Solving 0-1 Semidefinite Programs for Distributionally Robust Allocation of Surgery Blocks

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Outline

Allocation of Surgery Blocks

Operating rooms (ORs):

- **40%** of a hospital's total revenues; BUT, a **similarly large** proportion of its total expenses[1]
- Average OR runs at only **68%** capacity[1]
- Uncertain service duration of surgical procedure

[1] Healthcare Financial Management Association 2003

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Works on allocation of surgery blocks:

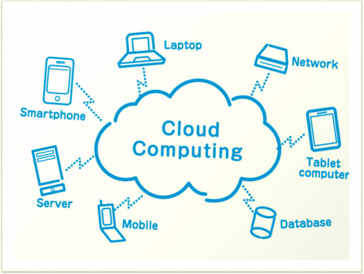
- Blake and Donald (2002): MILP
- Denton, Miller, Balasubramanian, and Huschka (2010): two-stage stochastic integer program
- Shylo, Prokopyev, and Schaefer (2012): chance-constrained formulation
- Deng, Shen, and Denton (2016): distributionally robust formulation
- ...

[1] Healthcare Financial Management Association 2003

Applications

Applications with similar settings (bin packing structure):

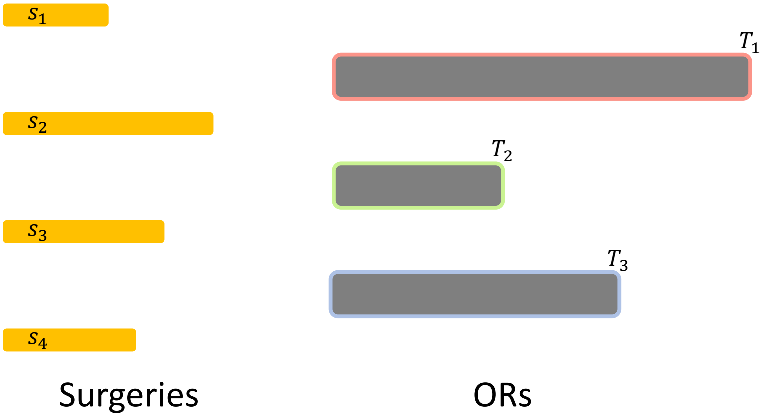
- Cloud computing server planning: uncertain job hours requested
  - Shen and Wang (2014)
- Machine scheduling: uncertain task duration
  - Skutella and Uetz (2005)



cloudcomputingcafe.com

theideasmith.net

Stochastic OR Allocation Problem



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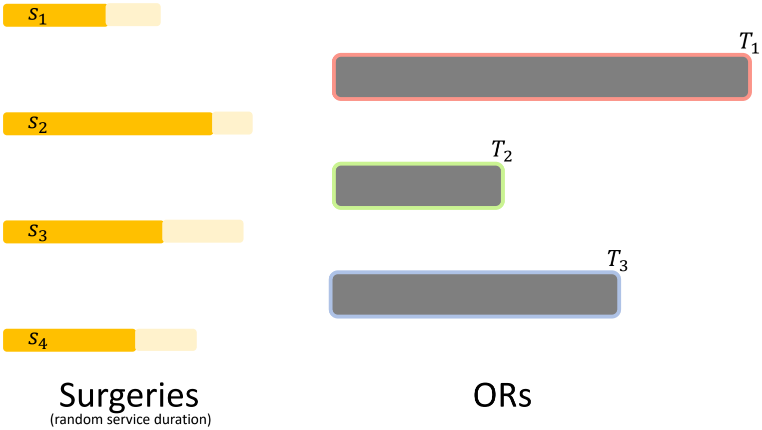

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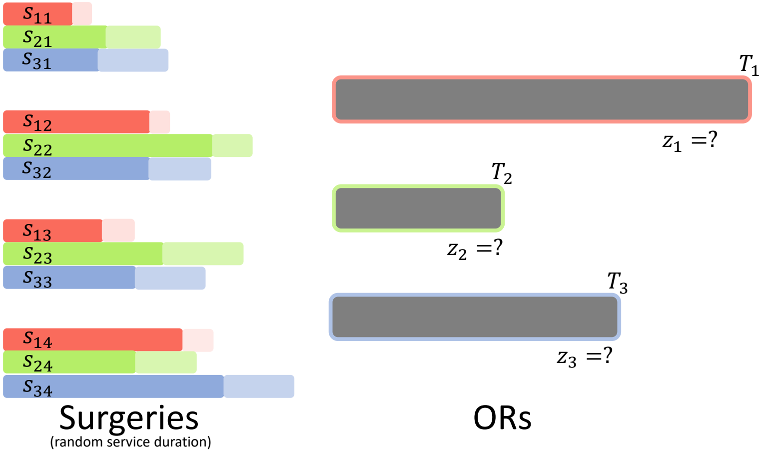

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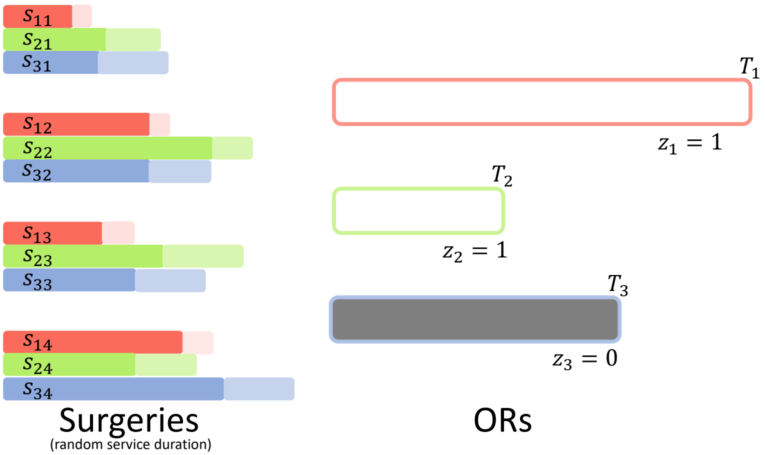
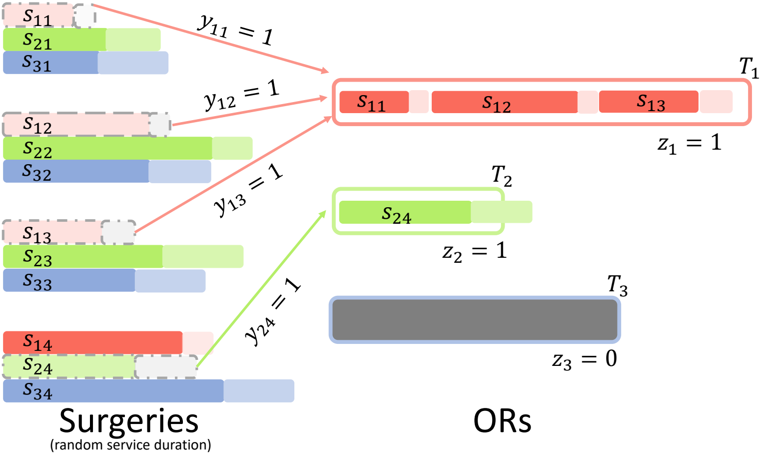


Decisions:

- $z_i \in \{0, 1\}$: $z_i = 1$ if we open OR i, and $= 0$ if not.

Stochastic OR Allocation Problem



Decisions:

- $z_i \in \{0, 1\}$: $z_i = 1$ if we open OR i, and $= 0$ if not.
- $y_{ij} \in \{0, 1\}$: $y_{ij} = 1$ if allocate surgery j to OR i

A Chance-Constrained Formulation

Let $s_i = [s_{ij},\ j \in J]^\mathsf{T}$, $y_i = [y_{ij},\ j \in J]^\mathsf{T}$

$$\min_{\mathbf{z},\mathbf{y}} \quad \sum_{i \in I} c_i^z z_i + \sum_{i \in I} \sum_{j \in J} c_{ij}^y y_{ij}$$

- Objective: Minimize the cost of opening ORs

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\text{s.t.} \quad & y_{ij} \le \rho_{ij} z_i \quad \forall i \in I,\ j \in J \\
& \sum_{i \in I} y_{ij} = 1 \quad \forall j \in J \\
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- Objective: Minimize the cost of opening ORs
- Deterministic constraints: Feasible surgery allocation

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& \mathbb{P}_{f_s} \left\{ s_i^\mathsf{T} y_i \le T_i \right\} \ge 1 - \alpha_i,\ \forall i \in I
\end{aligned}
$$

- Objective: Minimize the cost of opening ORs
- Deterministic constraints: Feasible surgery allocation
- Chance constraint: "Total operating time $\le$ time available in OR i" at $1 - \alpha_i$ probability, given the distribution f_s

Outline

Distributionally Robust (DR) Model

$$\min_{\mathbf{z},\mathbf{y}} \quad \sum_{i \in I} c_i^z z_i + \sum_{i \in I} \sum_{j \in J} c_{ij}^y y_{ij} \tag{2}$$

$$\text{s.t.} \quad y_{ij} \le \rho_{ij} z_i \quad \forall i \in I,\ j \in J \tag{3}$$

$$\sum_{i \in I} y_{ij} = 1 \quad \forall j \in J \tag{4}$$

$$y_{ij}, z_i \in \{0, 1\} \quad \forall i \in I,\ j \in J \tag{5}$$

$$\inf_{f_s \in \mathcal{D}_i} \mathbb{P}_f \left\{ s_i^{\mathsf{T}} y_i \le T_i \right\} \ge 1 - \alpha_i,\ \forall i \in I \tag{6}$$

- (6): The worst-case probability given by any $f_s \in D_i$ is guaranteed at least $1 - \alpha_i$ (a DR chance constraint).

Literature Review

Distributionally robust optimization

- Scarf, Arrow, and Karlin (1958); Delage and Ye (2010); Bertsimas, Doan, Natarajan, and Teo (2010); Goh and Sim (2010), Wiesemann, Kuhn, and Sim (2014), Esfahani and Kuhn (2016)...

Distributionally robust chance-constrained programming

- Zymler, Kuhn, and Rustem (2013); Jiang and Guan (2015)

Jointly chance-constrained binary packing

- Song, Luedtke, and Küçükyavuz (2014)

DR chance-constrained knapsack/bin packing

- Zhang, Denton, and Xie (2015): mean + variance
- Wagner (2008): mean + covariance
- Cheng, Delage, and Lisser (2014): mean + covariance

Outline

Moment-based Ambiguity Set

- Ambiguity set (Delage and Ye, 2010):

$$\mathcal{D}_i = \mathcal{D}_i^M(\mu_i^0, \Sigma_i^0, \gamma_1, \gamma_2) = \left\{ f(s_i) : \begin{array}{l} \int_{s_i \in \Xi_i^*} f(s_i) ds_i = 1 \\ (\mathbb{E}[s_i] - \mu_i^0)^\top (\Sigma_i^0)^{-1} (\mathbb{E}[s_i] - \mu_i^0) \leq \gamma_1 \\ \mathbb{E}[(s_i - \mu_i^0)(s_i - \mu_i^0)^\top] \preceq \gamma_2 \Sigma_i^0 \end{array} \right\}$$

$^*\Xi_i = \mathbb{R}^{|J|}$

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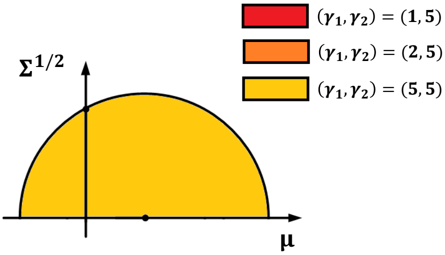
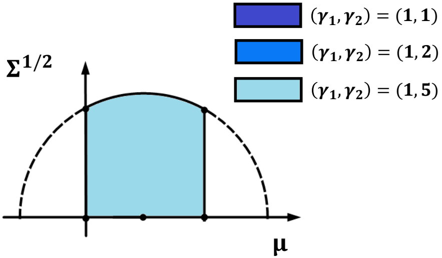
- decrease γ_1 with fixed γ_2

$\gamma_1 = 5$

- decrease γ_2 with fixed γ_1

$\gamma_2 = 5$





$^*(\mu, \Sigma)$: **True** mean and covariance pair

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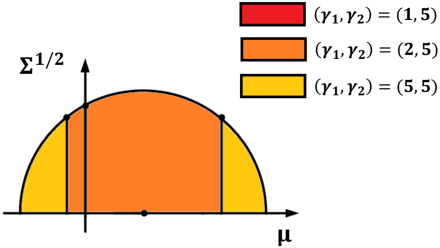
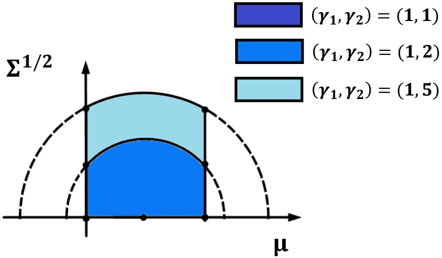
- decrease γ_1 with fixed γ_2

$\gamma_1 = 2$



- decrease γ_2 with fixed γ_1

$\gamma_2 = 2$



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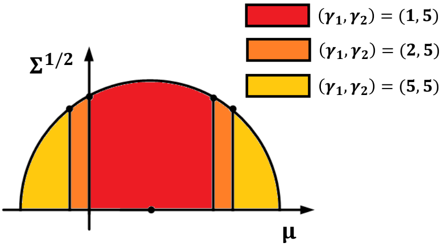
$^*\Xi_i = \mathbb{R}^{|J|}$

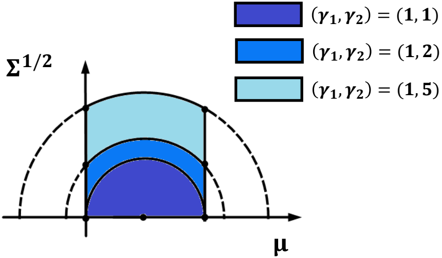
- decrease γ_1 with fixed γ_2

  $\gamma_1 = 1$

- decrease γ_2 with fixed γ_1

  $\gamma_2 = 1$





$^*(\mu, \Sigma)$: **True** mean and covariance pair

Outline

0-1 SDP Reformulation with $\mathcal{D}_i = \mathcal{D}_i^M$

Jiang and Guan (2015) show that

- By introducing the dual variables, the DR chance constraints (6) $\Leftrightarrow$ SDP constraints (exact):

$$\gamma_2 \Sigma_i^0 \cdot G_i + 1 - r_i + \Sigma_i^0 \cdot H_i + \gamma_1 q_i - \alpha_i \lambda_i \leq 0 \tag{7a}$$

$$\begin{bmatrix} G_i & -p_i \\ -p_i^{\mathsf{T}} & 1 - r_i \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{2} y_i \\ \frac{1}{2} y_i^{\mathsf{T}} & \lambda_i + y_i^{\mathsf{T}} \mu_i^0 - T_i z_i \end{bmatrix} \succeq 0 \tag{7b}$$

$$\begin{bmatrix} G_i & -p_i \\ -p_i^{\mathsf{T}} & 1 - r_i \end{bmatrix} \in \mathbb{S}_+^{(|J|+1)\times(|J|+1)}, \ \begin{bmatrix} H_i & p_i \\ p_i^{\mathsf{T}} & q_i \end{bmatrix} \in \mathbb{S}_+^{(|J|+1)\times(|J|+1)},$$

$$\lambda_i \geq 0. \tag{7c}$$

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$$\lambda_i \geq 0. \tag{7c}$$

However, 0-1 SDP CANNOT be directly solved in solvers.

Outline

Master Problem

Recall the DR model:

$$
\begin{aligned}
\min_{\mathbf{z},\mathbf{y}} \quad & \sum_{i\in I} c_i^z z_i + \sum_{i\in I}\sum_{j\in J} c_{ij}^y y_{ij} \\
\text{s.t.} \quad & y_{ij} \le \rho_{ij} z_i \quad \forall i\in I,\ j\in J \\
& \sum_{i\in I} y_{ij} = 1 \quad \forall j \in J \\
& y_{ij}, z_i \in \{0,1\} \quad \forall i\in I,\ j\in J \\
& \inf_{f_s\in\mathcal{D}_i} \mathbb{P}_{f_s}\left\{ s_i^{\mathsf{T}} y_i \le T_i \right\} \ge 1-\alpha_i,\ \forall i\in I
\end{aligned}
$$

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& \sum_{i\in I} y_{ij} = 1 \quad \forall j \in J \\
& y_{ij}, z_i \in \{0,1\} \quad \forall i \in I,\ j \in J \\
& \mathcal{C}_i^\ell y_i \leq c_i^\ell z_i,\ \ell = 1,\ldots,k_i,\ i \in I
\end{aligned}
\tag{8}
$$

- (8): set of linear cuts with OR i, $i \in I$

Subproblem

- Given a solution $(\hat{y}_i, \hat{z}_i)$ from the master problem

$$
\begin{aligned}
&\gamma_2 \Sigma_i^0 \cdot G_i + 1 - r_i + \Sigma_i^0 \cdot H_i + \gamma_1 q_i - \alpha_i \lambda_i \leq 0 \\
&\begin{bmatrix} G_i & -p_i \\ -p_i^\mathsf{T} & 1 - r_i \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{2}\hat{y}_i \\ \frac{1}{2}\hat{y}_i^\mathsf{T} & \lambda_i + \hat{y}_i^\mathsf{T} \mu_i^0 - T_i \hat{z}_i \end{bmatrix} \succeq 0 \\
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&\lambda_i \geq 0.
\end{aligned}
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$$\lambda_i \geq 0.$$

Is it feasible for the SDP constraints?

Subproblem

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$$
\begin{aligned}
V_P = \min \quad & \gamma_2 \Sigma_i^0 \cdot G_i + 1 - r_i + \Sigma_i^0 \cdot H_i + \gamma_1 q_i - \alpha_i \lambda_i \quad \leq 0 \\
\text{s.t.} \quad & \begin{bmatrix} G_i & -p_i \\ -p_i^\mathsf{T} & 1 - r_i \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{2}\hat{y}_i \\ \frac{1}{2}\hat{y}_i^\mathsf{T} & \lambda_i + \hat{y}_i^\mathsf{T} \mu_i^0 - T_i \hat{z}_i \end{bmatrix} \succeq 0 \\
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& \begin{bmatrix} H_i & p_i \\ p_i^\mathsf{T} & q_i \end{bmatrix} \in \mathbb{S}_+^{(|J|+1)\times(|J|+1)}, \lambda_i \geq 0.
\end{aligned}
$$

Subproblem

- The **dual** of the SDP problem:

$$
\begin{aligned}
V_D = \max_{Q_i, d_i, u_i, v_i} \quad & \hat{y}_i^{\mathsf{T}} d_i + (\hat{y}_i^{\mathsf{T}} \mu_i^0 - T_i \hat{z}_i) u_i \quad \leq 0 && \text{(9a)} \\
\text{s.t.} \quad & \begin{bmatrix} \gamma_2 \Sigma_i^0 & v_i \\ v_i^{\mathsf{T}} & 1 \end{bmatrix} - \begin{bmatrix} Q_i & d_i \\ d_i^{\mathsf{T}} & u_i \end{bmatrix} \succeq 0 && \text{(9b)} \\
& u_i - \alpha_i \geq 0 && \text{(9c)} \\
& \begin{bmatrix} \Sigma_i^0 & -v_i \\ -v_i^{\mathsf{T}} & \gamma_1 \end{bmatrix} \succeq 0 && \text{(9d)} \\
& v_i \in \mathbb{R}^{|J|}, \ \begin{bmatrix} Q_i & d_i \\ d_i^{\mathsf{T}} & u_i \end{bmatrix} \in \mathbb{S}_+^{(|J|+1)\times(|J|+1)} && \text{(9e)}
\end{aligned}
$$

- **Strong duality holds**: $V_D = V_P \leq 0$

Subproblem

- The **dual** of the SDP problem:

$$V_D = \max_{Q_i, d_i, u_i, v_i} \quad \hat{y}_i^{\mathsf{T}} d_i + (\hat{y}_i^{\mathsf{T}} \mu_i^0 - T_i \hat{z}_i) u_i \quad \leq 0 \tag{9a}$$

$$\text{s.t.} \quad \begin{bmatrix} \gamma_2 \Sigma_i^0 & v_i \\ v_i^{\mathsf{T}} & 1 \end{bmatrix} - \begin{bmatrix} Q_i & d_i \\ d_i^{\mathsf{T}} & u_i \end{bmatrix} \succeq 0 \tag{9b}$$

$$u_i - \alpha_i \geq 0 \tag{9c}$$

$$\begin{bmatrix} \Sigma_i^0 & -v_i \\ -v_i^{\mathsf{T}} & \gamma_1 \end{bmatrix} \succeq 0 \tag{9d}$$

$$v_i \in \mathbb{R}^{|J|}, \ \begin{bmatrix} Q_i & d_i \\ d_i^{\mathsf{T}} & u_i \end{bmatrix} \in \mathbb{S}_+^{(|J|+1)\times(|J|+1)} \tag{9e}$$

- **Strong duality holds**: $V_D = V_P \leq 0$
- The linear **CUT**: $y_i^{\mathsf{T}} \hat{d}_i + (y_i^{\mathsf{T}} \mu_i^0 - T_i z_i) \hat{u}_i \quad \leq 0$
  The dual solution: $\hat{d}_i, \hat{u}_i$

Outline

0-1 SOCP Reformulation with $\mathcal{D}_i = \mathcal{D}_i^C$

- Exactly match the given μ_i^0 and Σ_i^0:

$$\mathcal{D}_i = \mathcal{D}_i^C(\mu_i^0, \Sigma_i^0) = \left\{ f(s_i) : \quad \begin{array}{l} \int_{s_i \in \Xi_i} f(s_i) ds_i = 1, \\ \mathbb{E}[s_i] = \mu_i^0 \\ \mathbb{E}[(s_i - \mu_i^0)(s_i - \mu_i^0)^\mathsf{T}] = \Sigma_i^0 \end{array} \right\}$$

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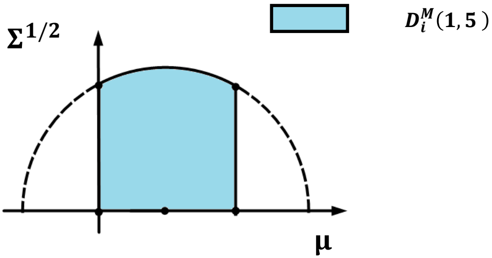

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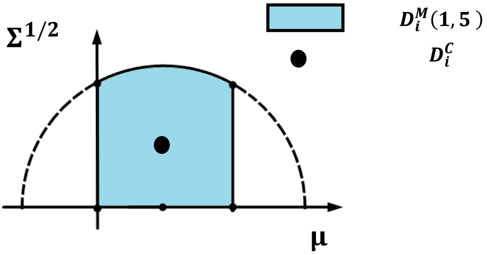
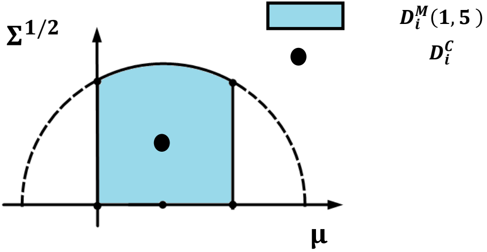


0-1 SOCP Reformulation with $\mathcal{D}_i = \mathcal{D}_i^C$

Following a variant of Chebyshev's inequality (Wagner, 2008), the DR chance constraint (6) is equivalent to

$$\sqrt{y_i^\mathsf{T} \Sigma_i^0 y_i} \leq \sqrt{\frac{\alpha_i}{1-\alpha_i}} \left(T_i - (\mu_i^0)^\mathsf{T} y_i \right), \ \forall i \in I \qquad (10)$$

That is, the DR model is equivalent to a 0-1 SOCP.



0-1 SOCP Reformulation with $\mathcal{D}_i = \mathcal{D}_i^C$

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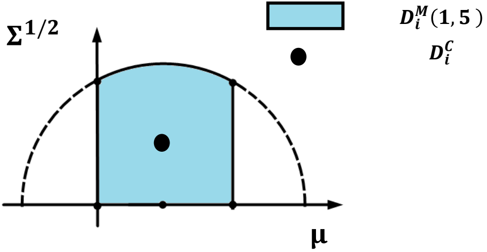

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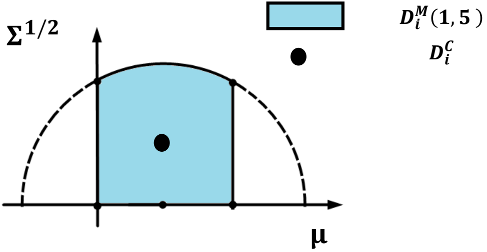

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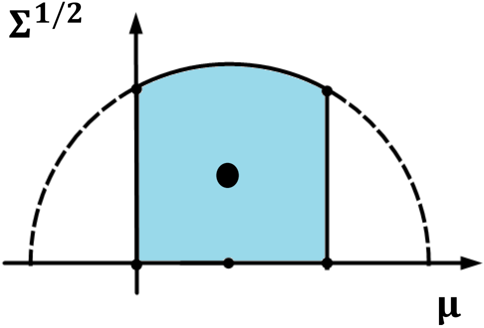

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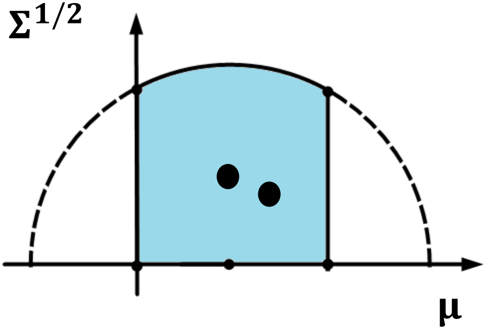

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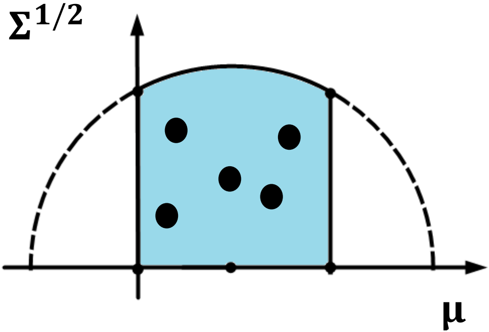

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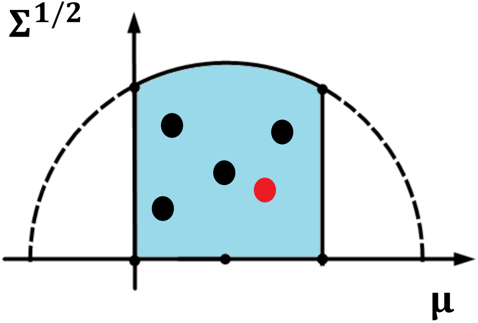

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- 0-1 SOC constraint:

$$\sqrt{\frac{1}{1-a-b}} \left(1 + \sqrt{\frac{\alpha_i b}{1-\alpha_i}} \right) \sqrt{y_i^\mathsf{T} \Sigma_i^0 y_i} \leq \sqrt{\frac{\alpha_i}{1-\alpha_i}} \left(T_i z_i - (\mu_i^0)^\mathsf{T} y_i \right)$$

Outline

Computational Setup

Approaches $(\gamma_1, \gamma_2) = (0, 1)$

- "Cutting-plane" approach
- "0-1 SOCP" approximation approach
- "MILP" –Sample Average Approximation approach

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  - In-sample: mix (8 hMhV, 8 hMℓV, 8 ℓMℓV, 8 ℓMhV)
  - Out-of-sample: hMhV

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Computational Results I

Table: CPU time (in second) and optimal solutions

$1-\alpha_i$	Approach	CPU (sec)	Obj. Cost	# of open ORs
95%	Cutting-plane	10.86	4.50	4
	0-1 SOCP	124.50	3.66	3
	MILP	107.47	2.95	2
90%	Cutting-plane	7.77	3.65	3
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- Reliability of each open OR i =

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Table: Average reliability performance in out-of-sample data with only "**hMhV**" surgeries

$1 - \alpha_i$	Approach	OR #1	OR #2	OR #3	OR #4
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	0-1 SOCP	**0.98**	**0.98**	N/A	**0.99**
	MILP	0.81	N/A	N/A	0.82
90%	Cutting-plane	0.96	0.98	N/A	0.99
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- Cutting-plane approach for 0-1 SDP reformulations
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