \mathcal{A}^{k}$, $\forall q \in Q$.

# $\mathrm{SP}^q(x)$-Model 2

$$\text{min:} \quad \sum_{k \in K} \left( \sum_{(i,j) \in \mathcal{A}^k} d_{ij}^k f_{ij}^{kq} + \sum_{i \in \mathcal{N}_+^k} h_i^k g_i^{kq} + \sum_{i \in \mathcal{N}_-^k} p_i^k u_i^{kq} \right) + \sum_{k_1, k_2 \in K, k_1 \neq k_2} \left\{ \sum_{(i,j) \in P(k_1,k_2)} s_{ij}^{k_1 k_2} z_{ijq}^{k_1 k_2} \right\} \tag{9}$$

s.t. flow balance at all nodes in $\mathcal{N}^k$ of all infrastructure $k \in K$.

$$\sum_{j:(i,j) \in \mathcal{A}^k} f_{ij}^{kq} + \sum_{j:(j,i) \in \mathcal{A}^k} f_{ji}^{kq} \leq 2 \min \left\{ \sum_{i \in \mathcal{N}_+^k} S_i^k, \sum_{i \in \mathcal{N}_-^k} D_i^k \right\} (1 - e_{liq}^{k' k})$$

$$\forall i \in \overline{N}^k, \ k, \ k' \in K, \ k' \neq k, \ (l,i) \in P(k',k) \tag{10}$$

$$u_i^{k_1 q} \leq D_i^{k_1} (z_{ijq}^{k_1 k_2} + e_{ijq}^{k_1 k_2}) \quad \forall k_1, \ k_2 \in K, \ k_1 \neq k_2, \ (i,j) \in P(k_1, k_2) \tag{11}$$

$$f_{ij}^{kq} \leq a_{ij}^k l_{ij}^{kq} x_{ij}^k \quad \forall k \in K, \ (i,j) \in \mathcal{A}^k \tag{12}$$

$$z_{ijq}^{k_1 k_2}, \ e_{ijq}^{k_1 k_2} \in \{0,1\}, \quad \forall k_1, \ k_2 \in K, \ k_1 \neq k_2, \ (i,j) \in P(k_1, k_2)$$

$$g_i^{kq} \geq 0, \ \forall i \in \mathcal{N}_+^k, \ u_i^{kq} \geq 0, \ \forall i \in \mathcal{N}_-^k, \ f_{ij}^{kq} \geq 0, \ \forall k \in K, \ (i,j) \in \mathcal{A}^k \tag{13}$$

# Feasible solutions to Model 2

Solving Model 2 (optimally):

- Combining $\text{SP}^q(x)$-Model 2 with the master problem, we obtain an MIP model for Model 2.
- The MIP model is hard to compute due to scales of sets $P(k_1, k_2)$ for all combinations of $k_1$ and $k_2$.

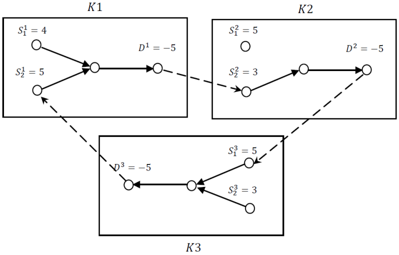
Approaches for computing lower and upper bounds of Model 2.

- Heuristic 1: generates an upper bound by disconnecting all infrastructure interdependencies *a priori* and then minimizing demand loss penalties in each infrastructure.
- Heuristic 2: aims to minimize disconnections, but will result in higher potential demand losses.
- A lower bound: by optimizing $K$ individual infrastructure design and recovery problems.

# An Example

Infrastructures $K1$, $K2$, and $K3$, in which $S_1^i$, $S_2^i$, $D^i$ represent two suppliers and one consumer in each infrastructure for $i = 1, 2, 3$.

- The interdependency sets are indicated by dash lines.
- Disconnection costs of $(D^1, S_2^2)$, $(D^2, S_1^3)$, and $(D^3, S_2^1)$ are 1, 10, and 100.
- Assume zero flow cost, zero generation cost, and \$1 penalty cost for each unit of demand losses.



# Demonstrations of Heuristics 1 and 2 I

Heuristic 1 for solving the Example:

1. Delete all interdependent arcs $(D^1, S_2^2)$, $(D^2, S_1^3)$, and $(D^3, S_2^1)$, and the disconnection cost is $1 + 10 + 100 = 111$.
2. By solving a minimum-cost flow problem in each infrastructure, we only lost two units of demand at node $D^2$, yielding a penalty cost as 2.
3. Demands $D^1$ and $D^3$ are fully satisfied, we cancel the disconnections $(D^1, S_2^2)$ and $(D^3, S_2^1)$, and the total cost of Heuristic 1 is $111 + 2 - 1 - 100 = 12$.

# Demonstrations of Heuristics 1 and 2 II

Heuristic 2 for solving the Example:

1. $\overline{N}^k \neq \emptyset$ for all $k = K1, K2, K3$.
2. Select $K2$ as an initial $k^0$ as it is the cheapest to delete $(D^1, S_2^2)$.
3. By minimizing demand loss penalties in $K2$, we obtain a solution having 2 units of unsatisfied demand at $D^2$. Given an existing interdependency $(D^2, S_1^3)$, the loss at $D^2$ sets $S_1^3$ dysfunctional.
4. Because $\overline{N}^{K3} = \emptyset$, choose $k^1 = K3$. However, as $S_1^3$ becomes dysfunctional, we again lost 2 units of demand at $D^3$, which disables $S_2^1$.
5. This further leads to one unit of demand loss at $D^1$, and the total cost is $1 + 2 + 2 + 1 = 6$.

# An Approach to Find an Objective Lower Bound

A lower bound can be found by solving a LP relaxation of Model 2.

Alternatively, we compute a lower bound by optimizing $K$ individual infrastructure design and recovery problems.

1. Set $z = 0$ for all node pairs, and minimize the total demand loss within each individual infrastructure $k \in K$.
2. That is, we ignore system interdependency, and only optimize demand losses by assuming that all nodes are functional.
3. The result has two cases:
   - If the solution conflicts with $z = 0$, it yields a lower bound of the real optimal objective cost.
   - Otherwise, i.e., the current solution is also feasible for $z = 0$, we attain optimum.

# Computations and Results

Model 1 is tested on an IEEE 118-bus system.

- Compare the effectiveness of Benders cuts (2) and cuts (8) (referred to as “BAC cuts”) for solving Model 1.
- Test a hybrid method by incorporating Benders and BAC cuts.

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For Model 2, test two- and three-infrastructure systems.

- Preserve the 118-bus system, representing a major power grid, whose demand losses might affect node functions in other smaller-scale systems (20- or 50-node networks).
- Compare MIP-Model-2, Heuristic 1, Heuristic 2, and lower-bound approaches.
- Use MIP-Model-2 to solve instances having three infrastructures, and the topology of their interdependencies varies as *Tree*, *Chain*, and *Cycle*.

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All models and algorithms use default CPLEX 12.3 with C++.

# Computing Model 1: Result Analyses

The hybrid approach randomly decides to either generate a Benders or a BAC cut, following Bernoulli trails. Time limit = 600 seconds.

| Instance | Benders (in \$1000) | | | BAC (in \$1000) | | | Hybrid (in \$1000) | | |
|---|---|---|---|---|---|---|---|---|---|
| | LB | UB | Gap (%) | LB | UB | Gap (%) | LB | UB | Gap (%) |
| Ins-2-(-2) | 31.915 | 37.456 | 17.361 | 12.855 | 38.026 | 195.816 | 30.693 | 37.404 | 21.867 |
| Ins-2-(-1) | 30.184 | 37.451 | 24.076 | 4.278 | 38.195 | 792.875 | 30.337 | 37.359 | 23.146 |
| Ins-2-(0) | 29.078 | 37.521 | 29.039 | 13.001 | 38.291 | 194.518 | 29.708 | 37.468 | 26.124 |
| Ins-2-(+1) | 27.341 | 37.930 | 38.731 | 22.173 | 38.263 | 72.563 | 27.267 | 37.860 | 38.850 |
| Ins-2-(+2) | 26.207 | 38.064 | 45.242 | 30.850 | 37.952 | 23.023 | 28.856 | 38.033 | 31.801 |
| Ins-3-(-2) | 39.861 | 41.556 | 4.254 | 7.027 | 42.369 | 502.968 | 40.344 | 41.612 | 3.144 |
| Ins-3-(-1) | 40.080 | 41.567 | 3.710 | 8.469 | 42.419 | 400.896 | 40.025 | 41.660 | 4.085 |
| Ins-3-(0) | 38.550 | 41.660 | 8.068 | 22.294 | 42.645 | 91.282 | 38.288 | 42.178 | 10.160 |
| Ins-3-(+1) | 35.720 | 42.461 | 18.870 | 39.163 | 42.453 | 8.399 | 39.367 | 42.453 | 7.838 |
| Ins-3-(+2) | 38.572 | 42.554 | 10.324 | 41.220 | 44.262 | 7.379 | 41.146 | 44.224 | 7.481 |

- Decomposition becomes more effective as the arc construction cost and the arc repair cost increase as compared to the generation cost and the flow cost.
- Benders and BAC cuts are unstable, while the hybrid cut is stable under various parameter settings.

# Computing Model 2: Setup and Parameter Design

For every two-infrastructure system,

- the 118-bus system is attached with either a 20-node or a 50-node system each having two different network layouts (i.e., "20-1," "20-2," "50-1," and "50-2").
- Any demand losses in the 118-bus system might dysfunction some nodes in the attached system.

For every three-infrastructure system,

- Attach combinations of (20-1, 20-2), (20-1, 50-1), and (50-1, 50-2) to the 118-bus system.
- Vary the topology of system interdependency as Chain, Tree, and Cycle.

# Computing Model 2: Result Analyses I

Optimizing two-infrastructure systems via different approaches:

| Instance | CPU seconds | | | | Cost (in $1000) | | | |
|---|---|---|---|---|---|---|---|---|
| | MIP-Model-2 | Heuristic 1 | Heuristic 2 | LB | MIP-Model-2 | Heuristic 1 | Heuristic 2 | LB |
| 20-1-Ins1 | 5.086 | 2.215 | 1.295 | 0.999 | 1048.800 | 100.663% | 101.101% | 99.953% |
| 20-1-Ins2 | 4.774 | 1.732 | 1.217 | 0.858 | 1059.361 | 100.657% | 101.102% | 99.984% |
| 20-1-Ins3 | 4.664 | 1.841 | 1.248 | 0.952 | 1044.174 | 100.666% | 101.170% | 99.977% |
| 20-1-Ins4 | 4.477 | 1.966 | 1.138 | 0.952 | 1058.106 | 100.656% | 101.093% | 98.661% |
| 20-2-Ins1 | 3.823 | 1.934 | 5.741 | 0.952 | 998.260 | 100.713% | 100.259% | 99.964% |
| 20-2-Ins2 | 3.338 | 1.716 | 3.089 | 0.983 | 987.793 | 100.724% | 100.240% | 99.973% |
| 20-2-Ins3 | 3.276 | 1.920 | 1.550 | 0.936 | 1010.901 | 100.708% | 100.248% | 99.978% |
| 20-2-Ins4 | 5.959 | 2.434 | 6.427 | 1.029 | 1032.633 | 100.683% | 100.264% | 99.970% |
| 50-1-Ins1 | 6.740 | 3.401 | 2.917 | 7.566 | 1062.289 | 100.726% | 100.289% | 99.974% |
| 50-1-Ins2 | 6.692 | 3.541 | 2.949 | 7.597 | 1068.586 | 100.727% | 100.152% | 99.979% |
| 50-1-Ins3 | 7.067 | 3.682 | 3.058 | 7.815 | 1072.775 | 100.723% | 100.133% | 99.978% |
| 50-1-Ins4 | 6.677 | 3.681 | 2.949 | 7.784 | 1078.481 | 100.716% | 100.190% | 99.982% |
| 50-2-Ins1 | 4.743 | 2.286 | 3.916 | 7.162 | 1052.067 | 100.695% | 100.319% | 99.982% |
| 50-2-Ins2 | 4.508 | 2.792 | 4.087 | 7.742 | 1060.748 | 100.688% | 100.317% | 99.952% |
| 50-2-Ins3 | 4.602 | 2.917 | 3.666 | 7.161 | 1053.092 | 100.692% | 100.291% | 99.979% |
| 50-2-Ins4 | 4.680 | 2.761 | 3.276 | 7.086 | 1089.856 | 100.668% | 100.361% | 99.735% |

# Computing Model 2: Result Analyses II

Optimizing three-infrastructure Tree, Chain, and Cycle:

| | CPU seconds | | | | Cost (in $1000) | | | |
|---|---|---|---|---|---|---|---|---|
| Tree | MIP-Model-2 | Heuristic 1 | Heuristic 2 | LB | MIP-Model-2 | Heuristic 1 | Heuristic 2 | LB |
| (20-1, 20-2) | 15.772 | 3.861 | 0.818 | 0.840 | 1050.067 | 101.229% | 102.331% | 99.869% |
| (20-1, 50-1) | 11.091 | 5.156 | 4.852 | 1.408 | 1070.717 | 101.254% | 100.624% | 99.936% |
| (50-1, 50-2) | 15.412 | 7.037 | 5.352 | 2.075 | 1130.676 | 101.229% | 100.261% | 99.979% |
| Chain | MIP-Model-2 | Heuristic 1 | Heuristic 2 | LB | MIP-Model-2 | Heuristic 1 | Heurisitc2 | LB |
| (20-1, 20-2) | 4.290 | 1.956 | 0.742 | 1.008 | 1049.434 | 100.690% | 102.393% | 99.930% |
| (20-1, 50-1) | 6.770 | 3.201 | 2.345 | 2.535 | 1070.645 | 100.716% | 100.365% | 99.943% |
| (50-1, 50-2) | 10.404 | 4.519 | 4.279 | 1.892 | 1130.626 | 100.653% | 100.190% | 99.983% |
| Cycle | MIP-Model-2 | Heuristic 1 | Heuristic 2 | LB | MIP-Model-2 | Heuristic 1 | Heurisitc2 | LB |
| (20-1, 20-2) | 5.640 | 2.175 | 0.571 | 0.819 | 1049.638 | 100.724% | 123.563% | 99.910% |
| (20-1, 50-1) | 6.650 | 3.565 | 2.395 | 1.698 | 1070.645 | 100.573% | 100.365% | 99.943% |
| (50-1, 50-2) | 11.889 | 6.004 | 4.523 | 2.331 | 1130.626 | 101.051% | 100.190% | 99.983% |

# Computing Model 2: Result Analyses III

- CPU time of Heuristics 1, 2, and the LB method are much shorter than MIP-Model-2.
- Both (20-1, 50-1) and (50-1, 50-2) have the same cost in MIP-Model-2 for Chain and Cycle, indicating that the 118-bus system dominates all three systems, and the feedback interdependency in a Cycle from either 50-1 or 50-2 to the 118-bus system is negligible in our computations.
- Overall, we do not observe much solution difference among Tree, Chain, and Cycle-structured systems.
- Both Heuristic 1 and Heuristic 2 yield slightly worse bounds than testing two-infrastructure systems, because more interdependency variables are pre-fixed or relaxed by the heuristic approaches given more sub-networks.

# Conclusions

- Investigate problems of critical infrastructure design and recovery optimization under random network arc disruptions.
- Consider both small-scale failures in a single network, and large-scale cascading failures in multiple interdependent infrastructures.
- Model 1 (small networks): (i) complicated by Big-$M$ constraints yielded by the Kirchhoff's Voltage Law for specifically modeling power transmission networks; (ii) solved by LP-based Benders cuts and a Branch-and-Cut algorithm.
- Model 2 (multiple infrastructures): we develop an MIP and heuristic approaches for bounding the optimal objectives.

Future research:

- Risk variants of Model 1 and Model 2.
- Specially-structured topologies of interdependency among multiple infrastructures.