SECTIONS OF THE DIFFERENCE BODY

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Abstract. Let K be an n-dimensional convex body. Define the difference body by

$$K - K = \{x - y \mid x, y \in K\}.$$

We estimate the volume of the section of $K - K$ by a linear subspace F via the maximal volume of sections of K parallel to F. We prove that for any m-dimensional subspace F there exists $x \in \mathbb{R}^n$, such that

$$\operatorname{vol}((K-K) \cap F) \leq C^m \left(\min \left(\frac{n}{m}, \sqrt{m} \right) \right)^m \cdot \operatorname{vol}(K \cap (F + x)),$$

for some absolute constant C. We show that for small dimensions of F this estimate is exact up to a multiplicative constant.

1. Introduction.

Let K be an n-dimensional convex body. Define the difference body by

$$K - K = \{x - y \mid x, y \in K\}.$$

In 1957 Rogers and Shephard [R-S] proved that

$$\operatorname{vol}(K - K) \leq \binom{2n}{n} \operatorname{vol}(K).$$

A simpler proof was found later by Chakerian [C].

Let F be an m-dimensional linear subspace of $\mathbb{R}^n$ and let P_F be the orthogonal projection onto F. It follows from the inequality of Rogers and Shephard that

$$\left(\frac{\operatorname{vol}(P_F(K-K))}{\operatorname{vol}(P_F K)} \right)^{1/m} \leq \binom{2m}{m}^{1/m} < 4.$$

Here and later we denote by vol the volume in the relevant dimension.

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For some problems it would be interesting to obtain a similar estimate for the volumes of *sections* of $K-K$. In particular, would the expression

$$R(K,F)=\left(\frac{\operatorname{vol}\left((K-K)\cap F\right)}{\sup_{x\in\mathbb{R}^n}\operatorname{vol}\left(K\cap(F+x)\right)}\right)^{1/m}$$

be uniformly bounded?

Although, as it is shown below, the answer to this question is negative, some estimates of this ratio are possible. Our main result is the following

Theorem 1. *Let $K\subset\mathbb{R}^n$ be a convex body and let $F\subset\mathbb{R}^n$ be an m-dimensional subspace. Then*

$$\operatorname{vol}\left((K-K)\cap F\right)\le C^m\varphi^m(m,n)\cdot\sup_{x\in\mathbb{R}^n}\operatorname{vol}\left(K\cap(F+x)\right),$$

where

$$\varphi(m,n)=\min\left(\frac{n}{m},\sqrt{m}\right).$$

Here and later C denotes an absolute constant whose value may change from line to line.

This result can be applied to estimating the Banach – Mazur distance between two non-symmetric convex bodies. To use random rotations for such an estimate one has to put the bodies into some specific positions. This can be achieved by comparison of the positions of the difference body and the body itself. We are going to present the details in a separate paper.

It follows from Theorem 1 that $R(K,F)$ is bounded for m proportional to n and for a small m. This suggests that $R(K,F)$ should be bounded for all dimensions. Surprisingly, this is not the case. Namely, the following Theorem implies that for some body $K\subset\mathbb{R}^n$ and $F\subset\mathbb{R}^n$, $\dim(F)=m$

$$R(K,F)\ge c\sqrt{\log n}$$

when $c\log n\le m\le n^\alpha$ and $\alpha\in(0,1)$.

Theorem 2. *For any $m<n$ there exists a convex body $K\subset\mathbb{R}^n$ and a subspace $F\subset\mathbb{R}^n$ of dimension m such that for any $x\in\mathbb{R}^n$*

$$\operatorname{vol}\left((K-K)\cap F\right)\ge C^m\psi^m(m,n)\cdot\operatorname{vol}\left(K\cap(F+x)\right),$$

where

$$\psi(m,n)=\min\left(\sqrt{\log\left(\frac{n}{m}+1\right)},\sqrt{m}\right).$$

Notice that Theorem 2 implies that the estimate obtained in Theorem 1 is exact for $m\le c\log n$.

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2. Upper estimate.

The proof of Theorem 1 consists of two steps. First we reduce the problem to a question of comparing the volume of projection and the volume of parallel sections of a certain convex body. Then we use the Rogers – Shephard inequality and the John decomposition to complete the proof.

Denote by $V_m(D)$ the m-th intrinsic volume of a body D [S]. Consider the following integral

$$I(K,F) = \int_F V_m\big(K \cap (K+x)\big)dx.$$

To prove the Theorem we shall estimate $I(K,F)$ from above and from below.

For the lower estimate we apply the following Lemma due to Chakerian [C].

Lemma 1. *Let $B \subset \mathbb{R}^m$, $0 \in B$ be a convex body. Let $h : B \to \mathbb{R}$ be a non-negative concave function and let $f : \mathbb{R} \to \mathbb{R}$ be increasing. Then*

$$\int_B f(h(x))dx \geq m \cdot \mathrm{vol}\,(B) \cdot \int_0^1 f(t \cdot h(0))(1-t)^{m-1}dt. \qquad \square$$

For $x \in F$ let

$$h(x) = V_m^{1/m}\big(D_x(K)\big), \qquad f(t) = t^m,$$

where

$$D_x(K) = K \cap (K+x)$$

It follows from the Alexandrov – Fenchel inequality that the intrinsic volumes satisfy the General Brunn – Minkowski inequality. Namely, for any two bodies B, D and for any number $0 \leq \lambda \leq 1$

$$V_m^{1/m}(\lambda B + (1-\lambda)D) \geq \lambda V_m^{1/m}(B) + (1-\lambda)V_m^{1/m}(D) \tag{1}$$

[S, Th. 6.4.3, p.339]. Since for any $x, \bar{x}$

$$\lambda D_x(K) + (1-\lambda)D_{\bar{x}}(K) \subset D_{\lambda x + (1-\lambda)\bar{x}}(K),$$

it follows from (1) that $h(x)$ is a concave function. By Lemma 1,

$$\begin{aligned}\int_{(K-K)\cap F} h^m(x)\,dx &\geq m \cdot \mathrm{vol}\,((K-K)\cap F) \cdot \int_0^1 \big(t \cdot h(0)\big)^m \cdot (1-t)^{m-1}\,dt \\ &= \mathrm{vol}\,((K-K)\cap F) \cdot h^m(0) \cdot \binom{2m}{m}^{-1}.\end{aligned}$$

So, we get that

$$\begin{aligned}\mathrm{vol}\,((K-K)\cap F) &\leq \binom{2m}{m} \cdot V_m^{-1}(K) \cdot \int_{(K-K)\cap F} V_m\big(K \cap (K+x)\big)\,dx \\ &\leq 4^m \cdot V_m^{-1}(K) \cdot \int_F V_m\big(K \cap (K+x)\big)\,dx.\end{aligned} \tag{2}$$

To estimate $I(K,F)$ we apply Crofton's formula [S, formula (4.5.9), p. 235]. Let $\mathbb{A}(n,n-m)$ be the set of all $(n-m)$-dimensional affine subspaces of $\mathbb{R}^n$ and let μ be the Haar measure on $\mathbb{A}(n,n-m)$. By Crofton's formula, we get

$$V_m(K\cap(K+x)) = C_{n,m}\cdot\int_{\mathbb{A}(n,n-m)}\chi(K\cap(K+x)\cap E)\,d\mu(E),$$

where $C_{n,m}$ is a constant depending on n and m. By Fubini's theorem,

$$\begin{aligned}
I(K,F) &= \int_{(K-K)\cap F} V_m(K\cap(K+x))\,dx\\
&= C_{n,m}\cdot\int_F\int_{\mathbb{A}(n,n-m)}\chi(K\cap(K+x)\cap E)\,d\mu(E)\,dx\\
&= C_{n,m}\cdot\int_{\mathbb{A}(n,n-m)}\text{mes}\,\{x\in F\,|\,(K+x)\cap(K\cap E)\neq\emptyset\}\,d\mu(E), \qquad (3)
\end{aligned}$$

where mes is the Lebesgue measure on F. Let $\mathbb{A}_F$ be the set of all $(n-m)$-dimensional affine subspaces which are transversal to F:

$$\mathbb{A}_F = \{E\in\mathbb{A}(n,n-m)\mid \text{card}\,(E\cap F)=1\}.$$

Since $\mu(\mathbb{A}(n,n-m)\setminus\mathbb{A}_F)=0$, we can integrate in (3) only over $\mathbb{A}_F$. Then (3) can be estimated above by

$$\begin{aligned}
&C_{n,m}\cdot\int_{\mathbb{A}_F}\chi(K\cap E)\,d\mu(E)\cdot\sup_{E\in\mathbb{A}_F}\text{mes}\,\{x\in F\,|\,(K+x)\cap(K\cap E)\neq\emptyset\}\\
&= V_m(K)\cdot\sup_{E\in\mathbb{A}_F}\text{mes}\,\{x\in F\,|\,(K+x)\cap(K\cap E)\neq\emptyset\}.
\end{aligned}$$

To complete the proof of Theorem 1 we have to prove the following

Claim. *For any m-dimensional linear subspace $F\subset\mathbb{R}^n$ and any $(n-m)$-dimensional affine subspace $E\subset\mathbb{R}^n$, such that E and F intersect at one point only,*

$$\text{mes}\,\{x\in F\,|\,(K+x)\cap(K\cap E)\neq\emptyset\}\le C^m\varphi^m(m,n)\cdot\sup_{y\in\mathbb{R}^n}\text{vol}\,((K+y)\cap F)\,.$$

Proof of the Claim. Since the statement of the Claim is invariant under translations, we may assume that $E\cap F=\{0\}$. Also, let $T:\mathbb{R}^n\to\mathbb{R}^n$ be an invertible linear operator, such that $T|_F = id$ and $T|_E = F^\perp$. The Claim is invariant under T, so we may assume that E and F are orthogonal.

Define

$$Z = K\cap\big((K\cap E)+F\big).$$

Let P_E, P_F be orthogonal projections onto E and F respectively. We have

$$\begin{aligned}
&\text{mes}\,\{x\in F\,|\,(K+x)\cap(K\cap E)\neq\emptyset\}\\
&=\text{mes}\,\{x\in F\,|\,K\cap\big((K\cap E)-x\big)\neq\emptyset\}\\
&=\text{mes}\,\Big(P_F\big((K\cap(K\cap E)-F)\big)\Big) = \text{vol}\,(P_F(Z))\,.
\end{aligned}$$

By the construction of Z we have

$$Z \cap E \subset P_E Z \subset P_E\big((K \cap E) + F\big) = K \cap E = Z \cap E. \tag{4}$$

Since $Z \subset K$, and $\binom{n}{m} \leq e^m (n/m)^m$, it is enough to prove that

$$\mathrm{vol}\,(P_F(Z)) \leq \binom{n}{m} \cdot \sup_{y \in E} \mathrm{vol}\,((Z+y) \cap F) \tag{i}$$

and

$$\mathrm{vol}\,(P_F(Z)) \leq C^m m^{m/2} \cdot \sup_{y \in E} \mathrm{vol}\,((Z+y) \cap F)\,. \tag{ii}$$

Proof of (i). By (4),

$$\begin{aligned} \mathrm{vol}\,(Z) &\leq \mathrm{vol}\,(P_E Z) \cdot \sup_{y \in E} \mathrm{vol}\,((Z+y) \cap F) \\ &= \mathrm{vol}\,(Z \cap E) \cdot \sup_{y \in E} \mathrm{vol}\,((Z+y) \cap F)\,. \end{aligned} \tag{5}$$

From the other side, another inequality of Rogers and Shephard [R-S] implies that

$$\mathrm{vol}\,(Z) \geq \binom{n}{m}^{-1} \cdot \mathrm{vol}\,(P_F Z) \cdot \mathrm{vol}\,(Z \cap E)\,. \tag{6}$$

Now (i) follows from the combination of (5) and (6). $\square$

Remark. Using the inequality (6) of Rogers and Shephard in the proof of (i) leads to a gap between the upper and lower estimates of $\varphi(m,n)$. Although the Rogers and Shephard inequality is exact, it holds as an equality for the bodies of the form $Z = \mathrm{conv}(Z \cap E, Z \cap F)$, while for such bodies $P_F(Z) = Z \cap F$.

Proof of (ii). Without loss of generality we may assume that the ellipsoid of minimal volume containing $P_F Z$ is B_2^m. Then there exists a John's decomposition of the identity operator. Namely, there exist $M \leq (n+3)n/2$ contact points $x_1, \ldots, x_M \in S^{m-1} \cap Z$ and M positive numbers $c_1, \ldots, c_M$ satisfying the following system of equations

$$id = \sum_{i=1}^{M} c_i \, x_i \otimes x_i$$

$$0 = \sum_{i=1}^{M} c_i \, x_i.$$

Here by id we denote the identity operator in $\mathbb{R}^m$.

Since $x_i \in P_F Z$, we can choose the points $y_i \in P_E Z$ so that $x_i + y_i \in Z$. Define

$$u = \sum_{i=1}^{M} \frac{c_i}{m} \, y_i.$$

Since

$$\sum_{i=1}^{M} \frac{c_i}{m} = 1,$$

$u \in P_E Z = Z \cap E$. Notice that $y_1, \dots, y_M \in Z$. So,

$$\begin{aligned} Z \cap (F+u) &\supset \sum_{i=1}^{M} \frac{c_i}{m} \cdot \big(Z \cap (F+y_i)\big) \\ &\supset \sum_{i=1}^{M} \frac{c_i}{m} \cdot [y_i, y_i + x_i] = \left(\sum_{i=1}^{M} \frac{c_i}{m} \cdot [0, x_i] \right) + u \\ &= \frac{1}{2} \cdot \left(\sum_{i=1}^{M} \frac{c_i}{m} \cdot [-x_i, x_i] \right) + u. \end{aligned}$$

Here $\sum$ means the Minkowski sum and $[x, y]$ denotes the segment joining x and y. Put

$$W = \sum_{i=1}^{M} \frac{c_i}{m} \cdot [-x_i, x_i].$$

Then, by [B, Lemma 4], we have

$$\operatorname{vol}(W) \geq 2^m m^{-m},$$

so

$$\operatorname{vol}(P_F Z) \leq \operatorname{vol}(B_2^m) \leq C^m m^{m/2} \cdot \operatorname{vol}(W). \quad \square$$

Notice that $\varphi(m, n) \leq n^{1/3}$. So, we have the following immediate

Corollary. *Let $K \subset \mathbb{R}^n$ be a convex body and let $F \subset \mathbb{R}^n$ be an m-dimensional subspace. Then*

$$\operatorname{vol}((K - K) \cap F) \leq \left(C \cdot n^{1/3} \right)^m \cdot \sup_{x \in \mathbb{R}^n} \operatorname{vol}(K \cap (F + x)). \quad \square$$

3. Lower estimate.

We now turn to the proof of Theorem 2.

Assume first that $n - m + 1 \geq 5^m$. In this case we have to prove Theorem 2 for $\psi(m, n) = \sqrt{m}$. The assumption guarantees that one can find points $z_1, \dots, z_{n-m+1}$ on the unit sphere of F which form a $(1/2)$-net. Let $j_1, \dots, j_{n-m+1}$ be the vertices of the standard simplex in the space $F^\perp$. Put

$$K = \operatorname{conv}(j_1 \pm z_1, \dots, j_{n-m+1} \pm z_{n-m+1}).$$

Since

$$(K - K) \cap F \supset 2\operatorname{conv}(\pm z_1, \dots, \pm z_{n-m+1}) \supset B_2^m,$$

we have to prove that for any $x \in \operatorname{conv}(j_1, \ldots, j_{n-m+1})$

$$\operatorname{vol}(K \cap (F+x)) \leq \left(\frac{c}{\sqrt{m}}\right)^m \operatorname{vol}(B_2^m).$$

Assume that

$$x = \sum_{i=1}^{n-m+1} \lambda_i j_i,$$

where

$$\lambda_i \geq 0 \text{ and } \sum_{i=1}^{n-m+1} \lambda_i = 1.$$

Then

$$K \cap (F+x) = \sum_{i=1}^{n-m+1} \lambda_i [j_i - z_i, j_i + z_i] = \sum_{i=1}^{n-m+1} \lambda_i [-z_i, z_i] + x.$$

Let $T_1, \ldots, T_N \in O(m)$ be random rotations in $\mathbb{R}^m$. By the Brunn – Minkowski inequality

$$\begin{aligned} \operatorname{vol}(K \cap (F+x)) &\leq \operatorname{vol}\left(\frac{1}{N}\sum_{s=1}^{N} T_s((K-x)\cap F)\right) \\ &= \operatorname{vol}\left(\frac{1}{N}\sum_{s=1}^{N}\sum_{i=1}^{n-m+1} \lambda_i[-T_s z_i, T_s z_i]\right). \end{aligned} \tag{7}$$

For a sufficiently large N

$$\frac{1}{N}\sum_{s=1}^{N} \lambda_i[-T_s z_i, T_s z_i] \subset \frac{2}{\sqrt{m}} B_2^m$$

so (7) does not exceed

$$\operatorname{vol}\left(\frac{2}{\sqrt{m}}\sum_{i=1}^{n-m+1} \lambda_i B_2^m\right) = \operatorname{vol}\left(\frac{2}{\sqrt{m}} B_2^m\right).$$

Now assume that $n - m + 1 < 5^m$ and let k be the largest integer such that $5^k \cdot (m/k) \leq n - m + 1$. Since in this case $k \leq c\log(n/m+1)$, it is enough to prove Theorem 2 for $\psi(m,n) = \sqrt{k}$. We shall use a construction which is similar to [F-J, p. 96–97]. Assume for simplicity that $L = m/k$ is an integer. Let $e_1, \ldots, e_m$ be an orthonormal basis of F. For $l = 1, \ldots, L$ put

$$F_l = \operatorname{span}\{e_i \mid i = k(l-1)+1, \ldots, kl\}.$$

Let $z_1^l, \dots, z_{N_l}^l$ be an 1/2-net on the unit sphere of F_l. Since $5^k \cdot (m/k) \le n-m+1$, we may assume that the total number of elements in these nets is $n-m+1$. Let us reorder the sequences $\{z_i^l\}$ into one sequence $\{z_i\}_{i=1}^{n-m+1}$. Let $j_1, \dots, j_{n-m+1}$ be the vertices of the standard simplex in $F^\perp$. Define as before

$$K = \operatorname{conv}(j_1 \pm z_1, \dots, j_{n-m+1} \pm z_{n-m+1}).$$

Then we have

$$(K-K) \cap F \supset 2\operatorname{conv}(\pm z_1, \dots, \pm z_{n-m+1}).$$

Since the sequence $z_1, \dots, z_{n-m+1}$ contains the (1/2)-nets for the unit spheres of the spaces F_l,

$$(K-K) \cap F \supset \operatorname{conv}(B_2^m \cap F_1, \dots, B_2^m \cap F_L).$$

Put $B_l = B_2^m \cap F_l$. We have to prove that for any $x \in \operatorname{conv}(j_1, \dots, j_{n-m+1})$

$$\operatorname{vol}(K \cap (F+x)) \le \left(\frac{c}{\sqrt{k}}\right)^m \cdot \operatorname{vol}(\operatorname{conv}(B_1, \dots, B_L)). \tag{8}$$

Assume that

$$x = \sum_{i=1}^{n-m+1} \lambda_i j_i,$$

where

$$\lambda_i \ge 0 \text{ and } \sum_{i=1}^{n-m+1} \lambda_i = 1.$$

Then as before we have

$$K \cap (F+x) = \sum_{i=1}^{n-m+1} \lambda_i [-z_i, z_i] + x.$$

Let $T_1^l, \dots, T_M^l : F_l \to F_l$, be random rotations of F_l. Denote by I_l the set of indexes i for which $z_i \in F_l$. Then

$$\frac{1}{M} \sum_{s=1}^{M} \sum_{i \in I_l} \lambda_i [-T_s^l z_i, T_s^l z_i] \subset \frac{2}{\sqrt{k}} \mu_l B_l.$$

Here $\mu_l = \sum_{i \in I_l} \lambda_i$, so

$$\sum_{l=1}^{L} \mu_l = 1.$$

Arguing as before, we prove that

$$\begin{aligned}\mathrm{vol}\,(K \cap (F+x)) &\le \mathrm{vol}\left(\sum_{l=1}^{L} \frac{1}{M} \sum_{s=1}^{M} \sum_{i \in I_l} \lambda_i [-T_s^l z_i, T_s^l z_i]\right) \\ &\le \mathrm{vol}\left(\sum_{l=1}^{L} \frac{2}{\sqrt{k}} \mu_l B_l\right) \\ &= \left(\frac{2}{\sqrt{k}}\right)^m \cdot \left(\prod_{l=1}^{L} \mu_l\right)^k \cdot (\mathrm{vol}\,(B_1))^L .\end{aligned}$$

By the inequality between the arithmetic and the geometric mean,

$$\mathrm{vol}\,(K \cap (F+x)) \le \left(\frac{2}{\sqrt{k}}\right)^m \cdot L^{-kL} \cdot (\mathrm{vol}\,(B_1))^L . \tag{9}$$

To complete the proof we apply the following easy Lemma, which can be proved by induction.

Lemma 3. *Let* $\mathbb{R}^{kL} = F_1 \oplus \ldots \oplus F_L$*, where* $F_1, \ldots, F_L$ *are mutually orthogonal subspaces of dimension* k*. Let* $B_l = B_2^{kL} \cap F_l$*. Then*

$$\mathrm{vol}\,(\mathrm{conv}(B_1, \ldots, B_L)) = \frac{(k!)^L}{(kL)!} \Big[\mathrm{vol}\,(B_1)\Big]^L . \qquad \square$$

Remark. A generalization of this formula appears in [M].

Since

$$\frac{(k!)^L}{(kL)!} \le C^L \cdot L^{kL} = C^L \cdot L^m$$

the inequality (8) follows from (9) and the Lemma.

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