Facility Location with Competition or Decision-dependent Uncertainty: Models, Algorithms and Extensions

Siqian Shen

Associate Professor
Department of Industrial and Operations Engineering
University of Michigan

Related Papers

Competitive Facility Location:

- Mingyao Qi, Ruiwei Jiang, Siqian Shen, "Sequential Competitive Facility Location: Exact and Approximate Algorithms," to appear in *Operations Research*, 2022. https://arxiv.org/abs/2103.04259

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Facility Location with Decision-dependent Stochastic Demand:

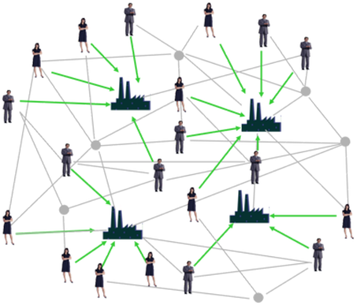
- Beste Basciftci, Shabbir Ahmed, Siqian Shen. "Distributionally robust facility location problem under decision-dependent stochastic demand," *European Journal of Operational Research*, 292(2), 548-561, 2021.

Outline

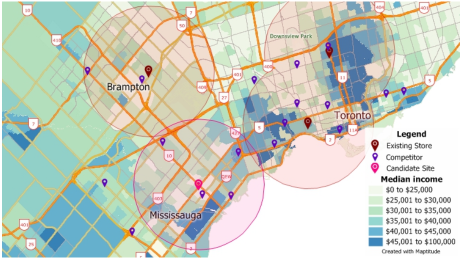
1. Facility Location Problem (FLP) with Competition
2. Formulations and Algorithms
   - Bilevel Program and Single-level Reformulation
   - Branch-and-Cut Framework
   - Two Types of Valid Inequalities
   - Model Extensions
3. S-CFLP: Computational Studies
   - Experimental Design
   - Computational Results
4. FLP with Endogenous Demand
   - Problem Description
   - Ambiguity Set
   - Formulation and Valid Inequalities
   - Numerical Studies

Facility Location Problem (FLP) with Competition

- FLP is fundamental for designing and operating transportation and logistics systems.
- **Traditional FLP:** Open facilities to
  - satisfy demand
  - minimize cost



Competitive FLP:



Source: `caliper.com`

Competitive Facility Location and Customer Choices

Competitive Facility Location

- Static - locate to maximize market share given others' existing facilities
  - Benati and Hansen (2002), Haase and Müller (2014), Ljubić and Moreno (2018), Mai and Lodi (2020)
- Sequential - a leader locates and then follower selects from remaining ones
  - Eiselt and Laporte (1997), Plastria and Vanhaverbeke (2008), Küçükaydn et al. (2011, 2012), Kress and Pesch (2012), Drezner et al. (2015), Gentile et al. (2018)
  - Static CFLP can be viewed as the follower's problem in sequential CFLP.
- Dynamic - competing firms play until a Nash equilibrium (if any) is established

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Customer Behavior (utility-based facility choices)

- Deterministic choice + sequential CFLP ⇒ mixed-integer linear program (MILP)
- Probabilistic choice + static CFLP ⇒ mixed-integer nonlinear program (MINLP) (well studied)
- Probabilistic choice + sequential CFLP ⇒ bilevel program with both levels MINLP (focus of this paper)

Literature Review

Table 1 Comparison with existing work on sequential CFLP

Reference	Choice Model	Formulation (Upper/Lower)	Solution Approach	Location Space
Drezner and Drezner (1998)	Probabilistic	Bilevel (NLP/NLP)	Heuristic	Planar
Serra and ReVelle (1994)	Deterministic	MILP	Heuristic	Discrete
Fischer (2002)	Deterministic	Bilevel (MINLP/MILP)	Heuristic	Discrete
Plastria and Vanhaverbeke (2008)	Deterministic	MILP	Exact; commercial solver	Discrete
Sáiz et al. (2009)	Probabilistic	Bilevel (NLP/NLP)	Exact; branch-and-bound	Planar
Küçükaydn et al. (2011)	Probabilistic	Bilevel (MINLP/NLP)	Exact; single-level MINLP reformulation	Discrete
Küçükaydn et al. (2012)	Probabilistic	Bilevel (MINLP/MINLP)	Heuristic	Discrete
Roboredo and Pessoa (2013)	Deterministic	MILP	Exact; branch-and-cut	Discrete
Alekseeva et al. (2015)	Deterministic	MILP	Exact; iterative method	Discrete
Drezner et al. (2015)	Deterministic	Bilevel (MINLP/MILP)	Heuristic	Discrete
Gentile et al. (2018)	Deterministic	MILP	Exact; branch-and-cut	Discrete
This paper	Probabilistic	Bilevel (MINLP/MINLP)	Exact and approximate; branch-and-cut	Discrete

Studies on solving bilevel integer programs:

- Tahernejad et al. (2020), Bolusani and Ralphs (2021), Bolusani et al. (2021)

Studies on cutting plane methods for CFLP:

- Ljubić and Moreno (2018), Mai and Lodi (2020)

Outline

Notation I

Sets

- J^{L}: Set of existing facilities by the leader;
- J^{F}: Set of existing facilities by the follower;
- J: Set of candidate facility locations;

Decision Variables

- x_j: binary variable modeling leader's decision $(j \in J)$
- y_j: binary variable modeling the follower's decision $(j \in J)$
- J^0: Sets of facilities open by either the leader or the follower.

Notation II

Parameter

- p: maximum # of facilities that can be located by the leader;
- r: maximum # of facilities located by the follower. $(p + r \leq |J|)$
- h_i: demand portion at customer location $i \in I$. $(\sum_{i \in I} h_i = 1)$
- P_{ij}: the probability that customer i patronizes facility j.
  - We assume that it follows a *multinomial logit (MNL) model*, and thus

$$P_{ij} = \frac{\exp\{\alpha_j - \beta d_{ij}\}}{\sum_{k \in J^0} \exp\{\alpha_k - \beta d_{ik}\}}$$

  - $w_{ij} = \exp\{\alpha_j - \beta d_{ij}\}$ is the utility of customer in i choosing j; it can depend on, e.g., distance between i and j.
- $U_i^{\mathrm{L}} := \sum_{j \in J^{\mathrm{L}}} w_{ij}$, $U_i^{\mathrm{F}} := \sum_{j \in J^{\mathrm{F}}} w_{ij}$: utility of the pre-existing facilities already open by leader and follower, respectively.

Bilevel Competition Game

Leader's market share:

$$L^{+}(x,y) = \sum_{i \in I} h_i \left(\frac{U_i^{\mathrm{L}} + \sum_{j \in J} w_{ij} x_j}{U_i^{\mathrm{L}} + U_i^{\mathrm{F}} + \sum_{j \in J} w_{ij}(x_j + y_j)} \right).$$

The following bilevel program describes sequential CFLP:

$$\text{(S-CFLP)} \quad \max_{x} \ L^{+}(x, y^*) \tag{1a}$$

$$\text{s.t.} \ \sum_{j \in J} x_j \le p, \ x_j \in \{0,1\}, \ \forall j \in J, \tag{1b}$$

$$\text{where} \quad y^* \in argmax \sum_{i \in I} h_i \left(\frac{U_i^{\mathrm{F}} + \sum_{j \in J} w_{ij} y_j}{U_i^{\mathrm{L}} + U_i^{\mathrm{F}} + \sum_{j \in J} w_{ij}(x_j + y_j)} \right) \tag{1c}$$

$$\text{s.t.} \ \sum_{j \in J} y_j \le r, \tag{1d}$$

$$y_j \le 1 - x_j, \ \forall j \in J, \tag{1e}$$

$$y_j \in \{0,1\}, \ \forall j \in J. \tag{1f}$$

The leader's feasible region: $\mathcal{X} := \left\{ x \in \{0,1\}^{|J|} : (1b) \right\}$ and the follower's feasible region $Y(x) := \left\{ y \in \{0,1\}^{|J|} : (1d)–(1f) \right\}$.

Max-min Reformulation

- The leader's and the follower's objective functions sum up to 1 $\Rightarrow$ for any $x \in \mathcal{X}$, follower finds $y^* \in argmin_{y \in Y(x)} \{L^+(x, y)\}$.
- (S-CFLP) is equivalent to (i.e., a robust optimization view):

$$\max_{x \in \mathcal{X}} \min_{y \in Y(x)} L^+(x, y).$$

  - Cannot take dual of the inner problem due to lack of strong duality (binary y).
  - If applying delayed constraint generation (DCG), cuts will be invalid because $Y(x)$ depends on specific x.

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  - Cannot take dual of the inner problem due to lack of strong duality (binary y).
  - If applying delayed constraint generation (DCG), cuts will be invalid because $Y(x)$ depends on specific x.

- **Key idea**: To revise $L^+(x,y)$ so that the feasible region for y in the inner problem can be independent of x.

Single-level Reformulation

For any $a, b \in \mathbb{R}$, we define $a \vee b := \max\{a, b\}$.

Theorem 2 (Qi, Jiang, S. (2021))

Define $\theta^+, \theta : \{0,1\}^{|J|} \to \mathbb{R}$ such that $\theta^+(x) := \min_{y \in Y(x)} L^+(x, y)$ and $\theta(x) := \min_y \{L(x, y) : \text{(1d), (1f)}\}$, where

$$L(x, y) := \sum_{i \in I} h_i \left(\frac{U^{\mathrm{L}} + \sum_{j \in J} w_{ij} x_j}{U^{\mathrm{L}} + U^{\mathrm{F}} + \sum_{j \in J} w_{ij}(x_j \vee y_j)} \right).$$

Then, $\theta^+(x) = \theta(x)$ for all $x \in \mathcal{X}$.

Therefore, the bilevel (**S-CFLP**) model is equivalent to

$$\max_{x \in \mathcal{X}, \theta} \quad \theta \tag{2a}$$

$$\text{s.t.} \quad \theta \le L(x, y), \quad \forall y \in \mathcal{Y} = \left\{ y \in \{0,1\}^{|J|} : \sum_{j \in J} y_j \le r \right\}. \tag{2b}$$

and can be solved via separation and cutting-plane procedures (i.e., DCG).

Outline

DCG algorithm $\Rightarrow$ Branch-and-Cut

Two challenges for applying DCG:

- The violated constraints we incorporate, $\theta \leq L(x, \hat{y})$, are nonlinear.
- Function $L(x, \hat{y})$ is non-concave in x in its current form. That is, the relaxed formulation remains a non-convex NLP even if integer variables are relaxed.

DCG algorithm ⇒ Branch-and-Cut

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Solutions:

- Two types of linear valid inequalities to generate a tight MILP relaxation of the nonlinear, non-convex model (2).
  - Submodular inequalities
  - Bulge Inequalities
- The separation problem $\min_y \{L(\hat{x}, y) : (1d), (1f)\}$ for any given $\hat{x}$ is MINLP ⇒ apply a fast approximation using single-round sorting.

The valid inequalities and approximate separation are integrated in a branch-and-cut framework for solving S-CFLP.

Outline

Submodular Inequalities

Definitions: Submodular Functions (diminishing returns)

A function $f : 2^J \to \mathbb{R}$ is submodular if

$$f(S \cup \{j\}) - f(S) \geq f(R \cup \{j\}) - f(R)$$

for all subsets $S \subseteq R \subseteq J$ and all element $j \in J \setminus R$.

- For any fixed $y \in \mathcal{Y}$, the function $L(x, y)$ is submodular with respect to the index set X of x, i.e., $X := \{j \in J : x_j = 1\}$ (Proposition 1 in Qi, Jiang, S. (2021)).

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- Then we can represent the nonlinear constraint $\theta \leq L(x, y)$ as a set of linear inequalities (Proposition 2; Nemhauser and Wolsey (1981)).
- However, the number of submodular cuts can be exponential $\Rightarrow$ we show that we can find the most violated submodular cut for any given $\hat{x}$ efficiently, in polynomial time (Proposition 3).

Bulge Inequalities

To develop cuts based on outer approximation, define:

$$\tilde{L}(x,y) := \sum_{i \in I} h_i \left(\frac{U_i^{\mathrm{L}} + \sum_{j \in J} w_{ij} x_j}{U_i^{\mathrm{L}} + U_i^{\mathrm{F}} + \sum_{j \in J} w_{ij} \left[(1 - y_j) x_j + y_j \right]} \right),$$

- by replacing $x_j \vee y_j$ in $L(x,y)$ with $(1 - y_j)x_j + y_j$.
- Note: $L(x,y)$ coincides with $\tilde{L}(x,y)$ whenever (x,y) are binary-valued, but $\tilde{L}(x,y)$ is well-defined on $[0,1]^{2|J|}$.

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- So, for fixed $y \in \mathcal{Y}$, we can replace $\theta \leq L(x,y)$ with a supporting hyperplane if $\tilde{L}(x,y)$ is concave in x. $\Rightarrow$ Unfortunately, $\tilde{L}(x,y)$ is not concave in x! (Example 1)

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We "bulge up" $\tilde{L}(x,y)$ to obtain a concave hypograph.

Proposition 4 (Qi, Jiang, S. (2021))

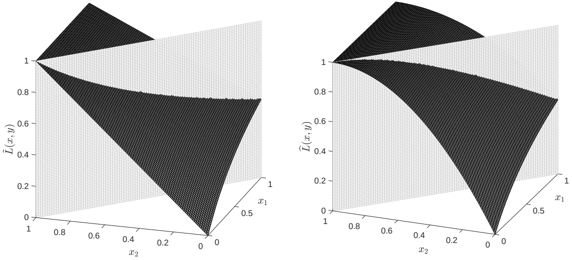
Define:

$$\widehat{L}(x,y) := \sum_{i\in I} h_i \left(\frac{U_i^{\text{L}} + \sum_{j\in J} w_{ij}\left[-y_j x_j^2 + (1+y_j)x_j\right]}{U_i^{\text{L}} + U_i^{\text{F}} + \sum_{j\in J} w_{ij}\left[(1-y_j)x_j + y_j\right]} \right). \quad (3)$$

Then, $\widehat{L}(x,y)$ is concave in x. In addition, $\widehat{L}(x,y) = L(x,y)$ for all $x \in \{0,1\}^{|J|}$.

Illustration

$\widehat{L}$ bulge up L to make it concave while retaining exactness at any binary x.



(a) $\tilde{L}(x, y)$ (b) $\widehat{L}(x, y)$

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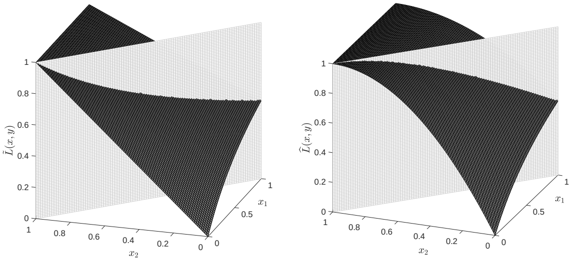


(a) $\tilde{L}(x,y)$ (b) $\widehat{L}(x,y)$

In DCG, given $(\hat{x},\hat{y})$, replace constraints $\theta \le L(x,y)$ with a supporting hyperplane of $\widehat{L}(x,y)$:

$$\theta \le \widehat{L}(\hat{x},\hat{y}) + \sum_{j\in J} g_j(\hat{x},\hat{y})(x_j - \hat{x}_j), \tag{4}$$

$$\text{where } g_j(\hat{x},\hat{y}) := \frac{\partial \widehat{L}(x,\hat{y})}{\partial x_j}\Big|_{x=\hat{x}} = \sum_{i\in I} h_i \left(\frac{-w_{ik}(1-\hat{y}_k)Q}{P^2} + \frac{w_{ik}(-2\hat{y}_k\hat{x}_k + 1 + \hat{y}_k)}{P} \right),$$

with $P = U_i^{\mathrm{L}} + U_i^{\mathrm{F}} + \sum_{j\in J} w_{ij}\left[(1-\hat{y}_j)\hat{x}_j + \hat{y}_j\right]$, and $Q = U_i^{\mathrm{L}} + \sum_{j\in J} w_{ij}\left[-\hat{y}_j\hat{x}_j^2 + (1+\hat{y}_j)\hat{x}_j\right]$.

Outline

S-CFLP: Model Extensions I

Heterogeneous setup costs:

- Replace cardinality budget constraints with general 0-1 knapsack constraints.
- All the solution approaches remain applicable in this extension.

Co-optimize locations and attractiveness levels:

- Replicate each site j, which now consists of a set N_j of potential facilities to build, and all these facilities share the same distances d_{ij} to demand nodes.
- The replicated facilities differ in the attractiveness level, denoted by α_{jn}, and setup cost, denoted by c_{jn}^{L} (for the leader) and c_{jn}^{F} (for the follower), $\forall n \in N_j$.
- Accordingly, we extend decision variables (x_j, y_j) to be (x_{jn}, y_{jn}) to reflect the attractiveness level choice and add constraints $\sum_{n \in N_j} x_{jn} \leq 1, \ \sum_{n \in N_j} y_{jn} \leq 1, \ \forall j \in J$.
- The bilevel program can still be reformulated as a single level and the B&C algorithm still works.

S-CFLP: Model Extensions II

Outside competitors:

- When introducing > 2 competitors to the model, the leader and the follower's total market shares do not add up to 1, and therefore we do not have a max-min equivalence of the bilevel program.
- We can show

$$\max_{x \in \mathcal{X}} \min_{y \in \mathcal{Y}(x)} L^{+}(x,y) \leq z^{\circ} \leq \max_{x \in \mathcal{X}} \min_{y \in \mathcal{Y}(x)} \left\{1 - F^{+}(x,y)\right\}.$$

  where

$$F^{+}(x,y) = \sum_{i \in I} h_i \left(\frac{U_i^{\mathrm{F}} + \sum_{j \in J} w_{ij} y_j}{U_i^{\mathrm{L}} + U_i^{\mathrm{F}} + \sum_{j \in J} w_{ij}(x_j + y_j) + U_i^{\circ}} \right).$$

- The two max-min upper- and lower-bounds can be computed via the B&C algorithm.

Outline

Instances Overview

- CPLEX 12.6; C++; PC with Intel CORE (TM) i7-8550 1.8GHz CPU, 16G RAM running 64-bit Windows 10.
- Consider a $[0, 50] \times [0, 50]$ square on a planar surface.
- locations of demand and candidate facility sites are randomly generated.
- MNL model with Euclidean distances, $\alpha_j = 0$ and $\beta = 0.1$ in the base setting.
- $J^{\mathrm{L}} = J^{\mathrm{F}} = \emptyset$. (No existing facilities built by either leader or follower)

Table: Parameter settings and instant sizes in state-of-the-art literature

Reference	$\mid I \mid$	$\mid J \mid$	p	r
Küçükaydn et al. (2011)	30	5	*	0
Küçükaydn et al. (2012)	100	20	*	*
Roboredo and Pessoa (2013)	100	50	4	4
Alekseeva et al. (2015)	100	100	20	20
Gentile et al. (2018)	225	225	5	5

Note: except for Küçükaydn et al. (2011) , none of the above can guarantee optimal solutions. We later challenge to solve up to 100 facilities and 2000 customer locations to global optimum.

Outline

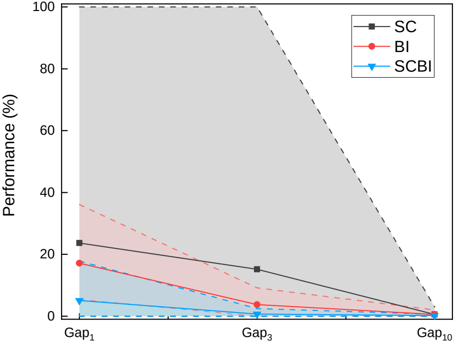
B&C with Cuts I

Table: Effectiveness of valid inequalities used in the B&C algorithm

Instance	SC			BI			SCBI			Enumeration
	Time(s)	#Cuts	#Nodes	Time(s)	#Cuts	#Nodes	Time(s)	#Cuts	#Nodes	Time(s)
20-20-2-2	0.94	38	129	2.14	33	88	**0.75**	50	110	3.47
20-20-3-2	1.66	76	450	3.09	59	361	**0.97**	83	311	19.58
20-20-2-3	2.00	58	151	2.42	44	117	**1.30**	73	120	5.48
40-40-2-2	13.23	220	839	4.06	56	576	**3.75**	114	514	46.42
40-40-3-2	68.09	1192	6339	15.28	196	2155	**11.16**	369	2323	496.77
40-40-2-3	64.44	335	964	20.88	60	448	**11.02**	115	538	146.52
60-60-2-2	69.95	514	1694	29.36	116	1172	**9.67**	124	1527	220.33
60-60-3-2	777.56	5631	33312	79.88	334	4006	**39.33**	549	11678	3630.78
60-60-2-3	640.11	755	2290	211.22	121	1268	**94.92**	156	1462	1399.27
80-80-2-2	353.55	1240	3601	65.49	122	2119	**25.75**	175	3134	817.75
80-80-3-2	13655.10	15236	93558	147.78	345	10278	**146.78**	941	24219	LIMIT
80-80-2-3	5181.42	1538	4480	384.99	142	2421	**228.08**	207	3387	6989.31
100-100-2-2	636.63	1573	5741	57.97	89	2834	**44.95**	176	3656	2087.59
100-100-3-2	13418.00	22628	155384	233.02	323	7972	**190.53**	772	33046	LIMIT
100-100-2-3	5469.91	2143	6943	384.00	105	3124	**273.86**	194	4066	LIMIT
Average	2690.17	3545	21058	109.44	143	2596	**72.19**	273	6006	N/A

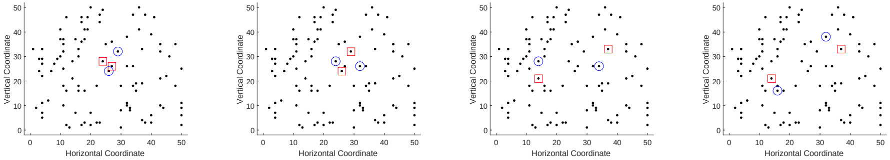
B&C with Cuts II

"$\texttt{Gap}_1$", "$\texttt{Gap}_3$", "$\texttt{Gap}_{10}$": the gaps between the final optimal obj and the best integer solution's obj found after the 1st, 3rd and 10th `lazy callbacks` in CPLEX.



Solid line: average of the gaps of all instances solved by different valid inequality combinations.

Sensitivity Analysis: Effects of β and $|I|$ I



(a) $\beta = 0.05$ (b) $\beta = 0.08$ (c) $\beta = 0.1$ (d) $\beta = 0.2$

• Customer · Leader's facility · Follower's facility

The optimal locations are clustered when β is small (i.e., lower spatial impedance effect) and spread out when β increases. The follower tends to locate its facilities near the leader's (see (a)-(c)), demonstrating the economies of agglomeration.

Sensitivity Analysis: Effects of β and $|I|$ II

Table: Computational results of instances $|I|$-100-2-2 with varying $|I|$

$\lvert I \rvert$	20	40	60	80	100	200	400	800	1200	1600	2000
Time(s)	19.00	49.52	29.23	54.80	44.95	65.69	85.70	130.50	284.80	254.91	279.97
Obj	0.5021	0.5004	0.5013	0.5003	0.5014	0.5011	0.5007	0.5009	0.5001	0.5002	0.5072
#Cuts	161	224	135	225	176	197	184	176	228	187	164

Outline

Motivation: Location-dependent demand

- Customers rent cars for a certain amount of time by
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- **Increase number of stations** ⇒ Increased service availability and convenience ⇒ Higher customer confidence ⇒ **Higher demand**

FLP with Endogenous Demand

Consider FLP where stochastic demand depends on location choices.

- How can we model the **impact of the facility location decisions** on the **customer demand distribution**? (Moment Ambiguity Set)
- How can we strategically **determine** the facility locations under **decision-dependent** demand uncertainty with **partial information**? (Distributionally Robust Optimization)

Outline

Ways to describe uncertainty under partial information

- Distance-based ambiguity sets: ϕ-divergence (Jiang and Guan, 2015), Wasserstein distance (Esfahani and Kuhn, 2017; Gao and Kleywegt, 2016)
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Under facility location decisions y with finite support $\{d_1, \cdots, d_K\}$, we define the ambiguity set for the set of customers J

$$
\begin{aligned}
U(y) = \{\pi_j \in \mathbb{R}_+^{|K|} : & \sum_{k=1}^{K} \pi_{jk} = 1 \quad \forall j \in J, \\
& \left| \sum_{k=1}^{K} \pi_{jk} d_k - \mu_j(y) \right| \leq \epsilon_j^{\mu} \quad \forall j \in J, \\
& (\sigma_j^2(y) + (\mu_j(y))^2)\underline{\epsilon}_j^{\sigma} \leq \sum_{k=1}^{K} \pi_{jk} d_k^2 \leq (\sigma_j^2(y) + (\mu_j(y))^2)\overline{\epsilon}_j^{\sigma} \quad \forall j \in J\}
\end{aligned}
$$

- Parameters ϵ_j^{μ}, $\underline{\epsilon}_j^{\sigma}$, $\overline{\epsilon}_j^{\sigma}$ for adjusting the robustness level

Decision-dependent moment information

- How to define the mean $\mu_j(y)$ and variance $\sigma_j^2(y)$?
  - Mean increases by the opening of close-by facilities until a market threshold value μ_j^{UB}.
  - Variance decreases by the opening of close-by facilities until an inherent uncertainty value of market $(\sigma_j^{LB})^2$.

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$$\mu_j(y) = \min \left\{ \bar{\mu}_j(1 + \sum_{j' \in I} \lambda_{jj'}^{\mu} y_{j'}), \mu_j^{UB} \right\}$$

$$\sigma_j^2(y) = \max \left\{ \bar{\sigma}_j^2(1 - \sum_{j' \in I} \lambda_{jj'}^{\sigma} y_{j'}), (\sigma_j^{LB})^2 \right\}$$

  - $\bar{\mu}_j$ and $\bar{\sigma}_j^2$ can be estimated from historical data.
  - λ_j parameters for adjusting the effect of the locations.

Outline

Optimization model

- Minimize expected total cost $-$ revenue
  - under any demand distribution in $U(y)$;
  - with respect to demand satisfaction and capacity constraints.

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$$\min_{y \in \mathcal{Y} \subseteq \{0,1\}^{|I|}} \left\{ \sum_{i \in I} f_i y_i + \max_{\pi \in U(y)} \mathrm{E}_\pi [h(y, d(y))] \right\}, \tag{5}$$

where

$$h(y, d(y)) = \min_{x,s} \quad \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \sum_{j \in J} (p_j s_j - r_j d_j(y)) \tag{6a}$$

$$\text{s.t.} \quad \sum_{i \in I} x_{ij} + s_j = d_j(y) \quad \forall j \in J \tag{6b}$$

$$x_{ij} \leq C_i y_i \quad \forall i \in I, j \in J \tag{6c}$$

$$s_i, x_{ij} \geq 0 \quad \forall i \in I, j \in J. \tag{6d}$$

Tractable reformulation

Proposition 1 (Basciftci, Ahmed, S. (2021))

The optimal objective function value of the problem (6) can be computed as

$$h(y, d(y)) = \sum_{j \in J} \left(\max_{i^*=0,1,\cdots,|I|} \left\{ c_{i^*j} d_j(y) + \sum_{i \in I : c_{ij} < c_{i^*j}} C_i y_i (c_{ij} - c_{i^*j}) \right\} - r_j d_j(y) \right),$$

where $c_{0j} := p_j$.

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- Apply duality

  $\Rightarrow$ Single stage mixed-integer nonlinear program.
- Assume that market capacity is large enough to omit μ_j^{UB} and $(\sigma_j^{LB})^2$.
- Still nonlinear due to bilinear and trilinear terms.
- Apply McCormick envelopes to obtain exact convex reformulations.

Optimization model reformulation

Theorem 2 (Basciftci, Ahmed, S. (2021))

Problem (5) is equivalent to problem (7).

$$\min \quad f^\top y + \sum_{j \in J} \Big(\alpha_j + \delta_j^1(\bar{\mu}_j + \epsilon_j^\mu) - \delta_j^2(\bar{\mu}_j - \epsilon_j^\mu) + \bar{\mu}_j \sum_{j' \in I} \lambda_{jj'}^\mu (\Delta_{jj'}^1 - \Delta_{jj'}^2) +$$

$$(\bar{\sigma}_j^2 + \bar{\mu}_j^2)(\bar{\epsilon}_j^\sigma \gamma_j^1 - \underline{\epsilon}_j^\sigma \gamma_j^2) + \sum_{j' \in I} \Lambda_{jj'} (\bar{\epsilon}_j^\sigma \Gamma_{jj'}^1 - \underline{\epsilon}_j^\sigma \Gamma_{jj'}^2) + 2\bar{\mu}_j^2 \sum_{l=1}^{|I|} \sum_{m=1}^{l-1} \lambda_{jl}^\mu \lambda_{jm}^\mu (\bar{\epsilon}_j^\sigma \Psi_{jlm}^1 - \underline{\epsilon}_j^\sigma \Psi_{jlm}^2) \Big) \tag{7a}$$

$$\text{s.t.} \quad \alpha_j + (\delta_j^1 - \delta_j^2) d_k + (\gamma_j^1 - \gamma_j^2) d_k^2 \geq (c_{i^*j} - r_j) d_k + \sum_{i \in I : c_{ij} < c_{i^*j}} C_i y_i (c_{ij} - c_{i^*j})$$

$$\forall i^* \in I \cup \{0\}, j \in I, k = 1, \cdots, K \tag{7b}$$

$$(\Delta_{jj'}^h, \delta_j^h, y_{j'}) \in M^q_{(0, \overline{\delta_j^h})}, (\Gamma_{jj'}^h, \gamma_j^h, y_{j'}) \in M^q_{(0, \overline{\gamma_j^h})} \quad \forall j \in J, j' \in I, h = 1, 2 \tag{7c}$$

$$(\Psi_{jlm}^h, \gamma_j^h, y_l, y_m) \in M^t_{(0, \overline{\gamma_j^h})} \quad \forall j \in J, l = 1, \ldots, |I|, l > m \tag{7d}$$

$$y \in \mathcal{Y} \subseteq \{0,1\}^{|I|}, \delta_j^1, \gamma_j^1, \delta_j^2, \gamma_j^2 \geq 0 \quad \forall j \in J. \tag{7e}$$

Optimization model reformulation

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$$
\begin{aligned}
\min \quad & f^\top y + \sum_{j \in J} \Big(\alpha_j + \delta_j^1(\bar{\mu}_j + \epsilon_j^\mu) - \delta_j^2(\bar{\mu}_j - \epsilon_j^\mu) + \bar{\mu}_j \sum_{j' \in I} \lambda_{jj'}^\mu (\Delta_{jj'}^1 - \Delta_{jj'}^2) + \\
& (\bar{\sigma}_j^2 + \bar{\mu}_j^2)(\bar{\epsilon}_j^\sigma \gamma_j^1 - \underline{\epsilon}_j^\sigma \gamma_j^2) + \sum_{j' \in I} \Lambda_{jj'} (\bar{\epsilon}_j^\sigma \Gamma_{jj'}^1 - \underline{\epsilon}_j^\sigma \Gamma_{jj'}^2) + 2\bar{\mu}_j^2 \sum_{l=1}^{|I|} \sum_{m=1}^{l-1} \lambda_{jl}^\mu \lambda_{jm}^\mu (\bar{\epsilon}_j^\sigma \Psi_{jlm}^1 - \underline{\epsilon}_j^\sigma \Psi_{jlm}^2) \Big) && \text{(7a)} \\
\text{s.t.} \quad & \alpha_j + (\delta_j^1 - \delta_j^2) d_k + (\gamma_j^1 - \gamma_j^2) d_k^2 \geq (c_{i^* j} - r_j) d_k + \sum_{i \in I : c_{ij} < c_{i^* j}} C_i y_i (c_{ij} - c_{i^* j}) \\
& \qquad \forall i^* \in I \cup \{0\}, j \in I, k = 1, \cdots, K && \text{(7b)} \\
& (\Delta_{jj'}^h, \delta_j^h, y_{j'}) \in M^q_{(0, \overline{\delta_j^h})}, (\Gamma_{jj'}^h, \gamma_j^h, y_{j'}) \in M^q_{(0, \overline{\gamma_j^h})} \quad \forall j \in J, j' \in I, h = 1, 2 && \text{(7c)} \\
& (\Psi_{jlm}^h, \gamma_j^h, y_l, y_m) \in M^t_{(0, \overline{\gamma_j^h})} \quad \forall j \in J, l = 1, \ldots, |I|, l > m && \text{(7d)} \\
& y \in \mathcal{Y} \subseteq \{0,1\}^{|I|}, \delta_j^1, \gamma_j^1, \delta_j^2, \gamma_j^2 \geq 0 \quad \forall j \in J. && \text{(7e)}
\end{aligned}
$$

$\Rightarrow$ Single-stage mixed-integer linear program (MILP)

Valid Inequalities

Can we further improve the formulation?

Valid Inequalities

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Proposition 2 (Basciftci, Ahmed, S. (2021))

The following inequalities are valid for the problem (5):

$$d_{(1)}d_{(2)} - (d_{(1)} + d_{(2)})(\mu_j(y) - \epsilon_j^{\mu}) + (\sigma_j^2(y) + (\mu_j(y))^2)\overline{\epsilon}_j^{\sigma} \geq 0 \quad \forall j \in J$$

$$d_{(K-1)}d_{(K)} - (d_{(K-1)} + d_{(K)})(\mu_j(y) - \epsilon_j^{\mu}) + (\sigma_j^2(y) + (\mu_j(y))^2)\overline{\epsilon}_j^{\sigma} \geq 0 \quad \forall j \in J$$

$$-d_{(1)}d_{(K)} + (d_{(1)} + d_{(K)})(\mu_j(y) + \epsilon_j^{\mu}) - (\sigma_j^2(y) + (\mu_j(y))^2)\underline{\epsilon}_j^{\sigma} \geq 0 \quad \forall j \in J$$

Outline

Numerical Studies

Compare the facility location decisions of three approaches:

1. Decision-Dependent Distributionally Robust (DDDR)
2. Distributionally Robust (DR)
3. Stochastic Programming (SP)*

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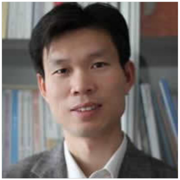
Average objective function and unmet demand values under 10 instance with different sizes:

	$\|I\|$	SP (20)	SP (100)	DR	**DDDR**
average objective	10	−63281.7	−63337.3	−67084.4	−75164.8
	9	−48247.1	−48790.2	−51503.5	−59816.9
	8	−40914.9	−39206.7	−35810.4	−48027.2
	7	−28201.8	−27810.6	−24911.6	−36608.5
	6	−21460.7	−21483.7	−16425.1	−30298.1
	5	−12806.7	−12763.3	−7618.55	−22554.2
average unmet demand	10	47.0	46.9	11.1	**0.4**
	9	58.1	53.9	38.7	**0.2**
	8	56.8	67.8	82.1	**0.1**
	7	71.7	75.1	86.2	**0.0**
	6	75.7	74.7	110.6	**3.9**
	5	95.4	95.9	128.2	**15.1**

*SP is trained with Normal distribution. All approaches are evaluated over test scenarios with Normal distribution.

Acknowledgement

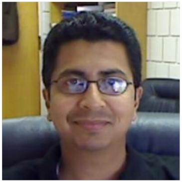
Collaborators:



Mingyao Qi (Tsinghua)

Ruiwei Jiang (UMich)

Beste Basciftci (U of Iowa)



Shabbir Ahmed (GT)

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Thank you!

Questions?